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Generalized mean transforms of EP and binormal operators

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Abstract *Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$, and $T = U|T|$ be its polar decomposition. For $\lambda \in [0, 1]$, the generalized mean transform of T is defined by*

$$\widehat{T}_\lambda = \frac{1}{2}(|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda).$$

The aim of this paper is to investigate the generalized mean transform of closed range operators, EP operators and binormal operators.

1 Introduction

let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* for the range, the null subspace and the adjoint operator of T, respectively. For any closed subspace M of $\mathcal{H}$, let P_M denote the orthogonal projection onto M.
An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, partial isometry if $TT^*T = T$, quasinormal if $[T, T^*T] = 0$ and binormal if $[TT^*, T^*T] = 0$, where $[A, B] = AB - BA$, for any operators A and B. We refer the reader to [3, 7, 14, 18], for more details about these classes of operators. Now, we recall the notion of the Moore-Penrose inverse that will be used in this work. For $T \in \mathcal{B}(\mathcal{H})$, the Moore-Penrose inverse of T is the unique operator $T^+ \in \mathcal{B}(\mathcal{H})$ which satisfies:

$$TT^+T = T,\ \ T^+TT^+ = T^+,\ \ (TT^+)^* = TT^+,\ \ (T^+T)^* = T^+T.$$

It is well known that the Moore-Penrose inverse of T exists if and only if $\mathcal{R}(T)$ is closed. In this case $\mathcal{R}(T^+) = \mathcal{R}(T^*)$, $TT^+ = P_{R(T)}$ and $T^+T = P_{R(T^*)}$. The operator T is said to be EP operator, if $\mathcal{R}(T)$ is closed and $TT^+ = T^+T$. Obviously, every normal operator with closed range is EP but the converse is not true even in a finite dimensional space.
For an operator $T \in \mathcal{B}(\mathcal{H})$, there is a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T and U is the associate partial isometry satisfying $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|)$. Then

$$UU^* = P_{\overline{R(T)}} = P_{\overline{R(|T^*|)}} \quad \text{and} \quad U^*U = P_{\overline{R(T^*)}} = P_{\overline{R(|T|)}}.$$

Moreover, $T^* = U^*|T^*|$ is the polar decomposition of T^* and the equality $U|T|^\alpha = |T^*|^\alpha U$ holds for any $\alpha \geq 0$. Related to the polar decomposition, the λ-Aluthge transform of $T \in \mathcal{B}(\mathcal{H})$ is defined by Okubo [16] as $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$, for $\lambda \in [0, 1]$. Notice that $\Delta_0(T) = T$ and $\Delta_{\frac{1}{2}}(T) = \Delta(T)$ is the classical Aluthge transform [1] and $\Delta_1(T) = |T|U$ is the Duggal's transform. One of the interests of λ-Aluthge transform lies in the fact that it respects many properties of the original operator. the λ-Aluthge transform is very useful, and many authors *have obtained results by using it (see for instance, [1, 10, 15, 16, 17, 20]). Later, the mean transform* $\widehat{T}$ *of the operator* T *was introduced by S.H. Lee et al in [12] and is given as*

$$\widehat{T} = \frac{1}{2}(U|T| + |T|U).$$

This definition has been generalized recently in [2]. For $\lambda \in [0,1]$, *the generalized mean transform* $\widehat{T}_\lambda$ *of* T *is defined by*

$$\widehat{T}_\lambda = \frac{1}{2}(|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda).$$

Especially, $\widehat{T}_0 = \widehat{T}$ *and* $\widehat{T}_{\frac{1}{2}} = \Delta(T)$. *In recent years, the relationship between operators on a Hilbert space and their mean transform have been investigated by many authors for example, S.H. Lee et al [12] showed that quasinormal operators are exactly the fixed points of the mean transform. Later, In [4, 5] Chabbabi and Mbekhta discussed when the mean transform of an operator in* $\mathcal{B}(\mathcal{H})$ *is invertible or self-adjoint or normal. Also they proved that the mean transform preservers the class of closed range operators [5]. The goal of this paper is to explore the generalized mean transform of EP operators and binormal operators.*
The paper is organized as follows. In section 2, at first we show that if T *is an operator such that the null subspace of its adjoint is contained in its own null subspace, then* T *has a closed range if and only if its generalized mean transform has a closed range too. Secondly, we prove that the generalized mean transform preserves the class of EP operators. Lastly, we provide various reverse-order laws for Moore-Penrose inverse of* T, $\widehat{T_\lambda}$ *and* $\Delta_\lambda(T)$.
In Section 3, we investigate when an operators and its generalized mean transform both are binormal. Afterwards, we proved that if $T \in B(H)$ *is a binormal operator with closed range, then the range of it's generalized mean transform is also closed. Finally, we discuss the polar decomposition of generalized mean transform. Precisely, we prove that if* $T \in B(H)$ *is a binormal operator with closed range, and* $T = U|T|$ *is its polar decomposition, then* $\widehat{T_\lambda} = U^*U^2|\widehat{T_\lambda}|$ *is the polar decomposition of* $\widehat{T_\lambda}$. *This result was also proved in [13, Theorem 4.14], for the generalized Aluthge transform.*
To achieve our main results we have to use some auxiliary lemmas as follows.

Lemma 1.1. *[19] Let* $T \in \mathcal{B}(\mathcal{H})$ *be positive and* $\alpha > 0$. *Then* $\mathcal{R}(T)$ *is closed if and only if* $\mathcal{R}(T^\alpha)$ *is closed. In this case* $\mathcal{R}(T) = \mathcal{R}(T^\alpha)$.

Lemma 1.2. *[9] Let* $T \in \mathcal{B}(\mathcal{H})$. *Then the reduced minimum modulus of T is defined by:*

$$\gamma(T) := \begin{cases} \inf\{\|Tx\|; \|x\| = 1,\, x \in \mathcal{N}(T)^\perp\} & if\ T \neq 0 \\ +\infty & if\ T = 0. \end{cases}$$

Thus, $\gamma(T) > 0$ *if and only if* T *has a closed range .*

In the following Lemma, F. Chabbabi et al. proved that T *and* $\widehat{T}$ *have the same null subspaces.*

Lemma 1.3. *[4] For any operator* $T \in \mathcal{B}(\mathcal{H})$, *we have*

$$\mathcal{N}(\widehat{T}) = \mathcal{N}(T).$$

Later, in 2020, C. Benhida et al. [2] give this generalisation.

Lemma 1.4. *[2] Let* $T \in B(H)$ *and* $\lambda \in]0, \frac{1}{2}]$. *If* $N(T) \subset N(T^*)$, *then*

$$\mathcal{N}(\widehat{T}_\lambda) = \mathcal{N}(T).$$

2 EP operators and the generalized mean transform

Recently, the authors in [5] proved that if the range of $T \in B(H)$ *is closed then the range of it's mean transform is also closed. Our first theorem in this section, shows that if the null subspace of* T *is contained in that of its adjoint, then* T *has closed range if and only if so has its generalized mean transform.*

Theorem 2.1. *Let $\lambda \in]0, \frac{1}{2}]$ and let $T \in \mathcal{B}(\mathcal{H})$, such that $N(T) \subset N(T^*)$. Then*

$$\mathcal{R}(T) \textit{ is closed } \iff \mathcal{R}(\widehat{T}_\lambda) \textit{ is also closed.}$$

Proof. $(\Rightarrow)$. for $\lambda = \frac{1}{2}$ (see [5]). Let $\lambda \in]0, \frac{1}{2}[$. Suppose that $R(T)$ is closed and $\mathcal{R}(\widehat{T}_\lambda)$ is not closed. Since $N(T) \subset N(T^*)$, By Lemmas 1.2 and 1.4, there exists a sequence of unit vectors $x_n \in (\mathcal{N}(\widehat{T}_\lambda))^\perp = N(T)^\perp$ such that

$$\widehat{T}_\lambda x_n = \frac{1}{2}(|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda)x_n \longrightarrow 0.$$

Thus

$$|T|^\lambda(U|T|^{1-2\lambda} + |T|^{1-2\lambda}U)|T|^\lambda x_n \longrightarrow 0.$$

Therefore

$$(|T|^\lambda)^+|T|^\lambda(U|T|^{1-2\lambda} + |T|^{1-2\lambda}U)|T|^\lambda x_n \longrightarrow 0.$$

Also, since the condition $N(T) \subset N(T^*)$ is is equivalent to $R(T) \subset R(T^*)$, according to Lemma 1.1, we have

$$R(U) = R(T) \subset R(T^*) = R(|T|^\lambda) \text{ and } R(|T|^{1-2\lambda}) = R(|T|^\lambda).$$

It follows that

$$(U|T|^{1-2\lambda} + |T|^{1-2\lambda}U)|T|^\lambda x_n \longrightarrow 0.$$

Put $y_n = |T|^\lambda x_n$, it follows that

$$U^*(U|T|^{1-2\lambda} + |T|^{1-2\lambda}U)y_n \longrightarrow 0.$$

Thus

$$(|T|^{1-2\lambda} + U^*|T|^{1-2\lambda}U)y_n \longrightarrow 0.$$

Since $|T|^{1-2\lambda}$ and $U^*|T|^{1-2\lambda}U$ are positive operators. Then

$$|T|^{1-2\lambda}y_n \longrightarrow 0 \text{ and } U^*|T|^{1-2\lambda}Uy_n \longrightarrow 0$$

Hence $|T|^{1-2\lambda}|T|^\lambda x_n \longrightarrow 0$. Then $|T|x_n \longrightarrow 0$, implies $U|T|x_n \longrightarrow 0$ which is a contradiction with $R(T)$ is closed.

$(\Leftarrow)$. Let $\lambda \in]0, \frac{1}{2}]$. Suppose that $R(\widehat{T}_\lambda)$ is closed and $R(T)$ is not closed. Then by Lemma 1.1, $R(|T|)$ is not closed. Thus there exists a sequence of unit vectors $x_n \in \mathcal{N}(|T|)^\perp = \mathcal{N}(\widehat{T}_\lambda)^\perp$, such that $|T|x_n \longrightarrow 0$. Hence

$$|T|^\lambda x_n \longrightarrow 0, \text{ and } |T|^{1-\lambda}x_n \longrightarrow 0.$$

It follows that

$$|T|^\lambda U|T|^{1-\lambda}x_n \longrightarrow 0 \textit{ and } |T|^{1-\lambda}U|T|^\lambda x_n \longrightarrow 0.$$

Therefore, $\widehat{T}_\lambda x_n \longrightarrow 0$. This contradicts the fact that $R(\widehat{T}_\lambda)$ is closed. □

The condition $N(T) \subset N(T^*)$ is necessary in the previous theorem as shown by the following example.

Example 2.2. Let $A = \begin{pmatrix} Q & 0 \\ (I - Q^*Q)^{\frac{1}{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, where Q is a contraction and $\mathcal{R}(Q)$ is not closed. it follows that

$$A^*A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

is an orthogonal projection. Hence A is a partial isometry. Then $\mathcal{R}(A)$ is closed and $A = A|A| = AA^*A$ is the polar decomposition of A. Since

$$AA^*A = \begin{pmatrix} Q & 0 \\ (I - Q^*Q)^{\frac{1}{2}} & 0 \end{pmatrix} \neq \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} = A^*AA,$$

Then A is not quasinormal. Using [14, Proposition 2.1.], we obtain that $N(A) \nsubseteq N(A^*)$.

On the other hand, for $\lambda \in]0, \frac{1}{2}]$, we have

$$\widehat{A}_\lambda = (A^*A)^\lambda A(A^*A)^{1-\lambda} + (A^*A)^{1-\lambda}A\,(A^*A)^\lambda = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix},$$

so $\mathcal{R}(\widehat{A}_\lambda)$ is not closed.

As a consequence of the previous Theorem, we get the following result

Corollary 2.3. *Let $T \in B(H)$ be hyponormal. Then*

$$\mathcal{R}(T) \text{ is closed} \iff \mathcal{R}(\widehat{T}_\lambda) \text{ is also closed.}$$

Proof. Since T is hyponormal, then $N(T) \subset N(T^*)$. So, from the previous theorem, we deduce the equivalence above. □

Recently in [20], we showed that an operator $T \in \mathcal{B}(\mathcal{H})$ is EP if and only if its λ-Aluthge transform $\Delta_\lambda(T)$ is EP and $\mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T))$, for $\lambda \in]0, 1]$. The following theorem gives the same result in the case of the generalized mean transform .

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ and $T = U|T|$ be its polar decomposition. Then for $\lambda \in [0, \frac{1}{2}]$, we have*

$$T \text{ is an EP operator} \iff \widehat{T_\lambda} \text{ is EP and } R(T) = R(\widehat{T_\lambda}).$$

Proof. $(\Rightarrow)$. Suppose that T is EP operator, then $N(T) = N(T^*)$, according to Theorem 2.1, $\mathcal{R}(\widehat{T_\lambda})$ is closed. Hence, to prove $\widehat{T_\lambda}$ is EP, it is enough to prove that $\mathcal{N}(\widehat{T_\lambda}) = \mathcal{N}(\widehat{T_\lambda}^*)$. Let us distinct the following two cases:

Case 1: $\lambda = 0$. Let $x \in \mathcal{N}(\widehat{T})$, by Lemma 1.3, we get $x \in \mathcal{N}(T) = \mathcal{N}(T^*)$. Then $U^*x = |T|x = 0$ and thus $\widehat{T}^*x = \dfrac{1}{2}(|T|U^* + U^*|T|)x = 0$. This shows that $\mathcal{N}(\widehat{T}) \subset \mathcal{N}(\widehat{T}^*)$. Conversely, if $x \in \mathcal{N}(\widehat{T}^*)$, then

$$|T|U^*x + U^*|T|x = 0,$$

and hence

$$U|T|U^*x + UU^*|T|x = U(|T|U^*x + U^*|Tx|) = 0$$

Since $\mathcal{N}(T) = \mathcal{N}(T^*)$, then U is normal. So

$$|T^*|x + |T|x = 0.$$

It follows that

$$\langle |T^*|x, x\rangle + \langle |T|x, x\rangle = \langle |T^*|x + |T|x, x\rangle = 0.$$

Since $|T^*|$ and $|T|$ are both positive, we have

$$|||T|^{\frac{1}{2}}x|| = \langle |T|x, x\rangle = 0.$$

Therefore, $x \in \mathcal{N}(|T|) = \mathcal{N}(T)$. As a consequence, we obtain

$$\mathcal{N}(\widehat{T}^*) \subseteq \mathcal{N}(T) = \mathcal{N}(\widehat{T}).$$

On the other hand, taking orthogonal complements in the relation $\mathcal{N}(T) = \mathcal{N}(\widehat{T})$, and since T and $\widehat{T}$ are both EP operators, we deduce that $\mathcal{R}(T) = \mathcal{R}(\widehat{T})$.

Case 2. $\lambda \in]0, \frac{1}{2}[$. Let $x \in N(\widehat{T_\lambda})$. Since $N(T) = N(|T|^\lambda) = N(|T|^{1-\lambda})$, then

$$\widehat{T_\lambda}^* x = \frac{1}{2}(|T|^{1-\lambda}U^*|T|^\lambda + |T|^\lambda U^*|T|^{1-\lambda})x = 0.$$

This shows that $N(\widehat{T_\lambda}) \subset N(\widehat{T_\lambda}^*)$.

Conversly, if $x \in N(\widehat{T_\lambda}^*)$, then

$$(|T|^{1-\lambda}U^*|T|^\lambda + |T|^\lambda U^*|T|^{1-\lambda})x = 0,$$

and hence

$$|T|^{\lambda}(|T|^{1-2\lambda}U^* + U^*|T|^{1-2\lambda})|T|^{\lambda}x = 0.$$

So

$$U(|T|^{1-2\lambda}U^* + U^*|T|^{1-2\lambda})|T|^{\lambda}x = 0.$$

Put $S = |T|^{1-2\lambda}U^* + U^*|T|^{1-2\lambda}$ and $y = |T|^{\lambda}x$. Since $\mathcal{N}(T) = \mathcal{N}(T^*)$, then U is normal. So

$$\begin{aligned} < USy, y > &=< U(|T|^{1-2\lambda}U^* + U^*|T|^{1-2\lambda})y, y > \\ &=< (|T^*|^{1-2\lambda} + |T|^{1-2\lambda})y, y > \\ &=< |T^*|^{1-2\lambda}y, y > + < |T|^{1-2\lambda}y, y > \\ &= |||T^*|^{\frac{1-2\lambda}{2}}y||^2 + |||T|^{\frac{1-2\lambda}{2}}y||^2 = 0. \end{aligned}$$

Thus $|T|^{1-2\lambda}y = 0$. Therefore $|T|^{1-\lambda}x = 0$. Hence $x \in N(T) = N(\widehat{T_\lambda})$.
$(\Leftarrow)$. Using the assumptions, we get that $\mathcal{R}(T)$ is closed and

$$\mathcal{N}(T^*) = \mathcal{N}(\widehat{T_\lambda}^*) = N(\widehat{T_\lambda}.).$$

In case $\lambda = 0$, by Lemma 1.3, we get $\mathcal{N}(T^*) = N(\widehat{T}) = \mathcal{N}(T)$, and so T is EP.
Now, In case $\lambda \in]0, \frac{1}{2}[$, we have

$$N(T) = N(|T|^{\lambda}) = N(|T|^{1-\lambda}) \subset N(\widehat{T_\lambda}^*) = N(T^*),$$

which implies $\mathcal{N}(T^*) = \mathcal{N}(\widehat{T}) = \mathcal{N}(T)$. Hence T is EP opertator. This completes the proof.
For $\lambda = \frac{1}{2}$ see [17]. □

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an EP operator. It holds that*

$$T \ is \ quasinormal \implies \widehat{T_\lambda}^+ = \widehat{T_\lambda^+}.$$

Proof. Since T is an EP operator, then $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T)$. Using this sum, T has the following matrix form :

$$T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where the operator $A : \mathcal{R}(T) \longrightarrow \mathcal{R}(T)$ is invertible. Then, the polar decompositions of T can be decomposed as follows :

$$T = U|T| = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix},$$

such that $A = V|A|$ is the polar decomposition of A. Therefore,

$$\widehat{T_\lambda} = \begin{pmatrix} \widehat{A} & 0 \\ 0 & 0 \end{pmatrix}, \quad (\widehat{T_\lambda})^+ = \begin{pmatrix} (\widehat{A_\lambda})^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$T^+ = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{T_\lambda^+} = \begin{pmatrix} \widehat{A_\lambda^{-1}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence,

$$(\widehat{T_\lambda})^+ = \widehat{T_\lambda^+} \iff (\widehat{A_\lambda})^{-1} = \widehat{A_\lambda^{-1}}.$$

If T is quasinormal, then $(\widehat{T_\lambda})^{-1} = T^{-1} = \widehat{T_\lambda^{-1}}$, by [2, Theorem 1]. This last equality implies $(\widehat{T_\lambda})^+ = \widehat{T_\lambda^+}$. □

The assumption T is an EP operator is necessary in the previous result as shown by the following example.

Example 2.6. Consider the right shift operator A, defined on the Hilbert space $\ell^2(\mathbb{N})$ by

$$A(x_1, x_2, ...) = (0, x_1, x_2, ...).$$

Then,

$$A^*(x_1, x_2, ...) = (x_2, x_3, ...),$$

and so $A^*A = I$. Hence A is isometry, which implies that A is quasinormal and $A^* = A^+$. But A is not EP because $A^*A \neq AA^*$. On the other hand a simple calculation shows that

$$A^*AA^* \neq AA^*A.$$

Thus, A^* is not quasinormal. So

$$(\widehat{A})^+ = A^+ = A^* \neq (\widehat{A^*}) = (\widehat{A^+}).$$

We close this section with the following result that gives various reverse-order laws for Moore-Penrose inverse of T, $\widehat{T_\lambda}$ and $\Delta_\lambda(T)$, where T is EP.

Proposition 2.7. *If $T \in \mathcal{B}(\mathcal{H})$ is an EP operator, then we have*

(1) $(T\widehat{T_\lambda})^+ = \widehat{T_\lambda}^+ T^+$ *and* $(\widehat{T_\lambda}T)^+ = T^+\widehat{T_\lambda}^+$, *for all* $\lambda \in [0, \frac{1}{2}]$.

(2) $(\Delta_\lambda(T)T)^+ = T^+(\Delta_\lambda(T))^+$ *and* $(T\Delta_\lambda(T))^+ = (\Delta_\lambda(T)^+T^+$, *for all* $\lambda \in [0, 1]$.

(3) $(\Delta_\lambda(T)\widehat{T_\lambda})^+ = (\widehat{T_\lambda})^+(\Delta_\lambda(T))^+$ *and* $(\widehat{T_\lambda}\Delta_\lambda(T))^+ = (\Delta_\lambda(T))^+(\widehat{T_\lambda})^+$, *for all* $\lambda \in [0, \frac{1}{2}]$.

Proof. It follows from [6], [20, Theorem 3.4] and Theorem 2.4. □

3 The generalized mean transform of binormal operators

The goal of this section is to explore some properties of the generalized mean transform of binormal operators. First, we gave an example of a binormal operator T such that its generalized mean transform is not binormal.

Example 3.1. Consider $T = \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix} \in \mathbb{C}^3$ and let $T = U|T|$ be its polar decomposition. Then T is binormal since

$$TT^*.T^*T = T^*T.TT^* = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{pmatrix}.$$

We also have

$$|T| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

so that $U = T|T|^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix}$. Therefore, for $\lambda \in]0, 1[$, we obtain

$$|T|^\lambda U|T|^{1-\lambda} = \begin{pmatrix} 0 & 0 & 5^{1-\lambda} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{5^\lambda\sqrt{3}}{2} & -\frac{5}{2}^\lambda & 0 \end{pmatrix}, \text{ and } |T|^{1-\lambda}U|T|^\lambda = \begin{pmatrix} 0 & 0 & 5^\lambda \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{5^{1-\lambda}\sqrt{3}}{2} & -\frac{5}{2}^{1-\lambda} & 0 \end{pmatrix}.$$

Hence,

$$\widehat{T}_\lambda = \begin{pmatrix} 0 & 0 & \frac{5^{1-\lambda}+5^\lambda}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{4}(5^\lambda+5^{1-\lambda}) & -\frac{5^{1-\lambda}+5^\lambda}{4} & 0 \end{pmatrix}.$$

It follows that

$$\widehat{T}_\lambda \widehat{T}_\lambda^* = \begin{pmatrix} \frac{(5^{1-\lambda}+5^\lambda)^2}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{(5^{1-\lambda}+5^\lambda)^2}{4} \end{pmatrix}$$

and

$$\widehat{T}_\lambda^* \widehat{T}_\lambda = \begin{pmatrix} \frac{1}{4}+\frac{3}{16}(5^{1-\lambda}+5^\lambda)^2 & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{16}(5^{1-\lambda}+5^\lambda)^2 & 0 \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{16}(5^{1-\lambda}+5^\lambda)^2 & \frac{3}{4}+\frac{1}{16}(5^{1-\lambda}+5^\lambda)^2 & 0 \\ 0 & 0 & \frac{(5^{1-\lambda}+5^\lambda)^2}{4} \end{pmatrix}.$$

So

$$\widehat{T}_\lambda^* \widehat{T}_\lambda \widehat{T}_\lambda \widehat{T}_\lambda^* = \begin{pmatrix} \frac{(5^{1-\lambda}+5^\lambda)^2}{16}+\frac{3}{64}(5^{1-\lambda}+5^\lambda)^4 & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{16}(5^{1-\lambda}+5^\lambda)^2 & 0 \\ \frac{\sqrt{3}}{16}(5^{1-\lambda}+5^\lambda)^2-\frac{\sqrt{3}}{64}(5^{1-\lambda}+5^\lambda)^4 & \frac{3}{4}+\frac{1}{16}(5^{1-\lambda}+5^\lambda)^2 & 0 \\ 0 & 0 & \frac{(5^{1-\lambda}+5^\lambda)^4}{16} \end{pmatrix}$$

and

$$\widehat{T}_\lambda \widehat{T}_\lambda^* \widehat{T}_\lambda^* \widehat{T}_\lambda = \begin{pmatrix} \frac{(5^{1-\lambda}+5^\lambda)^2}{16}+\frac{3}{64}(5^{1-\lambda}+5^\lambda)^4 & \frac{\sqrt{3}}{16}(5^{1-\lambda}+5^\lambda)^2-\frac{\sqrt{3}}{64}(5^{1-\lambda}+5^\lambda)^4 & 0 \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{16}(5^{1-\lambda}+5^\lambda)^2 & \frac{3}{4}+\frac{1}{16}(5^{1-\lambda}+5^\lambda)^2 & 0 \\ 0 & 0 & \frac{(5^{1-\lambda}+5^\lambda)^4}{16} \end{pmatrix}$$

Then $\widehat{T}_\lambda$ is not binormal.

Next, we prove that under certain conditions the generalized mean transform of binormal operator is binormal too. To do so, we use a special class denoted by

$$\delta(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) \,/ U^2|T| = |T|U^2\}.$$

This class was introduced in [12], in order to study the relationship between a hyponormal operator and its mean transform.

Theorem 3.2. *Let $T \in \delta(\mathcal{H})$ and $T = U|T|$ be its polar decomposition. If U is unitary, then for $\lambda \in]0,1[$, we have*

$$T \ \ is \ \ binormal \implies \widehat{T_\lambda} \ \ is \ \ binormal.$$

Proof. First if $T \in \delta(\mathcal{H})$, then by the functional calculus, we obtain $U^2|T|^\lambda = |T|^\lambda U^2$ for all $\lambda > 0$. This implies $U|T^*|^\lambda U = |T|^\lambda U^2$. Multiplying this equality by U^* on the right side and since U is unitary, we get

$$U|T^*|^\lambda = |T|^\lambda U \qquad \text{for all } \lambda \in]0,1[. \tag{3.1}$$

It follows that

$$|T^*|^\lambda U^* = U^*|T|^\lambda \qquad \text{for all } \lambda \in]0,1[. \tag{3.2}$$

Also, since T is binormal, we obtain that

$$[|T|^\alpha, |T^*|^\beta] = [U^*|T|^\alpha U, |T|^\beta] = 0, \qquad \text{for all } \alpha, \beta > 0. \tag{3.3}$$

Hence, we have

$$\begin{aligned}
4|\widehat{T_\lambda}|^2 &= \left(|T|^{1-\lambda}U^*|T|^\lambda + |T|^\lambda U^*|T|^{1-\lambda}\right)\left(|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda\right)\\
&= |T|^{1-\lambda}U^*|T|^{2\lambda}U|T|^{1-\lambda} + |T|^{1-\lambda}U^*|T|U|T|^\lambda + |T|^\lambda U^*|T|U|T|^{1-\lambda} + |T|^\lambda U^*|T|^{2(1-\lambda)}U|T|^\lambda\\
&= U^*|T|^{2\lambda}U|T|^{2(1-\lambda)} + 2U^*|T|U|T| + U^*|T|^{2(1-\lambda)}U|T|^{2\lambda}\\
&= (U^*|T|^\lambda U|T|^{1-\lambda})^2 + 2U^*|T|U|T| + (U^*|T|^{(1-\lambda)}U|T|^\lambda)^2\\
&= \left(U^*|T|^\lambda U|T|^{1-\lambda} + U^*|T|^{(1-\lambda)}U|T|^\lambda\right)^2.
\end{aligned}$$

Then,

$$|\widehat{T_\lambda}| = \frac{1}{2}(U^*|T|^\lambda U|T|^{1-\lambda} + U^*|T|^{(1-\lambda)}U|T|^\lambda). \tag{3.4}$$

Also, since $|T^*|^\alpha = U|T|^\alpha U^*$, we have

$$\begin{aligned}
4|\widehat{T_\lambda}^*|^2 &= \left(|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda\right)\left(|T|^{1-\lambda}U^*|T|^\lambda + |T|^\lambda U^*|T|^{1-\lambda}\right)\\
&= |T|^\lambda U|T|^{2(1-\lambda)}U^*|T|^\lambda + |T|^\lambda U|T|U^*|T|^{1-\lambda} + |T|^{1-\lambda}U|T|U^*|T|^\lambda + |T|^{1-\lambda}U|T|^{2\lambda}U^*|T|^{1-\lambda}\\
&= |T|^\lambda|T^*|^{2(1-\lambda)}|T|^\lambda + |T|^\lambda|T^*||T|^{1-\lambda} + |T|^{1-\lambda}|T^*||T|^\lambda + |T|^{1-\lambda}|T^*|^{2\lambda}|T|^{1-\lambda}\\
&= |T|^{2\lambda}|T^*|^{2(1-\lambda)} + |T||T^*| + |T^*||T| + |T|^{2(1-\lambda)}|T^*|^{2\lambda}.\\
&= (|T|^\lambda|T^*|^{(1-\lambda)} + |T|^{1-\lambda}|T^*|^\lambda)^2
\end{aligned}$$

We deduce that

$$|\widehat{T_\lambda}^*| = \frac{1}{2}(|T|^\lambda|T^*|^{(1-\lambda)} + |T|^{1-\lambda}|T^*|^\lambda). \tag{3.5}$$

As a consequence, we have

$$\begin{aligned}
4|\widehat{T_\lambda}^*||\widehat{T_\lambda}| &= (|T|^\lambda|T^*|^{1-\lambda} + |T|^{1-\lambda}|T^*|^\lambda)(U^*|T|^\lambda U|T|^{1-\lambda} + U^*|T|^{1-\lambda}U|T|^\lambda)\\
&= |T|^\lambda|T^*|^{1-\lambda}U^*|T|^\lambda U|T|^{1-\lambda} + |T|^\lambda|T^*|^{1-\lambda}U^*|T|^{1-\lambda}U|T|^\lambda + |T|^{1-\lambda}|T^*|^\lambda U^*|T|^\lambda U|T|^{1-\lambda}\\
&\quad + |T|^{1-\lambda}|T^*|^\lambda U^*|T|^{1-\lambda}U|T|^\lambda\\
&= |T||T^*|^{1-\lambda}U^*|T|^\lambda U + |T|^{2\lambda}|T^*|^{1-\lambda}U^*|T|^{1-\lambda}U + |T|^{2(1-\lambda)}|T^*|^\lambda U^*|T|^\lambda U\\
&\quad + |T||T^*|^\lambda U^*|T|^{1-\lambda}U\\
&= |T|U^*|T|U + |T|^{2\lambda}U^*|T|^{2(1-\lambda)}U + |T|^{2(1-\lambda)}U^*|T|^{2\lambda}U + |T|U^*|T|U \qquad \text{by (3.2)}\\
&= 2U^*|T|U|T| + U^*|T|^{2(1-\lambda)}U|T|^{2\lambda} + U^*|T|^{2\lambda}U|T|^{2(1-\lambda)} \qquad \text{by (3.3)}
\end{aligned}$$

And

$$\begin{aligned}
4|\widehat{T_\lambda}||\widehat{T_\lambda}^*| &= (U^*|T|^\lambda U|T|^{1-\lambda} + U^*|T|^{1-\lambda}U|T|^\lambda)(|T|^\lambda|T^*|^{1-\lambda} + |T|^{1-\lambda}|T^*|^\lambda) \\
&= (U^*|T|^\lambda U|T|^{1-\lambda}|T|^\lambda|T^*|^{1-\lambda} + U^*|T|^\lambda U|T|^{1-\lambda}|T|^{1-\lambda}|T^*|^\lambda + U^*|T|^{1-\lambda}U|T|^\lambda|T|^\lambda|T^*|^{1-\lambda} \\
&+ U^*|T|^{1-\lambda}U|T|^\lambda|T|^{1-\lambda}|T^*|^\lambda \\
&= U^*|T|^\lambda U|T||T^*|^{1-\lambda} + U^*|T|^\lambda U|T|^{2(1-\lambda)}|T^*|^\lambda + U^*|T|^{1-\lambda}U|T|^{2\lambda}|T^*|^{1-\lambda} \\
&+ U^*|T|^{1-\lambda}U|T||T^*|^\lambda \\
&= U^*|T|^\lambda U|T^*|^{1-\lambda}|T| + U^*|T|^\lambda U|T^*|^\lambda|T|^{2(1-\lambda)} + U^*|T|^{1-\lambda}U|T^*|^{1-\lambda}|T|^{2\lambda} \\
&+ U^*|T|^{1-\lambda}U|T^*|^\lambda|T| \qquad \text{by (3.3)} \\
&= U^*|T|U|T| + U^*|T|^{2(1-\lambda)}U|T|^{2\lambda} + U^*|T|^{2\lambda}U|T|^{2(1-\lambda)} + U^*|T|U|T| \qquad \text{by (3.1)} \\
&= 2U^*|T|U|T| + U^*|T|^{2(1-\lambda)}U|T|^{2\lambda} + U^*|T|^{2\lambda}U|T|^{2(1-\lambda)}.
\end{aligned}$$

Hence, $\widehat{T_\lambda}$ is binormal. □

The following result shows that the generalized Aluthge and mean transform of a binormal operator with closed range have the same null subspace.

Proposition 3.3. *Let $T \in B(H)$ be binormal with closed range. Then*

(1) $N(|T|^\alpha|T^*|^\beta) = N(P_{R(T)}R_{R(T^*)})$, *for all* $\alpha, \beta > 0$.

(2) $N(\Delta_\lambda(T)) = \mathcal{N}(\widehat{T_\lambda})$ *for all* $\lambda \in]0,1[$.

Proof. (1) Since T is binormal, $U^*U|T^*|^\beta x = |T^*|^\beta U^*Ux = 0$ (see [8, Theorem 2]). Then, we have

$$\begin{aligned}
|T|^\alpha|T^*|^\beta x = 0 &\iff U|T^*|^\beta x = 0, \qquad (\text{ because } N(|T|^\alpha) = N(U)) \\
&\iff U^*U|T^*|^\beta x = 0 \\
&\iff |T^*|^\beta U^*Ux = 0 \\
&\iff (|T^*|^\beta)^+|T^*|^\beta U^*Ux = 0 \\
&\iff P_{R(T)}R_{R(T^*)}x = 0 \qquad (\text{ since } R(|T^*|^\beta) = R(T)).
\end{aligned}$$

Hence $N(|T|^\alpha|T^*|^\beta) = N(P_{R(T)}R_{R(T^*)})$.

(2) Suppose that $x \in N(\Delta_\lambda(T))$, then

$$|T|^\lambda U|T|^{1-\lambda}x = 0.$$

It follows that

$$|T|^\lambda|T^*|^{1-\lambda}Ux = 0,$$

Since $U|T|^{1-\lambda} = |T^*|^{1-\lambda}U$. Thus, by (1),

$$Ux \in N(|T|^\lambda|T^*|^{1-\lambda}) = N(P_{R(T)}R_{R(T^*)}) = N(|T|^{1-\lambda}|T^*|^\lambda).$$

So

$$|T|^{1-\lambda}|T^*|^\lambda Ux = 0.$$

Consequetly

$$(|T|^\lambda|T^*|^{1-\lambda}U + |T|^{1-\lambda}|T^*|^\lambda U)x = (|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda)x = 0.$$

Hence $x \in \mathcal{N}(\widehat{T_\lambda})$, which implies that $N(\Delta_\lambda(T)) \subset \mathcal{N}(\widehat{T_\lambda})$. Conversly, Let $x \in \mathcal{N}(\widehat{T_\lambda})$, then

$$(|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda)x = 0.$$

Thus

$$(|T|^\lambda |T^*|^{1-\lambda} + |T|^{1-\lambda}|T^*|^\lambda)Ux = 0,$$

It follows that

$$\langle |T|^\lambda |T^*|^{1-\lambda}Ux, Ux\rangle + \langle |T|^{1-\lambda}|T^*|^\lambda Ux, Ux\rangle = 0$$

As T is binormal, then $|T|^\lambda |T^*|^{1-\lambda}$ and $|T|^{1-\lambda}|T^*|^\lambda$ are positive operators and so

$$\| \left(|T|^\lambda |T^*|^{1-\lambda}\right)^{\frac{1}{2}} Ux\| = \langle |T|^\lambda |T^*|^{1-\lambda}Ux, Ux\rangle = 0.$$

Therefore,

$$|T|^\lambda U|T^*|^{1-\lambda}x = |T|^\lambda |T^*|^{1-\lambda}Ux = 0.$$

Then

$$x \in \mathcal{N}(\Delta_\lambda(T)).$$

Hence,

$$\mathcal{N}(\widehat{T_\lambda}) = \mathcal{N}(\Delta_\lambda(T)).$$

□

In [20, Proposition 3.9], we proved that if $T \in B(H)$ is binormal with closed range, then the range of it's λ-Aluthge transform is closed. It is equally true for the generalized mean transfom as the next theorem shows.

Theorem 3.4. *Let $\lambda \in]0, \frac{1}{2}[$. If $T \in B(H)$ is binormal with closed range, then $\mathcal{R}(\widehat{T_\lambda})$ is closed .*

Proof. Suppose that T is binormal with closed range and $\mathcal{R}(\widehat{T_\lambda})$ is not closed. Then there exists a sequence of unit vectors $x_n \in (\mathcal{N}(\widehat{T_\lambda}))^\perp$ such that

$$(|T|^\lambda U|T|^{1-\lambda} + |T|^{1-\lambda}U|T|^\lambda)x_n \longrightarrow 0,$$

Hence,

$$|T|^\lambda |T^*|^{1-\lambda}Ux_n + |T|^{1-\lambda}|T^*|^\lambda Ux_n \longrightarrow 0.$$

Using Proposition 3.3, $x_n \in \mathcal{N}(\Delta_\lambda(T))^\perp$, for all n. Moreover, as T is binormal, then $|T|^\lambda |T^*|^{1-\lambda}$ and $|T|^{1-\lambda}|T^*|^\lambda$ are positive operators and so

$$|T|^\lambda |T^*|^{1-\lambda}Ux_n \longrightarrow 0 \text{ and } |T|^{1-\lambda}|T^*|^\lambda Ux_n \longrightarrow 0.$$

Thus,

$$\Delta_\lambda(T)x_n = |T|^\lambda U|T^*|^{1-\lambda}x_n \longrightarrow 0.$$

Therefore, $x_n \in \mathcal{N}(\Delta_\lambda(T))^\perp$ such that $\Delta_\lambda(T)x_n \longrightarrow 0$. This contradicts the fact that $R(\Delta_\lambda(T))$ is closed (see [20]). □

In [10], the authors discussed the polar decomposition of the Aluthge transform and obtained the following result

Proposition 3.5. *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then $\Delta(T) = U^*U^2|\Delta(T)|$ is also the polar decomposition of $\Delta(T)$.*

This result was also proved in [13, Theorem 4.14], for the generalized Aluthge transform. Now, we show the same result in the case of the generalized mean transform as follows :

Theorem 3.6. *Let $\lambda \in]0, \frac{1}{2}[$ and let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. If T is binormal with closed range, then*

(1) $\widehat{T_\lambda} = U|\widehat{T_\lambda}|$.

(2) $\widehat{T_\lambda} = U^*U^2|\widehat{T_\lambda}|$ *is the polar decomposition of* $\widehat{T_\lambda}$.

Proof. (1). From (3.4), we have

$$|\widehat{T_\lambda}| = \frac{1}{2}(U^*|T|^\lambda U|T|^{1-\lambda} + U^*|T|^{(1-\lambda)}U|T|^\lambda).$$

Since $[UU^*, |T|^r] = 0$, for all $r > 0$, then we get

$$\begin{aligned} U|\widehat{T_\lambda}| &= \frac{1}{2}(UU^*|T|^\lambda U|T|^{1-\lambda} + UU^*|T|^{(1-\lambda)}U|T|^\lambda) \\ &= \frac{1}{2}(|T|^\lambda UU^*U|T|^{1-\lambda} + |T|^{(1-\lambda)}UU^*U|T|^\lambda) \\ &= \frac{1}{2}(|T|^\lambda U|T|^{1-\lambda} + |T|^{(1-\lambda)}U|T|^\lambda) \\ &= \widehat{T_\lambda}. \end{aligned}$$

(2). Firstly, we show that $\widehat{T_\lambda} = U^*U^2|\widehat{T_\lambda}|$. Using (1) and Lemma 1.1, we have

$$\begin{aligned} U^*U^2|\widehat{T_\lambda}| = U^*U\widehat{T_\lambda} &= \frac{1}{2}U^*U\left(|T|^\lambda U|T|^{1-\lambda} + |T|^{(1-\lambda)}U|T|^\lambda\right). \\ &= \frac{1}{2}\left(|T|^\lambda U|T|^{1-\lambda} + |T|^{(1-\lambda)}U|T|^\lambda\right) \\ &= \widehat{T_\lambda} \end{aligned}$$

Secondly, since T is binormal, then by putting $\alpha = \lambda$ and $\beta = 1-\lambda$, $(\lambda \in]0,1[)$ in [13, Theorem 4.14], we obtain that

$$\Delta_\lambda(T) = U^*U^2|\Delta_\lambda(T)|$$

is the polar decomposition of $\Delta_\lambda(T)$. So, U^*U^2 is a partial isometry such that $N(\Delta_\lambda(T)) = N(U^*U^2)$, which gives by Proposition 3.3 that $N(\widehat{T_\lambda}) = N(U^*U^2)$. Hence, $\widehat{T_\lambda} = U^*U^2|\widehat{T_\lambda}|$ is also the polar decomposition of $\widehat{T_\lambda}$. □

The following Corollary is a special case of Theorem 3.6 for which the binormal operator is invertible.

Corollary 3.7. *Let* $T \in B(H)$ *be an invertible binormal operator and* $T = U|T|$ *be its polar decomposition. Then* $U|\widehat{T_\lambda}|$ *is also the polar decomposition of* $\widehat{T_\lambda}$.

Proof. As T is invertible, U is unitary. Using Theorem 3.6, the proof follows. □

The last result of this section gives the polar decomposition for the Moore-Penrose inverse of $\widehat{T_\lambda}$ *when* T *is a binormal operator with closed range.*

Proposition 3.8. *Let* $T = U|T|$ *be the polar decomposition of* $T \in \mathcal{B}(\mathcal{H})$. *If* T *is binormal operator with closed range, then*

$$\widehat{T_\lambda}^+ = \frac{1}{2}(U^*)^2U\left(|T|^\lambda|T^*|^{1-\lambda} + |T|^{1-\lambda}|T^*|^\lambda\right)^+$$

is the polar decomposition of $\widehat{T_\lambda}^+$.

Proof. Since T is binormal with closed range, by Theorem 3.4, $R(\widehat{T_\lambda})$ is closed and so $\widehat{T_\lambda}^+$ exists. According to Theorem 3.6 and [11, Proposition 2.2], the polar decomposition of $\widehat{T_\lambda}^+$ is given by

$$\widehat{T_\lambda}^+ = (U^*)^2U|\widehat{T_\lambda}^*|^+$$

Therefore, from (3.5) we conclude that

$$\widehat{T_\lambda}^+ = \frac{1}{2}(U^*)^2U\left(|T|^\lambda|T^*|^{1-\lambda} + |T|^{1-\lambda}|T^*|^\lambda\right)^+$$

is the polar decomposition of $\widehat{T_\lambda}^+$. □

4 Conclusion

In this paper, we investigate when an operator $T \in \mathcal{B}(\mathcal{H})$ and its generalized mean transform $\widehat{T_\lambda}$ both have closed ranges, and show that this transform preserves the class of EP operators. Also, we study some relationships between a binormal operator and its generalized mean transform.

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