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Commutative weak idempotent nil-neat rings

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Abstract. *In this paper, we introduce the notion of a weak idempotent nil-neat ring which is a generalization of weakly nil-neat ring. We give certain characterizations of weak idempotent nil-neat rings in terms of semiprime ideals, maximal ideals, Jacobson radicals, and reduced weak idempotent nil-neat rings. Moreover, we obtain a ring R is weak idempotent nil-neat if and only if exactly R is a field; or $J(R) \neq 0$ and $R/J(R)$ is isomorphic to any of the special rings namely Boolean ring B or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times B$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$. Finally, we obtain that every nonzero prime ideal of R is maximal.*

1 Introduction

Throughout this paper, all rings are commutative with unity. We denote the set of nilpotents, the set of idempotents, the set of weak idempotents, and the set of units by $Nil(R)$, $Id(R)$, $wi(R)$ and $U(R)$, respectively.

N. Bisht [2] introduced the notion of semi-nil clean ring which is defined by a ring in which all its elements are a sum of a nilpotent and a periodic element. D.A. Yuwaningsih, I.E. Wijayanti, and B. Surodjo [3] introduced the notion of 2-nil-regular rings whose elements are a sum of 2- regular elements and a nilpotent element. D.A. Yuwaningsih, I.E. Wijayanti, and B. Surodjo [13] investigated r-clean rings in which all their elements are a sum of an idempotent element and a regular element. Danchev [11] introduced the notion of weakly nil-neat rings as a subclass of neat rings. A weakly nil-neat ring is a ring in which all its proper homomorphic images are weakly nil-clean rings whereas rings whose all proper homomorphic images are clean are called neat rings. Every weakly nil-neat ring is a neat ring.

The focus of this paper is to initiate the study of the class of rings in which the elements of the homomorphic image can be expressed as a sum of nilpotent element and weak idempotent element. This class is a generalization of the class of weakly nil-neat rings. We extend many of the properties of weakly nil-neat rings to the class of weak idempotent nil-neat rings.

In section 2, we recall definitions and basic properties of weak idempotent nil-clean rings from [1] and weakly nil-neat rings from [11]. In section 3, we introduce the notion of a weak idempotent nil-neat ring and furnish certain examples. Further, we characterize weak idempotent nil-neat rings in terms of direct product of Boolean ring and Z_3. Also, we establish the relationship between weak idempotent nil-clean and weak idempotent nil-neat rings. Finally, we obtain a complete classification of the weak idempotent nil-neat rings.

2 Preliminaries

Definition 2.1 ([10]). A ring R is called a weakly unipotent unit, WUU, if every unit can be represented as $n+1$ or $n-1$, where $n \in Nil(R)$.

Definition 2.2 ([7]). A ring R is called UNI if for each $u \in U(R)$, there are $n \in Nil(R)$ and $i \in Inv(R) \cap Z(R)$ such that $u = n + i$.

Definition 2.3 ([8]). The unit group $U(R)$ of a ring R is strongly invo-fine if, for every $u \in U(R)$, there are $v \in Inv(R)$ and $q \in Nil(R)$ such that $u = v + q$ with $vq = qv$.

Definition 2.4 ([1]). An element w of a ring R is called a weak idempotent if $w^2 = w^4$. A ring R is called weak idempotent nil-clean ring if every element of R can be expressed as a sum of a nilpotent and a weak idempotent element.

Proposition 2.5 ([1]). *The homomorphic image of every weak idempotent nil-clean ring is weak idempotent nil-clean.*

Proposition 2.6 ([1]). *Let R be a ring and $\{R_i : i \in I\}$ be a family of rings. Then,*

(1) R is weak idempotent nil-clean if and only if $R/Nil(R)$ is weak idempotent nil-clean.

(2) $R = \prod R_i$ is weak idempotent nil-clean ring if and only if each R_i is weak idempotent nil-clean.

(3) Let I be a nil ideal of a ring R. Then R is a weak idempotent nil-clean ring if and only if R/I is weak idempotent nil-clean.

Proposition 2.7 ([1]). *The following statements are equivalent.*

(i) $R = wi(R)$.

(ii) R is isomorphic to either a Boolean ring, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$.

(iii) For all $x \in R$, $x^4 = x^2$.

(iv) The ring R is reduced weak idempotent nil-clean ring.

Corollary 2.8 ([1]). *Let R be a ring. The following statements are true:*

(i) A reduced indecomposable ring is weak idempotent nil-clean if and only if it is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In particular, any weak idempotent nil-clean domain is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

(ii) A weak idempotent nil-clean ring is zero-dimensional and hence it is a clean ring.

Recall the following definitions from [10, 12].

Definition 2.9. A ring R is said to be

(i) neat if every proper homomorphic image of R is clean ring.

(ii) weakly nil-neat if every non-trivial homomorphic image of R is weakly nil-clean ring.

Proposition 2.10. *Let R be a ring. The following statements are equivalent:*

(i) R is a weak idempotent nil-clean ring.

(ii) R is zero-dimensional and $R/M \cong \mathbb{Z}_3$ for at least one maximal ideal M or $R/N \cong \mathbb{Z}_2$ for at least one other maximal ideal N.

(iii) $R/Nil(R)$ is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$.

(iv) $J(R)$ is nil and $R/J(R)$ is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $B \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$.

Proof. From Proposition 2.5 and Proposition 2.7 (ii) follows that $(i) \iff (iii)$. Also $(iii) \iff (iv)$ since $Nil(R) = J(R)$.
$(i) \implies (ii)$. Let R be a weak idempotent nil-clean ring. By Corollary 2.8(ii), R is zero-dimensional. For any maximal ideal M, R/M is reduced weak idempotent nil-clean domain so that either $R/M \cong \mathbb{Z}_2$ or $R/M \cong \mathbb{Z}_3$. By Chinese remainder theorem, for any two maximal ideals M and N of R, we have $R/(M \times N) \cong (R/M) \times (R/N)$ which is weak idempotent nil-clean ring. By Proposition 2.6 (2), R/M and R/N are weak idempotent nil-clean rings. We can apply Proposition 2.6(2) to finish the proof.
$(ii) \implies (i)$: Assume that R is zero-dimensional and also there is at least one maximal ideal of R, say M, which satisfies $R/M \cong \mathbb{Z}_3$ or there is at least one maximal ideal N of R such that $R/N \cong \mathbb{Z}_2$. It follows that $R/Nil(R) = R/J(R)$ is embeddable inside of $\prod_{M \in Max(R)}(R/M)$, which is isomorphic to either a product of copies of $\mathbb{Z}_2$ or a product of copies of $\mathbb{Z}_2$ and copies of $\mathbb{Z}_3$ or a product of copies of $\mathbb{Z}_3$. In all cases we have that $R/Nil(R)$ is a subring of a reduced weak idempotent nil-clean ring and hence reduced weak idempotent nil-clean ring. □

3 Commutative weak idempotent nil-neat rings

Definition 3.1. A ring R is called a weak idempotent nil-neat if every proper homomorphic image of R is a weak idempotent nil-clean.

Example 3.2. $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ are weak idempotent nil-neat rings.

Example 3.3. $\mathbb{Z}_5$ is a weak idempotent nil-neat ring but not a weak idempotent nil-clean ring.

Remark 3.4. $\mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_{10} \times \mathbb{Z}_2 \times \mathbb{Z}_3$ are not weak idempotent nil-neat rings because they contain homomorphic images $\mathbb{Z}_5 \times \mathbb{Z}_3$ and $\mathbb{Z}_{10} \times \mathbb{Z}_2$ that are not weak idempotent nil-clean rings, respectively.

Proposition 3.5. *A homomorphic image of a weak idempotent nil-neat ring is again a weak idempotent nil-neat ring.*

Proof. Let R be a weak idempotent nil-neat ring and let $f : R \to S$ be a ring homomorphism such that $f(R) = S$. Then $f(R/I) = S/f(I)$ is proper homomorphic image of R for every proper ideal I of R. Thus $f(I)$ is proper ideal of S and hence $S/f(I)$ is the proper homomorphic image of S. By Definition 3.1, R/I is a weak idempotent nil-clean ring. So $S/f(I)$ is a weak idempotent nil-clean ring by Proposition 2.5. Hence, S is a weak idempotent nil-neat ring. □

Proposition 3.6. *Let R be a weak idempotent nil-neat ring. If R is not weak idempotent nil-clean, then it is a reduced ring.*

Proof. Assume on the contrary and $Nil(R) \neq 0$. Then by Definition 3.1, $R/Nil(R)$ is weak idempotent nil-clean. By Proposition 2.6(1), R is weak idempotent nil-clean. □

Proposition 3.7. *Let R be a decomposable ring. Then R is a weak idempotent nil-neat ring if and only if R is a weak idempotent nil-clean.*

Proof. Given R is a decomposable ring. Then there are ideals I and J such that $R = I \times J$. If R is a weak idempotent nil-neat, then $I \cong R/J$ and $J \cong R/I$ are weak idempotent nil-clean by Definition 3.1. Thus R is a direct product of weak idempotent nil-clean rings. Hence, by Proposition 2.6(2), R is weak idempotent nil-clean. Conversely, assume that R is a weak idempotent nil-clean ring and I is the nonzero ideal of R. Then R/I is weak idempotent nil-clean by Proposition 2.5. Therefore, R is the weak idempotent nil-neat ring. □

Lemma 3.8. *A ring R is indecomposable weak idempotent nil-clean if and only if for every element r in R, either $r \in Nil(R)$ or $r \in Uni(R)$ or $r \in -Uni(R)$.*

Proof. ($\implies$). Suppose R is an indecomposable weak idempotent nil-clean and $r \in R$. Then $Id(R) = \{0, 1\}$ and $r = n + w$ for some nilpotent n and weak idempotent w. For any weak idempotent w, we have $(w^2)^2 = w^2$ and $(1 - w^2)^2 = 1 - w^2$. So w^2 and $1 - w^2$ are idempotents.

Thus $w^2 = 0$ or $w^2 = 1$ which implies that w is either nilpotent or a unit with its inverse. If w is nilpotent, then $r \in Nil(R)$. If w is unit, then $w+1, w-1 \in J(R) = Nil(R)$. Thus $r = n + w = (n + w - 1) + 1$ or $r = (n + w + 1) - 1$. Hence, $r \in Uni(R)$ or $r \in -Uni(R)$.
($\Longleftarrow$). Assume that $r = n$ or $r = n + 1$ or $r = n - 1$ for some nilpotent n and for every r in R. Then R is weak idempotent nil-clean and also $wi(R) = \{-1, 0, 1\}$. Since $Id(R) \subseteq wi(R)$, we have $Id(R) = \{0, 1\}$. □

Lemma 3.9. *If R is an indecomposable weak idempotent nil-clean ring, then so is every homomorphic image of R.*

Proof. Assume that R is an indecomposable weak idempotent nil-clean ring and S an arbitrary ring such that $f : R \to S$ is an epimorphism. Let $c \in S$. Then $c = f(n)$ or $c = f(n' + 1) = 1 + f(n')$ or $c = f(n' - 1) = f(n') - 1$ for some nilpotents n and n' by Lemma 3.8. Again, by Lemma 3.8, S is indecomposable. □

Proposition 3.10. *The following statements are equivalent for a ring R:*

(1) R is a local weak idempotent nil-clean ring.

(2) R is an indecomposable weak idempotent nil-clean ring.

(3) For all $x \in R$, either $x \in Nil(R)$, or $x \in Uni(R)$, or $x \in -Uni(R)$.

(4) R is a WUU ring and R has exactly one prime ideal.

Proof. (2) $\Longleftrightarrow$ (3). It follows from Lemma 3.8.
(1) $\Longleftrightarrow$ (3). Suppose R is a local weak idempotent nil-clean ring and $r \in R$. Then 0 and 1 are the only idempotent elements. So R is indecomposable. By Lemma 3.8, we have either $x \in Nil(R)$, or $x \in Uni(R)$, or $x \in -Uni(R)$. Conversely, for each $x \in R$, we have either $x \in Nil(R)$ or $x \in U(R)$. Thus $R/Nil(R)$ is a field and hence R is a local ring. As $\{-1, 0, 1\} \subseteq wi(R)$, R is local weak idempotent nil-clean ring.
(2) $\Longleftrightarrow$ (4). Suppose that R is an indecomposable weak idempotent nil-clean ring. Then every weak idempotent is either a nilpotent or a unit. So for every $x \in U(R)$, we have $x = n + w$ for some nilpotent n and weak idempotent w. In this case, w must be a unit otherwise $n + w$ will be nilpotent which is impossible. So $w \in U(R)$ implies that $w \pm 1 \in Nil(R)$ which in turn implies $x = n' \pm 1$ where $n' = n + w \pm 1$. Thus x is a WUU element and hence R is a WUU ring. Now we show that R has exactly one prime ideal. Assume that P_1 and P_2 are non-zero prime ideals of R. Then R/P_1 and R/P_2 are indecomposable weak idempotent nil-clean domains. By Corollary 2.8, R/P_1 and R/P_2 are isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Moreover, $R/P_1P_2 \cong R/P_1 \times R/P_2$ by Chinese remainder theorem. Thus $R/P_1P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R/P_1P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ or $R/P_1P_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. So R/P_1P_2 is not indecomposable since $(\bar{0}, \bar{1}) \in Id(R/P_1P_2)$. Hence, by contrapositive of Lemma 3.9, R is not indecomposable which is a contradiction. Conversely, assume that R is a WUU ring and P is the only prime ideal of R. Then all elements of P are nilpotents as $rad(R) = \bigcap P = P$ and also for any $r \in R$, we have either $r = n$ or $r = n' \pm 1$ where $n, n' \in Nil(R)$. Thus $R/P \cong \mathbb{Z}_3$. So $Id(R/P) = \{\bar{0}, \bar{1}\}$ implies that $Id(R) = \{0, 1\}$. Hence, R is an indecomposable weak idempotent nil-clean ring. □

Lemma 3.11. *Let R be a ring. Then the following statements are equivalent.*

(1) R is weak idempotent nil-neat ring.

(2) R/aR is weak idempotent nil-clean ring for every nonzero $a \in R$.

(3) For any collection of nonzero prime ideals $\{P_j\}_{j \in J}$ of R with $I = \cap_{j \in J} P_j \neq 0$, the factor ring R/I is weak idempotent nil-clean.

(4) R/aR is weak idempotent nil-neat for every $a \in R$.

(5) R/I is weak idempotent nil-clean for every nonzero semiprime ideal I.

(6) $R/I = wi(R/I)$ for every nonzero semiprime ideal I.

(7) R/I is isomorphic to either Boolean, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ for some Boolean ring B and for every nonzero semiprime ideal I.

(8) *For every nonzero semiprime ideal I of R, the factor ring R/I is zero dimensional and $R/P \cong \mathbb{Z}_3$ for at least one maximal ideal P containing I, or $R/Q \cong \mathbb{Z}_2$ for at least one other maximal ideal Q containing I.*

(9) *For every nonzero semiprime ideal I of R it must be that $J(R/I) = 0$ and R/I is isomorphic to either a Boolean ring B, or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$.*

Proof. $(1) \iff (2)$ follows from Definition 3.1 and also from the fact that the homomorphic image of weak idempotent nil-clean ring is weak idempotent nil-clean and any nontrivial ideal contains a principal nontrivial ideal.
$(1) \implies (4)$. For a nonzero element a in R, R/aR is the proper homomorphic image of R. So R/aR is a weak idempotent nil-clean ring.
$(4) \implies (1)$ is clear by choosing $a = 0$.
$(3) \iff (5)$ is instant since the intersection of any family of prime ideals is a semiprime ideal.
$(1) \implies (5)$ is obvious.
$(5) \implies (1)$. Suppose I is a nonzero ideal of R. Then $\sqrt{I}$ is semiprime ideal of R. By assumption, $R/\sqrt{I}$ is weak idempotent nil-clean ring and hence $R/\sqrt{I} \cong (R/I)/(\sqrt{I}/I)$ is weak idempotent nil-clean. Since $I \subseteq \sqrt{I}$ and $\sqrt{I} = \{a \in R : a + I \textit{ is nilpotent in } R/I\}$, $\sqrt{I}/I = \sqrt{I}$ and $\sqrt{I}/I$ is nil ideal of R/I. By Proposition 2.6(3), R/I is a weak idempotent nil-clean ring. Hence, R is a weak idempotent nil-neat ring.
$(5) \implies (6)$. Suppose R/I is weak idempotent nil-clean for every non-zero semiprime ideal I. Let $a + I$ be a nilpotent element of R/I. Thus, $(a + I)^k = I$ for some $k \in \mathbb{Z}$. So $a^k + I = I$ implies that $a^k \in I$ which in turn implies that $a \in \sqrt{I}$. As $I = \sqrt{I}$, $a \in I$, i.e., $a + I = I$. Hence, $R/I = wi(R/I)$. The converse is obvious. $(6) \iff (7)$ follows from Proposition 2.7.
$(1) \iff (8) \iff (9)$ is clear using Proposition 2.10. □

Example 3.12. Consider the ring of the localization of integers at the prime ideal (3), $R = \mathbb{Z}_{(3)}$. Then $0_{(3)} = \{0\}$, $2_{(3)}$ and $3_{(3)}$ are the only prime ideals of $\mathbb{Z}_{(3)}$. Now $\mathbb{Z}$ is an ideal of R such that $0_{(3)} \subset \mathbb{Z} \subset R$. So $0_{(3)}$ is not maximal ideal of R and hence R is not zero dimensional. By Corollary 2.8, R is not a weak idempotent nil-clean ring. Define a homomorphism $\alpha : \mathbb{Z}_{(3)} \to \mathbb{Z}_3$ by $\alpha(\frac{m}{n}) = \bar{0}$ if $m \equiv 0(mod\ 3)$; $\alpha(\frac{m}{n}) = \bar{1}$ if $m \equiv 1(mod\ 3)$ and $\alpha(\frac{m}{n}) = \bar{2}$ if $m \equiv 2(mod\ 3)$. Then it can be verified that α is an epimorphism. Thus $ker\alpha = 3_{(3)}$ and $\mathbb{Z}_{(3)}/3_{(3)} \cong \mathbb{Z}_3$ by first isomorphism theorem. Since every prime ideal is a semiprime ideal, $3_{(3)}$ is semiprime. Also, $\mathbb{Z}_3$ is a weak idempotent nil-clean ring. Hence, by Lemma 3.11, $\mathbb{Z}_{(3)}$ is weak idempotent nil-neat ring.

Corollary 3.13. *A ring R is weak idempotent nil-neat if and only if*

(i) *Every nonzero prime ideal of R is maximal.*

(ii) *For any nonzero semiprime ideal I of R, either $R/M \cong \mathbb{Z}_3$ for at least one maximal ideal M containing I or $R/N \cong \mathbb{Z}_2$ for at least one other maximal ideal N containing I.*

Proof. $(\implies)$. Suppose R is a weak idempotent nil-neat ring.
(i) Let P be a nonzero prime ideal of R. Then R/P is the weak idempotent nil-clean domain. By Corollary 2.8, either $R/P \cong \mathbb{Z}_2$ or $R/P \cong \mathbb{Z}_3$. This implies that R/P is a field. Hence, P is a maximal ideal.
(ii) Assume that I is a nonzero semiprime ideal of R. Then R/I is a weak idempotent nil-clean ring by Lemma 3.11(5). Hence, the result obtained from Lemma 3.11(8).
$(\impliedby)$. It is obvious by using Lemma 3.11 $((8) \implies (1))$ and Proposition 2.10. □

Proposition 3.14. *A ring R is weak idempotent nil-neat if and only if exactly one of the following is true:*

(1) *R is a field, or*

(2) *$J(R) \neq 0$ and $R/J(R)$ is isomorphic to either a Boolean ring (i.e., to a subring of a direct product of copies of $\mathbb{Z}_2$), or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ for some Boolean ring B.*

(3) *$J(R) = 0$, R is not a field, and R is isomorphic to either a Boolean ring B (i.e., to a subring of a direct product of copies of $\mathbb{Z}_2$), or $B \times \prod_\mu \mathbb{Z}_3$, or $\prod_\lambda \mathbb{Z}_3$ for some ordinals μ and λ. Moreover, every nonzero prime ideal of R is maximal.*

Proof. ($\Longrightarrow$). Suppose R is a weak idempotent nil-neat ring. If R is a field, then we are done. Now assume that R is a weak idempotent nil-neat ring which is not a field.
Let $J(R) \neq 0$. Then $rad(J(R)) = J(R)$ implies that $J(R)$ is semiprime ideal of R. By Lemma 3.11 (5) and (7), $R/J(R)$ is isomorphic to either a Boolean ring, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ for some Boolean ring B, as required.
Suppose $J(R) = 0$ and $Max(R) = \{M_i\}_{i\in I}$ for some index set I. Since R is not a field, $M_i \neq 0$. This implies that R has at least two maximal ideals. If $I = \{0, 1\}$, then by Lemma 3.11 (8), we have either $R/M_i \cong \mathbb{Z}_2$ or $R/M_i \cong \mathbb{Z}_3$ for $i \in I$. So R is isomorphic to a subring of either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Assume that $|I| > 2$. Then $i > 2$ and set $I_k = \cap_{i\neq k} M_i$. Again by Lemma 3.11(8), either $R/M_i \cong \mathbb{Z}_2$ or $R/M_i \cong \mathbb{Z}_3$. If $R/M_k \cong \mathbb{Z}_3$, then $R/Max(R) \cong \mathbb{Z}_2\times\mathbb{Z}_2\times\prod \mathbb{Z}_3$, or $R/Max(R) \cong \mathbb{Z}_2\times\prod \mathbb{Z}_3$, or $R/Max(R) \cong \prod \mathbb{Z}_3$ is weak idempotent nil-clean ring by Proposition 2.6 (2). If $R/M_k \cong \mathbb{Z}_2$, then $R/Max(R) \cong \prod \mathbb{Z}_2$, or $R/Max(R) \cong \prod \mathbb{Z}_2 \times \mathbb{Z}_3$, or $R/Max(R) \cong \prod \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is weak idempotent nil-clean ring by Proposition 2.6 (2). Hence, we conclude that R is isomorphic to either a subring of $\prod_\mu \mathbb{Z}_2 \times \prod_\lambda \mathbb{Z}_3$, or a subring of $\prod_\mu \mathbb{Z}_2$, or a subring of $\prod_\lambda \mathbb{Z}_3$.
($\Longleftarrow$). Assume that one of the statements (1), (2) and (3) holds true. By Proposition 2.8(ii), we have that every nonzero prime ideal of R is maximal. If R is a field, then R is a weak idempotent nil-neat ring since R has no proper ideal. Now assume that R is not a field. If $J(R) \neq 0$ and I is a non-zero semiprime ideal of R, then by assumption $R/J(R)$ is isomorphic to a boolean ring, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ for some Boolean ring B. Now $J(R) \subseteq I$ since for any $x \in J(R)$, we have $x^k = 0 \in I$ for some $n \in \mathbb{N}$ which implies that $x \in I$. So R/I is isomorphic to a Boolean ring, or $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3 \times B$, or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$ for some Boolean ring B by Lemma 3.11. If $J(R) = 0$, then two cases arise.
Case 1. Assume that R is isomorphic to a subring of a direct product of copies of $\mathbb{Z}_2$ and a direct product of copies of $\mathbb{Z}_3$. So $\phi : R \to \prod_\mu \mathbb{Z}_2 \times \prod_\lambda \mathbb{Z}_3$ is monomorphism. We know that the order of the element 1_R divides the order of $1_{\phi(R)}$. This implies that $O(1_R)$ is either 2 or 3 or 6 since $\prod_\mu \mathbb{Z}_2 \times \prod_\lambda \mathbb{Z}_3$ has characteristic exactly 6. Let I be a nonzero semiprime ideal of R and M_i be a maximal ideal of R containing I. Consider the epimorphism $\pi_i : R \to R/M_i$. Then $\pi_i(1_R) = 1_{R/M_i}$ as π_i is ring homomorphism. Since R/M_i is a field, 2 or 3 divides the order of the element $1_{R/M_i}$. Thus R/M_i is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Hence, R is a weak idempotent nil-neat ring by Corollary 3.13.
Case 2. Suppose R is isomorphic to $\prod_\lambda \mathbb{Z}_3$. Then $\alpha : R \to \prod_\lambda \mathbb{Z}_3$ is monomorphism. So R embeds into a ring of order 3^λ and hence R has either 1, or 3, or $\cdots$, or $3^{|\lambda|}$ elements. If R embeds in a one-element ring, then it is trivial. Assume that $|R| = 3$. Then it is an integral domain with no nontrivial ideal. So it must be isomorphic to $\mathbb{Z}_3$. For the case where $|R| = 3^{|\lambda|}$, we have $R \cong \prod_\lambda \mathbb{Z}_3$ which is weak idempotent nil-clean ring by Proposition 2.6(2). □

Corollary 3.15. *Let R be a ring such that $J(R) \neq 0$. Consider the following statements:*

(1) R is a weak idempotent nil-clean ring.

(2) R is a weak idempotent nil-neat ring.

(3) R is a clean UNI ring.
Then (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). Further, if $2 \in Nil(R)$, then the above three statements are equivalent to:

(4) R is a clean WUU ring.

(5) R is a weakly nil-clean ring.

(6) $J(R)$ is a nil ideal, and $R/J(R)$ is a Boolean ring;

(7) R is an exchange WUU ring.

(8) R is a weakly nil-neat ring.

(9) R is weakly clean WUU.

Proof. (1) $\Longrightarrow$ (2). Suppose R is a weak idempotent nil-clean ring. Then R is a weak idempotent nil-neat ring Since homomorphic images of weak idempotent nil-clean rings are weak idempotent nil-clean.

(2) $\implies$ (3). Assume that R is a weak idempotent nil-neat ring. Then by Proposition 3.14, $R/J(R)$ is isomorphic to either a Boolean ring B (i.e., to a subring of a direct product of copies of $\mathbb{Z}_2$), or a subring of $\prod_\lambda \mathbb{Z}_3$, or $B \times \prod_\mu \mathbb{Z}_3$ for some ordinals μ and λ. Thus R is clean UNI [by [7], Theorem 2.1].
(3) $\implies$ (2). Suppose R is clean UNI. Then $R/J(R)$ is isomorphic to either a Boolean ring B, or a subring of $\prod \mathbb{Z}_3$, or a direct product of two such rings. Hence, R is weak idempotent nil-neat ring by Proposition 3.14.
(4) $\implies$ (3). Suppose R is clean WUU. Then we show that every WUU ring is UNI. By hypothesis, we have $U(R) = \pm 1 + Nil(R)$. Since $(\pm 1)^2 = 1$, 1 and -1 are central involutions. So R is UNI.
(3) $\implies$ (4). Let $i \in R$ be involution. Then $(1-i)^2 = 2(1-i) \in Nil(R)$ which implies that $1 - i \in Nil(R)$. Let $r \in R$. Then by hypothesis, $r = n + i$ for some nilpotent n. Thus $r = [n - (1-i)] + 1$ or $r = [n + (1-i)] - 1$. Hence, R is the WUU ring.
(4) $\iff$ (6) $\iff$ (9) follows from [[6], Theorem 2.7]
(4) $\iff$ (5) $\iff$ (6) $\iff$ (7) $\iff$ (8) follows from [[10], Corollary 2.10]. □

Corollary 3.16. *The following statements are equivalent for a ring R.*

(1) R is weakly clean UNI;

(2) R is clean UNI;

(3) R is weak idempotent nil-neat;

(4) R is exchange with strongly invo-fine $U(R)$;

(5) $J(R)$ is nil with $R/J(R)$ is isomorphic to either $\prod_\lambda \mathbb{Z}_2$, or $\prod_\mu \mathbb{Z}_3$, or $\prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

Proof. (2) $\implies$ (1). It is obvious.
(1) $\implies$ (2). Suppose R is weakly clean UNI. Then $6 \in J(R)$ or $30 \in J(R)$ by [[8], Lemma 2.1].
Case 1: If $6 \in J(R)$, then R can be decomposed as $R \cong R_1 \times R_2$, where R_1 is a UU ring and R_2 is either $\{0\}$ or a UNI ring with $3 \in J(R_2)$ by [[7], Lemma 2.3]. Since homomorphic images of weakly clean rings is weakly clean, R_1 and R_2 are weakly clean rings. As $2 \in R_1$ is nilpotent, R_1 is clean by [[5], Proposition 2.6]. Thus, R_1 is clean UNI. Also, R_2 is clean UNI by [[7], Theorem 2.1]. Hence, R is clean because the direct product of clean UNI rings is clean UNI.
Case 2: If $30 \in J(R)$, then $(30)^n = 0$ for some natural number n. Thus either $(2^n, 3^n, 5^n) = 1$, i.e., there exist integers u, v and w such that either $2^n u + 3^n v + 5^n w = 1$. So this allows us to write that either $R = 2^n R + 3^n R + 5^n R$ and also $2^n R \cap 3^n R \cap 5^n R = \{0\}$. Thus $R = 2^n R \oplus 3^n R \oplus 5^n R$ and hence $R \cong (R/2^n R) \times (R/3^n R) \times (R/5^n R) = R_1 \times R_2 \times R_3$ with $R_1 = R/2^n R \cong (R/3^n R) \times (R/5^n R)$, $R_2 = R/3^n R \cong (R/2^n R) \times (R/5^n R)$ and $R_3 = R/5^n R \cong (R/2^n R) \times (R/3^n R)$. For the case $R_1 \cong R_2 \times R_3$, we have $3 \in J(R_2)$ and $5 \in J(R_3)$. By [7] of Theorem 2.1, R_2 is a clean UNI ring. Next, we claim that R_3 is a trivial ring. Now $6 = 1 + 5 \in 1 + J(R_3)$ is a unit in R_3. So 2 and 3 are units in R_3. But this contradicts Lemma 2.1 from [7]. Hence, the claim. In this case, $R_1 \cong R_2$ is a clean UNI ring. Therefore, R is clean UNI.
(2) $\iff$ (3) $\iff$ (5) follows from Corollary 3.15.
(4) $\iff$ (5) obtained from Corollary 3.15 and [[8], Theorem 2.3]. □

Proposition 3.17. *Let (R, M) be a local ring which is not a field. The following statements are equivalent:*

(1) R is a clean UNI ring and M is a nil ideal.

(2) R is a weak idempotent nil-clean ring.

(3) R is a weak idempotent nil-neat ring.

(4) R is a UNI ring.

Proof. Suppose (R, M) is a local ring with the nonzero maximal ideal M. Then $Nil(R) = J(R) = M$ is nil ideal, $Id(R) = \{0, 1\}$ and $U(R) = Nil(R) \pm 1$. By Corollary 3.15, we have

(1) $\iff$ (2) $\iff$ (3).
(4) $\implies$ (1) is obvious.
(1) $\implies$ (4). Let $r \in R$. Then $r = u + e$ where $u \in U(R)$ and $e \in Id(R)$. Thus $r = u$ or $r = u + 1 \in Nil(R)$ but $u = n \pm 1$ for some $n \in Nil(R)$. Hence, R is the UNI ring. □

4 Conclusion remarks

This paper aims is to obtain certain characterizations of weak idempotent nil-neat rings in terms of semiprime ideals, maximal ideals, Jacobson radicals, and reduced weak idempotent nil-neat rings. Moreover, we obtain a ring R is weak idempotent nil-neat if and only if exactly R is a field; or $J(R) \neq 0$ and $R/J(R)$ is isomorphic to any of the special rings namely Boolean ring B or $\mathbb{Z}_3$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times B$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times B$. There are many other properties of weak idempotent nil-neat rings that are not covered by these paper. Therefore, the results of this work are significant, interesting and capable to develop its study in the future.

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