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Controllability of fractional impulsive integro-differential control system

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Abstract In this paper we will study the results the controllability of fractional impulsive integro-differential control systems in Banach spaces using the fixed-point technique and the (α, θ)-resolvent operator.

1 Introduction

Over the years, the control theory has been highlighted in the scientific community. Controllability is one of the fundamental concepts in mathematical control theory. It is well known that the question of controllability plays a fundamental role in the design of engineering control problems [2, 1, 18, 33, 36]. In fact, the most important property of a control system is simply its controllability. It is possible to find numerous works on differential and integro-differential equations that discuss the controllability of solutions [3, 19, 21, 36] and references therein.

On the other hand, fractional calculus and fractional differential equations have been studied extensively, mainly because of their demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering such as physics, economics, medicine, control theory, aerodynamics and electromagnetic, [4, 8, 13, 32, 37, 38]. In this sense, it has been noted is that once you have control theory in hand, in particular controllability, it has been noted that the study of solutions of differential and integro-differential equations with non-instantaneous, evolution and abstract impulses, began to be the target of studies in these last decades [6, 14, 12, 30, 31, 33, 34]. For a reading of some works, see [1, 7, 21] and the references therein.

Controllability problems for different types of differential equations have been considered in many papers [5, 10, 15, 16, 17, 23, 35, 39]. In 2011, Debbouche and Baleanu [6], investigated the controllability result of a class of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems in a Banach space, using fixed point technique. Em 2015, Liu and Li [16], investigated the approximate controllability of the following fractional evolution control systems involving Riemann–Liouville fractional derivatives:

$$D_t^{\alpha} x(t) = Ax(t) + Bu(t) + f(t, x(t)),\ t \in (0, b],\ 0 < \alpha < 1,$$

$$I_t^{1-\alpha} x(t)|_{t=0} = x_0 \in X$$

where D_t^{α} is the Riemann-Liouville fractional derivative of order α with the lower limit zero. $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $T(t)_{t\geq 0}$ on a Banach space X. $f : [0, b] \times X \to X$ is a given function to be specified in the paper. The control function u takes value in $V = L^p([0, b], U)$, $p > \frac{1}{\alpha}$, and U is a Banach space. B is a linear operator from V to $L^p([0, b]; X)$.

Mu [20] investigated the existence of mild solutions for the impulsive fractional evolution equations of the form

$$\begin{cases} D^{\alpha}_{0+}u(t) + Au(t) &= f(t, u(t)), t \in I := [0, T], t \neq t_k \\ u(0) + g(u) &= u_0 \\ \Delta u|_{t=t_k} &= I_k(u(t_k)),\ k = 1, 2, ..., m \end{cases}$$

where D^{α}_{0+} is the Caputo fractional derivative with $0 < \alpha < 1$, $A : D(A) \subset X \to X$ is a linear closed densely defined operator, $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $(T(t)_{t\geq 0})$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $f : I \times X \to X$ is continuous, $g : PC(I, X) \to X$ is continuous, the impulsive function $I_k : X \to X$ is continuous, $\Delta u|_{t=t_k} = u(t^+) - u(t^-$, where $u(t^+)$, $u(t^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

In 2021, Kumar et al. [13], on necessary and sufficient conditions, studied the fractional stability damped differential system with non-instantaneous impulsive given by

$$\begin{cases} D^{\alpha}x(t) &= AD^{\beta}x(t) + M\left(t, x(t), \int_0^t H(t, r, x(t))dr\right),\ t \in \bigcup_{i=0}^{m}(r_i, t_{i+1}] \\ x(t) &= \mathcal{I}_i(t, x(t_i^-)),\ t \in (t_i, r_i],\ i = 1, 2, ..., m \\ x'(t) &= \mathcal{G}_i(t, x(t_i^-),,\ t \in (t_i, r_i],\ i = 1, 2, ..., m \\ x(0) &= x_0,\ x'(0) = x_1 \end{cases}$$

and for the controllability, the author consider the following system

$$\begin{cases} D^{\alpha}x(t) &= AD^{\beta}x(t) + Bu(t) + M\left(t, x(t), \int_0^t H(t, r, x(t))dr\right),\ t \in \bigcup_{i=0}^{m}(r_i, t_{i+1}] \\ x(t) &= \mathcal{I}_i(t, x(t_i^-)),\ t \in (t_i, r_i],\ i = 1, 2, ..., m \\ x'(t) &= \mathcal{G}_i(t, x(t_i^-),,\ t \in (t_i, r_i],\ i = 1, 2, ..., m \\ x(0) &= x_0,\ x'(0) = x_1 \end{cases}$$

where D^{α} and D^{β} denote the Caputo fractional derivative of order $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, respectively. For more details on the parameters A, B, H, $\mathcal{I}_i$ and $\mathcal{G}_i$, see the paper [13]. Other interesting papers on controllability can be found in [2, 3, 9, 11, 18, 19, 36, 40] and references therein.

However, there are some problems and open questions when discussing the controllability of mild solutions for fractional operators that involve the ψ-Hilfer fractional derivative. Over the years, Sousa and Oliveira [25], introduced the so-called ψ-Hilfer fractional derivative and due to the impact on the scientific community, this derivative has served as a motivation to discuss various problems of differential equations. Here we highlight the typo Leibniz I and II rule [26]; and the Laplace transform with respect to another function. In this sense, it allowed the discussion of mild solutions of fractional differential equations. However, there are still problems that prevent a closed form for the mild solution of fractional differential equations, which makes the theory still under construction.

Motivated by these questions and the work presented above, in this paper we consider the fractional impulsive integro-differential control system of the form

$${}^{C}\mathcal{D}^{\alpha}_{0+}\theta(t) + \mathcal{A}(t, \theta(t))\theta(t) = (\mathbf{B}\mu)(t) + \mathbf{\Phi}(t, \theta(t)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(t, s, \theta(s))ds \quad (1.1)$$

$$\theta(0) + \Xi(\theta) = \theta_0 \quad (1.2)$$

$$\Delta\theta(t_i) = I_i(\theta(t_i)),\ i = 1, ..., n, 0 < t_1 < ... < t_n < b \quad (1.3)$$

where ${}^{C}\mathcal{D}^{\alpha}_{0+}(\cdot)$ is the Caputo fractional derivative of order $0 < \alpha \leq 1, t \in J := [0, b]$, the state $\theta(\cdot)$ takes values in the Banach space Λ, $\theta_0 \in \Lambda$, $i = 1, 2, ..., n$ and Λ in Λ such that $D(A)$ is independent of t, it is also assumed that -$\mathcal{A}(t, \cdot)$ generates an addition in the Banach space Λ, the control function μ belongs to the spaces $L^2(J, U)$ a Banach of admissible control functions with U as a Banach space and $\mathbf{B} : U \to \Lambda$ is a bounded linear operator. We also, we have $\mathbf{\Phi} : [0, b] \times \Lambda \to \Lambda$, $g : \Omega \times \Lambda \to \Lambda$, $\Xi : \mathcal{PC}([0, b], \Lambda) \times \Lambda \to \Lambda$ and $\Delta\theta(t_i) = \theta(t_i^+) - \theta(t_i^-)$.

In 2020 Ramos et al. [22], investigated the existence and uniqueness of mild solutions for a fractional problem of the type Eq.(1.1)-Eq.(1.3), however, at that time it was not considered the control function μ. In this sense, we impose the μ control to discuss the main objective of this paper. This paper is a natural continuation of the paper [22].

For the discussion of the main result of this paper, we assume the following conditions:

H$_1$ The bound linear operator $W : L^2(J,U) \to \Lambda$ defined by

$$W_\mu = \int_0^b \mathcal{R}_{(\alpha,\theta)}(b,s)(\mathbf{B}\mu)(s)\, ds,$$

has an induced inverse operator $\tilde{W}^{-1}$ taking values in $L^2(J,U)/\ker W$ and there exist positive constants $\mathbf{M}_1$ and $\mathbf{M}_2$ such that $\|\mathbf{B}\| \ \leq {}_1$ and $\|\tilde{W}^{-1}\| \ \leq {}_2$.

H$_2$ $h : \mathcal{PC}(J;\Omega) \to Y$ is Lipschitz continuous in Λ and bounded in Y, i.e., there exist constants $\mathbf{K}_1 > 0$ and $\mathbf{K}_2 > 0$ such that

$$\|h(\theta)\|_Y \ \leq \mathbf{K}_1,$$
$$\|h(\theta) - h(v)\|_Y \ \leq \mathbf{K}_2 \max_{t\in J} \|\theta - v\|_{\mathcal{PC}}, \ \ \theta, v \in \mathcal{PC}(J;\Lambda).$$

For the conditions (**H**$_3$) - (**H**$_5$) let Z be taken as both Λ and Y.

H$_3$ $g : \Omega \times Z \to Z$ is continuous and there exist constants $\mathbf{K}_3 > 0$ and $\mathbf{K}_4 > 0$ such that

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|g(t,s,\theta) - g(t,s,v)\|_Z\, ds \ \leq \mathbf{K}_3 \|\theta - v\|_Z, \ \theta, v \in Z,$$

$$\mathbf{K}_4 = \max\left\{ \int_0^t \|g(t,s,0)\|_Z\, ds; \ (t,s) \in \Omega \right\}.$$

H$_4$ $f : J \times Z \to Z$ is continuous and there exist constants $\mathbf{K}_5 > 0$ and $\mathbf{K}_6 > 0$ such that

$$\|f(t,\theta) - f(t,v)\|_Z \ \leq \mathbf{K}_5 \|\theta - v\|_Z, \ \theta, v \in \Omega,$$
$$\mathbf{K}_6 = \max_{t\in J} \|f(t,0)\|_Z.$$

H$_5$ $I_i : \Lambda \to \Lambda$ are continuous and there exist constants $l_i > 0, \ i = 1, 2, ..., m$ such that

$$\|I_i(\theta) - I_i(v)\| \leq l_i \|\theta - v\|, \ \theta, v \in \Lambda.$$

Let us take $\mathbf{M}_0 = \max \|\mathcal{R}_{(\alpha,\theta)(t,s)}\|_{B(Z)}, \ 0 \leq s \leq t \leq b, \ \theta \in \Lambda$.

H$_6$ There exist positive constants $\delta_1, \delta_2, \delta_3 \in (0, \delta/3]$ such that

$$\delta_1 = \mathbf{M}_0 \|\theta_0\| + \mathbf{M}_0 \mathbf{K}_1,$$
$$\delta_2 = \mathbf{M}_0 \mathbf{M}_1 \mathbf{M}_2 b \left(\|\theta_1\| + \mathbf{M}_0 \|\theta_0\| + \mathbf{M}_0 \mathbf{K}_1 + \mathbf{M}_0 \tilde{\theta} + \mathbf{M}_0 \xi \right) \text{ and}$$
$$\delta_3 = \mathbf{M}_0 \tilde{\theta} + \mathbf{M}_0 \xi,$$

where $\xi = \sum_{i=1}^m (l_i \delta + \|I_i(0)\|)$.

The main contribution of this paper is to attack, through necessary and sufficient conditions, to attack the controllability of the mild solution of the fractional impulsive integro-differential control system given by Eq.(1.1)-(1.3). In other words, we will attack the following result:

Theorem 1.1. *Suppose that the operator $-\mathcal{A}(t,\theta)$ generates a (α,θ)-resolvent family whit $\|\mathcal{R}_{(\alpha,\theta)}(t,s)\| \leq \mathbf{M} e^{-\sigma(t-s)}$ for some constant $\mathbf{M}, \sigma > 0$. If the hypotheses (**H**$_1$)-(**H**$_6$) are satisfied, then the fractional control integro-differential system (1.1) with nonlocal condition (1.2) and impulsive condition (1.3) is controllable on J.*

The rest of the paper is structured as follows. Section 2 introduces the basic concepts of the Riemann-Liouville fractional integral and the ψ-Hilfer fractional derivative, as well as some special cases. In this sense, we introduce the concepts of mild solution and family (α,θ)-resolvent. In section 3, we investigate a fundamental lemma and attack the main objective of the paper, that is, the controllability of mild solutions of fractional impulsive integro-differential control systems.

2 Preliminaries

For the preparation of this paper, we will consider X and Y to be two Banach spaces such that Y is densely and continuously embedded in X. For any Banach space Z, the norm of Z is denoted by $||\cdot||_Z$.

Let $p \in [1,\infty) \subset \mathbb{R}$ and $\mathbf{J} = [a,b] \subset \mathbb{R}$. The space of the real p-integrable functions, in the Lebesgue sense, $L^p(J)$, equipped with its canonical norm, is given by [22, 26]

$$L^p(\mathbf{J}) := \left\{ f : \mathbf{J} \to \mathbb{R};\ \int_a^b |f(t)|^p dt < \infty \right\}$$

and

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p},$$

respectively. The pair $(L^p(\mathbf{J}), \|f\|_p)$ is a Banach space.

Consider the Banach space $(\mathbf{E}, \|\cdot\|)$ and $n \in \mathbb{N}$. The space of continuous functions and the space of continuously differentiable functions n-times:

$$\left(C(\mathbf{J},\mathbf{E}) := \{ f : J \to \mathbf{E};\ f : \text{continuous} \},\quad \|f\|_C := \sup_{t\in J} |f(t)| \right) \text{ and}$$

$$\left(C^n(\mathbf{J},\mathbf{E}) := \left\{ f : J \to \mathbf{E};\ f^{(n)} \in C(J,\mathbf{E}) \right\},\quad \|f\|_{C^n} := \sup_{t\in J} |f^{(n)}(t)| \right)$$

are Banach spaces.

Let $\mathbf{J} = [a,b] \subset \mathbb{R}$ be an interval, with $0 < a < b < \infty$, then the space of the n functions-absolutely continuous times is given by

$$AC^n(\mathbf{J},\mathbb{R}) = AC^n(J) = \left\{ f : \mathbf{J} \to \mathbb{R};\ f^{(n-1)} \in AC(\mathbf{J}) \right\}.$$

Let $\mathbf{E}$ be a Banach space and $a = t_0 < t_1 < \ldots < t_n = b$ a n-partition of the interval $\mathbf{J} \subset \mathbb{R}$. The scope of continuous functions by parts given by

$$\mathcal{PC}(\mathbf{J},\mathbf{E}) := \left\{ \begin{array}{c} f : \mathbf{J} \to \mathbf{E};\ f(t) \text{ be continuous in } t \neq t_k, \text{ left continuous} \\ \text{in } t = t_k \text{ there is the limit on the right, } f(t_k^+), \text{for } k = 1,2,\ldots n. \end{array} \right\},$$

equipped with the standard $\|f\|_{\mathcal{PC}} = \{\sup \|f(t)\|;\ t \in \mathbf{J}\}$ is a Banach space.

Definition 2.1. [25, 28, 29] Let (a,b) $(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and let $\alpha > 0$. Furthermore, let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, with a continuous derivative $\psi'(t)$ on (a,b). The left fractional integral of the function θ with respect to another function ψ on $[a,b]$ is defined by

$$I_{a^+}^{\alpha;\psi}\theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\theta(s)ds. \tag{2.1}$$

The right-sided fractional integral is defined in an analogous form [25, 28, 29].

If we choose $\psi(t) = t$ in Eq.(2.1), we have the Riemann-Liouville fractional integral given by [25, 28, 29]

$$I_{a^+}^{\alpha}\theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\theta(s)ds, \tag{2.2}$$

where $\Gamma(\cdot)$ is the gamma function and $f \in L^1(\mathbf{J},\mathbb{R})$.

If $a = 0$, we can write $I^\alpha\theta(t) = (g_\alpha * \theta)(t)$, where

$$g_\alpha(t) := \left\{ \begin{array}{ll} \dfrac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0 \\ 0, & t \le 0 \end{array} \right. \tag{2.3}$$

and as usual $*$ denotes convolution of functions, we also have $\lim_{\alpha\to 0} g_\alpha(t) = \delta(t)$. By choosing $\psi(\cdot)$, we have another fractional integral.

We will restrict ourselves here to the Riemann-Liouville fractional integral in order to discuss the results of this paper. However, other formulations of fractional integrals can be obtained by choosing $\psi(\cdot)$ [25, 28, 29].

We also start begin with the definition of the ψ-Hilfer fractional derivative.

Definition 2.2. [25, 28, 29] Let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, let $\mathbf{J}$ be an interval such that $-\infty \leq a < b \leq \infty$ and let $\theta, \psi \in C^n(\mathbf{J}, \mathbb{R})$ be two functions, such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in \mathbf{J}$. The left-sided ψ-Hilfer fractional derivative ${}^{\mathbf{H}}\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ of a function θ, of order α and type $0 \leq \beta \leq 1$ is defined by

$$ {}^{\mathbf{H}}\mathbb{D}_{a+}^{\alpha,\beta;\psi}\theta(t) := I_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi}\theta(t). \tag{2.4}$$

The right-sided ψ-Hilfer fractional derivative is defined in an analogous form [25, 28, 29]

Choosing $\psi(t) = t$ and taking the limit $\beta \to 1$, on both sides of the Eq.(2.4), we have the Caputo fractional derivative given by [25, 28, 29]

$$ {}^{C}\mathcal{D}_{a+}^{\alpha}\theta(t) = I_{a+}^{(n-\alpha);\psi}\left(\frac{d}{dt}\right)^n \theta(t) = I_{a+}^{(n-\alpha);\psi}\theta^{(n)}(t). \tag{2.5}$$

To investigate our main result, we use Caputo fractional derivative as in Eq.(2.5).

Definition 2.3. By a mild solution of the system (1.1)-(1.3) we wean a function $\theta \in \mathcal{PC}([0,b];\Lambda)$ with values in Ω which satisfy the integral equation

$$\begin{aligned} \theta_\mu(t) &= \mathcal{R}_{(\alpha,\theta)}(t,0)[\theta_0 - h(\theta)] \\ &+ \int_0^t \mathcal{R}_{(\alpha,\theta)}(t,s)\left[(B\mu)(s) + f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\eta)^{\alpha-1} g(s,\eta,\theta(\eta))\,d\eta\right] ds \end{aligned} \tag{2.6}$$

for all $t \in J$, for all $\theta_0 \in \Lambda$ and admissible control $\mu \in L^2(J,U)$.

Lemma 2.4. [22] *If the evolution family* $\left\{U_\theta(t,s)\right\}_{0\leq s\leq t\leq b}$ *is continuous and* $\eta \in \mathscr{L}(J,\mathbb{R}^+)$, *then the set* $\left\{\int_0^t U_\theta(t,s)\theta(s)ds\right\}$, $\|\theta(s)\| \leq \eta(s)$ *for a.e.* $s \in J$ *is equicontinuous for* $t \in J$.

From [22] we know that for any fixed $u \in \mathcal{PC}(J,\Lambda)$ there exists a unique continuous function $U_\theta : J \times J \to \mathbf{B}(\Lambda)$ defined on $J \times J$ such that

$$ U_\theta(t,s) = I + \int_s^t \mathcal{A}_\theta(w)U_\theta(w,s)dw, \tag{2.7}$$

where $\mathbf{B}(\Lambda)$ denotes the Banach space of a bounded linear operator from Λ to Λ with the norm $\|\Theta\| = \sup\{\|\Theta(\theta)\|; \|\theta\| = 1\}$ and I stands for the identity operator on Λ, $\mathcal{A}_\theta = \mathcal{A}(t,\theta(t))$, we have [22]

$$ U_\theta(t,t) = I,\ U_\theta(t,s)U_\theta(s,r) = U_\theta(t,r),\ (t,s,r) \in J \times J \times J$$

and

$$ \frac{\partial U_\theta(t,s)}{\partial t} = \mathcal{A}_\theta(t)U_\theta(t,s),\ \text{for almost all } t,s \in J.$$

Let $\mathbf{E}$ be the Banach space formed by $D(\mathcal{A})$ with the graph norm. Since, $\mathcal{A}(t)$ is a closed operator, it follows that $\mathcal{A}(t)$ is in the set bounded by $\mathbf{E}$ and Λ.

Definition 2.5. [8, 22] Let $\mathcal{A}(t,\theta)$ be a closed and linear operator with domain $D(\mathcal{A})$ defined on a Banach space Λ and $\alpha > 0$. Let $\rho(\mathcal{A}(t,\theta))$ be the resolvent set of $\mathcal{A}(t,\theta)$. We call $\mathcal{A}(t,\theta)$ the generator of an (α,θ)-resolvent family if there exists $w \geq 0$ and a strongly continuous function $\mathcal{R}_{(\alpha,\theta)} : \mathbb{R}_+^2 \to \mathscr{L}(\Lambda)$ such that $\{\lambda^\alpha : Re(\lambda) > w\} \subset \rho(\mathcal{A})$ and for $0 \leq s \leq t \leq \infty$,

$$ \left(\lambda^\alpha I - \mathcal{A}(s,\theta)\right)^{-1}\nu = \int_0^\infty e^{-\lambda(t-s)}\mathcal{R}_{(\alpha,\theta)}(t,s)\,\nu\,dt,\ Re(\lambda) > w,\ (\theta,\nu) \in \Lambda^2.$$

In this case, $\mathcal{R}_{(\alpha,\theta)}(t,s)$ is called the (α,θ)-resolvent family generated by $\mathcal{A}(t,\theta)$.

Remark 2.6.
- We can deduce that Eqs.(1.1)-(1.3) is well posed if and only if $\mathcal{A}(t,\theta)$ is the generator of the (α,θ)-resolvent family.
- Here $\mathcal{R}_{(\alpha,\theta)}(t,s)$ can be extracted from the evolution operator of the generator $\mathcal{A}(t,\theta)$.
- The (α,θ)-resolvent family is similar to the evolution for non-autonomous differential equations in a Banach space.

3 Controllability result

Definition 3.1. We shall say that the fractional system Eqs.(1.1)-(1.3) is controllable on the interval $J=[0,b]$ if for all $\theta_0,\theta_1\in\Lambda$, there exists a constant $\mu\in L^2(J,U)$, such that the mild solution $\theta(\cdot)$ of the systems (1.1)-(1.3) corresponding to μ is verified: $\theta(0)+h(\theta)=\theta_0, \Delta\theta(t_i)=I_i(\theta(t_i)),\ i=1,2,...,m$ and $\theta_\mu(b)=\theta_1$

Lemma 3.2. *Let $\mathcal{R}_{(\alpha,\theta)(t,s)}$ be the (α,θ)-resolvent family for the fractional problem (1.1)-(1.3). There exists a constant $K>0$ such that*

$$\|\mathcal{R}_{(\alpha,\theta)}(t,s)\omega-\mathcal{R}_{(\alpha,v)}(t,s)\omega\|\leq \mathbf{K}\|\omega\|_Y\int_s^t\|\theta(\tau)-v(\tau)\|\,d\tau,$$

for every $\theta,v\in\mathcal{PC}(J;\Lambda)$ whit values in Ω and every $\omega\in Y$.

Proof. See [17]. □

Lemma 3.3. $\|\varphi(t)\|\ \leq\ \tilde{\theta}$, *where* $\varphi(t)=\int_0^t\left(f(s,\theta(s))+\frac{1}{\Gamma(\alpha)}\int_0^s(s-\eta)^{\alpha-1}g(s,\eta,\theta(\eta))\,d\eta\right)\,ds$

Proof.

$$\begin{aligned}
&\|\varphi(t)\|\\
=&\left\|\int_0^t\left(f(s,\theta(s))+\frac{1}{\Gamma(\alpha)}\int_0^s(s-\eta)^{\alpha-1}g(s,\eta,\theta(\eta))\,d\eta\right)\,ds\right\|\\
=&\left\|\int_0^t(f(s,\theta(s))+\frac{1}{\Gamma(\alpha)}\int_0^s(s-\eta)^{\alpha-1}\left(g(s,\eta,\theta(\eta))\,d\eta-g(s,\eta,0)+g(s,\eta,0)\right)\right.\\
+&\left.\ f(s,0)-f(s,0))\right\|ds\\
\leq&\int_0^t\Bigg(\|f(s,\eta(s))-f(s,0)\|+\|f(s,0)\|\\
+&\frac{1}{\Gamma(\alpha)}\int_0^s(s-\eta)^{\alpha-1}\|g(s,\eta,\theta(\eta))-g(s,\eta,0)\|\,d\eta\\
+&\frac{1}{\Gamma(\alpha)}\int_0^s(s-\eta)^{\alpha-1}\|g(s,\eta,0)\|\,d\eta\Bigg)\,ds.
\end{aligned}$$

Using $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ yields

$$\begin{aligned}
\|\phi(t)\|&\leq\int_0^t(\mathbf{K}_5\|\theta(s)\|+\mathbf{K}_6+\mathbf{K}_3\|\theta(s)\|+\mathbf{K}_4)\,ds\\
&=\mathbf{K}_5\int_0^t\|\theta(s)\|\,ds+\mathbf{K}_6\int_0^t\,ds+\mathbf{K}_3\int_0^t\|\theta(s)\|ds+\mathbf{K}_4\int_0^t\,ds\\
&\leq K_5\int_0^t\|\theta(s)\|ds+\mathbf{K}_6b+\mathbf{K}_4b+\mathbf{K}_3\int_0^t\|\theta(s)\|ds.
\end{aligned}$$

Therefore, we concluded the proof. □

Now, let's attack our main result, that is, the proof of Theorem 1.1.

Proof. Using the hypothesis ($\mathbf{H}_1$), for any arbitrary function $\theta(\cdot)$, we define the control

$$\begin{aligned}
\mu(t) &= \tilde{W}^{-1}\Big[\theta_1 - \mathcal{R}_{(\alpha,\theta)}(b,0)\theta_0 + \mathcal{R}_{(\alpha,\theta)}(b,0)h(\theta) \\
&- \int_0^b \mathcal{R}_{(\alpha,\theta)}(b,s)\left(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\eta)^{\alpha-1} g(s,\eta,\theta(\eta))\, d\eta\right) ds \\
&- \sum_{i=1}^{m} \mathcal{R}_{(\alpha,\theta)}(b,t_i)\, I_i(\theta(t_i))\Big](t).
\end{aligned}$$

We define an operator $\mathbf{Q} : \mathbf{S}_\delta \to \mathbf{S}_\delta$ by

$$\begin{aligned}
(\mathbf{Q}\theta_\mu)(t) &= \mathcal{R}_{(\alpha,\theta)}(t,0)\theta_0 - \mathcal{R}_{(\alpha,\theta)}(t,0)h(\theta) \\
&+ \int_0^t \mathcal{R}_{(\alpha,\theta)}(t,\eta)\, \mathbf{B}\tilde{W}^{-1}\Big[\theta_1 - \mathcal{R}_{(\alpha,\theta)}(b,0)\theta_0 + \mathcal{R}_{(\alpha,\theta)}(b,0)h(\theta) \\
&- \int_0^b \mathcal{R}_{(\alpha,\theta)}(b,s)\left(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau))\, d\tau\right) ds \\
&- \sum_{i=1}^{m} \mathcal{R}_{(\alpha,\theta)}(b,t_i)\, I_i(\theta(t_i))\Big](\eta)\, d\eta \\
&+ \int_0^t \mathcal{R}_{(\alpha,\theta)}(t,s)\Big(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau))\, d\tau\Big) ds \\
&+ \sum_{0<t_i<t} \mathcal{R}_{(\alpha,\theta)}(t,t_i)\, I_i(\theta(t_i)).
\end{aligned}$$

Using this controller we shall that the operator $\mathbf{Q}$ has a fined point is then a solution of Eq.(1.1).

It is clear that $\mathbf{Q}\theta_\mu(b) = \theta_1$, which means that the control μ steers the system (1.1)-(1.3) from the initial state θ_0 to θ_1 in time b, provided that we can obtain a fixed value of the nonlinear operator $\mathbf{Q}$.

Now we show $\mathbf{Q}$ maps $\mathbf{S}_\delta$ into itself.

$$\begin{aligned}
\|(\mathbf{Q}\theta_\mu)(t)\| &\le \|\mathcal{R}_{(\alpha,\theta)}(t,0)\theta_0\| + \|\mathcal{R}_{(\alpha,\theta)}(t,0)h(\theta)\| \\
&+ \int_0^t \|\mathcal{R}_{(\alpha,\theta)}(t,\eta)\|\, \|\mathbf{B}\tilde{W}^{-1}\| \Big[\|\theta_1\| + \|\mathcal{R}_{(\alpha,\theta)}(b,0)\theta_0\| + \|\mathcal{R}_{(\alpha,\theta)}(b,0)h(\theta)\| \\
&+ \int_0^b \|\mathcal{R}_{(\alpha,\theta)}(b,s)\Big(\|f(s,\theta(s))\| + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\Big(\|g(s,\tau,\theta(\tau))\|\, d\tau\Big)\Big) \\
&+ \sum_{i=1}^{m} \|\mathcal{R}_{(\alpha,\theta)}(b,t_i)\|\|\{I_i(\theta(t_i)) - I_i(0)\| + \|I_i(0)\|\}\Big]\, d\eta \\
&+ \int_0^t \|\mathcal{R}_{(\alpha,\theta)}(t,s)\|\Big(\|f(s,\theta(s))\| + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|g(s,\tau,\theta(\tau))\|\, d\tau\Big) ds \\
&+ \sum_{0<t_i<t} \|\mathcal{R}_{(\alpha,\theta)}(t,t_i)\|\Big\{\|I_i(\theta(t_i)) - I_i(0)\| + \|I_i(0)\|\Big\}.
\end{aligned}$$

Using $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_5)$, $(\mathbf{H}_6)$ and Lemma 3.3, yields

$$\begin{aligned}
\|\left(\mathbf{Q}\theta_\mu\right)(t)\| &\leq \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \int_0^t \mathbf{M}_0\mathbf{M}_1\mathbf{M}_2\Bigg[\|\theta_1\| + \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 \\
&+ \int_0^b \mathbf{M}_0\Big(\|f(s,\theta(s))\|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|g(s,\tau,\theta(\tau))\|\,d\tau\Big)\,ds \\
&+ \mathbf{M}_0\sum_{i=1}^m (I_i\delta + \|I_i(0)\|)\Bigg]\,d\eta + \mathbf{M}_0\sum_{i=1}^m (I_i\delta + \|I_i(0)\|) \\
&+ \mathbf{M}_0\int_0^t \Big(\|f(s,\theta(s))\|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|g(s,\tau,\theta(\tau))\|\,d\tau\Big)\,ds \\
&\leq \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\mathbf{M}_1\mathbf{M}_2\int_0^t \Bigg[\|\theta_1\| + \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} \\
&+ \mathbf{M}_0\sum_{i=1}^m (I_i\delta + \|I_i(0)\|)\Bigg]\,d\eta + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\sum_{i=1}^m (I_i\delta + \|I_i(0)\|) \\
&= \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 \\
&+ \mathbf{M}_0\mathbf{M}_1\mathbf{M}_2\int_0^t \Big[\|\theta_1\| + \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi\Big]\,d\eta \\
&+ \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi \\
&\leq \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\mathbf{M}_1\mathbf{M}_2 \\
&\quad b\left(\|\theta_1\| + \mathbf{M}_0\|\theta_0\| + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi\right) + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi,
\end{aligned}$$

where $\xi = \sum_{i=1}^m (I_i\delta + \|I_i(0)\|)$.

From the assumption $(\mathbf{H}_6)$ we get $\|\left(\mathbf{Q}\theta_\mu\right)(t)\| \leq \delta$. So $\mathbf{Q}$ maps $\mathbf{S}_\delta$ into itself.

Now for $\theta, v \in \mathbf{S}_\delta$, we have

$$\|\left(\mathbf{Q}\theta_\mu\right)(t) - \left(\mathbf{Q}v_\mu\right)(t)\| \leq I_1 + I_2 + I_3 + I_4, \tag{3.1}$$

where

$$\begin{aligned}
I_1 &= \|\mathcal{R}_{(\alpha,\theta)}(t,0)\theta_0 - \mathcal{R}_{(\alpha,v)}(t,0)\theta_0\| + \|\mathcal{R}_{(\alpha,\theta)}(t,0)h(\theta) - \mathcal{R}_{(\alpha,v)}(t,0)h(v)\|, \\
I_2 &= \int_0^t \Bigg\{\Bigg\|\mathcal{R}_{(\alpha,\theta)}(t,\eta)\,B\tilde{W}^{-1}\Bigg[\theta_1 - \mathcal{R}_{(\alpha,\theta)}(b,0)\,\theta_0 + \mathcal{R}_{(\alpha,\theta)}(t,0)\,h(\theta) \\
&- \int_0^b \mathcal{R}_{(\alpha,\theta)}(b,s)\Big(f(s,\theta(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}g(s,\tau,\theta(\tau))\,d\tau\Big)\,ds \\
&- \sum_{i=1}^m \mathcal{R}_{(\alpha,\theta)}(b,t_i)I_i(\theta(t_i))\Bigg] - \mathcal{R}_{(\alpha,v)}(t,\eta)\,B\tilde{W}^{-1}\Bigg[\theta_1 - \mathcal{R}_{(\alpha,v)}(b,0)\,\theta_0 + \mathcal{R}_{(\alpha,v)}(b,0)\,h(v) \\
&- \int_0^b \mathcal{R}_{(\alpha,v)}(b,s)\Big(f(s,v(s))\,ds + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}g(s,\tau,v(\tau))\,d\tau\Big)\,ds \\
&- \sum_{i=1}^m \mathcal{R}_{(\alpha,v)}(b,t_i)I_i(v(t_i))\Bigg]\Bigg\|\Bigg\}, \\
I_3 &= \int_0^t \Bigg\|\mathcal{R}_{(\alpha,\theta)}(t,s)\Bigg(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}g(s,\tau,\theta(\tau))\,d\tau\Bigg) \\
&- \mathcal{R}_{(\alpha,v)}(t,s)\Big(f(s,v(s)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}g(s,\tau,v(\tau))\,d\tau\Big)\Bigg\|\,ds
\end{aligned}$$

and

$$I_4 = \sum_{i=1}^{m} \|\mathcal{R}_{(\alpha,\theta)}(t,t_i)I_i(\theta(t_i)) - \mathcal{R}_{(\alpha,v)}(t,t_i)I_i(v(t_i))\|.$$

Using the Lemma 3.2 and ($\mathbf{H}_2$), we have

$$\begin{aligned}
I_1 &\leq \|\mathcal{R}_{(\alpha,\theta)}(t,0)\theta_0 - \mathcal{R}_{(\alpha,v)}(t,0)\theta_0\| + \|\mathcal{R}_{(\alpha,\theta)}(t,0)h(\theta) - \mathcal{R}_{(\alpha,v)}(t,0)\, h(v)\| \\
&+ \|\mathcal{R}_{(\alpha,v)}(t,0)\, h(\theta) - \mathcal{R}_{(\alpha,v)}(t,0)\, h(v)\| \\
&\leq \mathbf{K}\, \|\theta_0\|_Y \int_0^t \|\theta(\tau) - v(\tau)\| d\tau \\
&+ \mathbf{K}\|h(\theta)\| \int_0^t \|\theta(\tau) - v(\tau)\| d\tau + \|\mathcal{R}_{(\alpha,v)}(t,0)\|\, \|h(\theta) - h(v)\| \\
&\leq \mathbf{K}\, \|\theta_0\| \max_{\tau\in J} \|\theta(\tau) - v(\tau)\| a + \mathbf{K} a \mathbf{K}_1 \max_{\tau\in J} \|\theta(\tau) - v(\tau)\| + \mathbf{M}_0\mathbf{K}_2 \max_{\tau\in J} \|\theta(\tau) - v(\tau)\| \\
&= (\mathbf{K}\|\theta_0\| a + \mathbf{K} a \mathbf{K}_1 + \mathbf{M}_0\mathbf{K}_2) \max_{\tau\in J} \|\theta(\tau) - v(\tau)\|. \qquad (3.2)
\end{aligned}$$

Consider $\tilde{\mathbf{A}}(\theta)$ and $\tilde{\mathbf{B}}(v)$ given by

$$\begin{aligned}
\tilde{\mathbf{A}}(\theta) &= \theta_1 - \mathcal{R}_{(\alpha,\theta)}(b,0)\, \theta_0 + \mathcal{R}_{(\alpha,\theta)}(b,0)\, h(\theta) \\
&- \int_0^b \mathcal{R}_{(\alpha,\theta)}(b,s)\Big(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau))\, d\tau \Big)\, ds \\
&- \sum_{i=1}^{m} \mathcal{R}_{(\alpha,\theta)}(b,t_i) I_i(\theta(t_i))
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathbf{B}}(v) &= \theta_1 - \mathcal{R}_{(\alpha,v)}(b,0)\, \theta_0 + \mathcal{R}_{(\alpha,v)}(b,0)\, h(v) \\
&- \int_0^b \mathcal{R}_{(\alpha,v)}(b,s)\Big(f(s,v(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,v(\tau))\, d\tau \Big)\, ds \\
&- \sum_{i=1}^{m} \mathcal{R}_{(\alpha,v)}(b,t_i) I_i(v(t_i)).
\end{aligned}$$

Using the Lemma 3.2 and the condition ($\mathbf{H}_1$), we have

$$\begin{aligned}
I_2 &\leq \int_0^t \|\mathcal{R}_{(\alpha,\theta)}(t,\eta)\mathbf{B}\tilde{W}^{-1}\tilde{\mathbf{A}}(\theta) - \mathcal{R}_{(\alpha,v)}(t,\eta)\mathbf{B}\tilde{W}^{-1}\tilde{\mathbf{A}}(v)\|\, d\eta \\
&\leq \|\mathbf{B}\|\|\tilde{W}^{-1}\|\mathbf{K}2\max\left\{\|\tilde{\mathbf{A}}(\theta), \tilde{\mathbf{B}}(v)\|\right\} \int_0^t \|\theta(\tau) - v(\tau)\|\, d\tau \\
&\leq \mathbf{M}_1\mathbf{M}_2\mathbf{K}2\max\left\{\|\tilde{\mathbf{A}}(\theta), \tilde{\mathbf{B}}(v)\|_Y\right\}\, a^2 \max_{\tau\in J} \|\theta(\tau) - v(\tau)\|. \qquad (3.3)
\end{aligned}$$

Note that,

$$\max \|\tilde{\mathbf{A}}(\theta), \tilde{\mathbf{B}}(v)\|_Y \leq \|\tilde{\mathbf{A}}(\theta)\|_Y + \|\tilde{\mathbf{B}}(v)\|_Y.$$

Using the Lemma 3.2, the conditions ($\mathbf{H}_2$), ($\mathbf{H}_5$) and ($\mathbf{H}_6$), yields

$$\begin{aligned}
\|\tilde{\mathbf{A}}(\theta)\|_Y &= \Big\|\theta_1 - \mathcal{R}_{(\alpha,\theta)}(b,0)\ \theta_0 + \mathcal{R}_{(\alpha,\theta)}(b,0)\ h(\theta) - \int_0^b \mathcal{R}_{(\alpha,\theta)}(b,s)\Big(f(s,\theta(s)) \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau))\ d\tau\Big)\ ds \\
&- \sum_{i=1}^{m} \mathcal{R}_{(\alpha,\theta)}(b,\theta)\Big[\Big(I_i(\theta(t_i)) - I_i(0)\Big) + I_i(0)\Big]\Big\| \\
&\le \|\theta_1\|_Y\ +\ \|\ \mathcal{R}_{(\alpha,\theta)}(b,0)\|_Y\ \|\theta_0\|_Y\ +\ \|\mathcal{R}_{(\alpha,\theta)}(b,0)\|_Y\|\ h(\theta)\|_Y \\
&+ \int_0^b \Big\|\mathcal{R}_{(\alpha,\theta)}(b,s)\Big\|\ \|f(s,\theta(s))\|\ ds \\
&+ \int_0^b \Big\|\mathcal{R}_{(\alpha,\theta)}(b,s)\Big\|\left(\frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}\|g(s,\tau,\theta(\tau))\| d\tau\right) ds \\
&+ \sum_{i=1}^{m} \mathbf{M}_0\Big(\|I_i(\theta(t_i)) - I_i(0)\| + \|I_i(0)\|\Big) \\
&\le \|\theta_1\|_Y + \mathbf{M}_0\|\theta_0\|_Y + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\left\{\sum_{i=1}^{m} I_i\delta + \|I_i(0)\|\right\} \\
&\le \|\theta_1\|_Y + \mathbf{M}_0\|\theta_0\|_Y + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\ \xi, \qquad (3.4)
\end{aligned}$$

where $\xi = \sum_{i=1}^{m}\left(I_i\delta + \|I_i(0)\|\right)$ and

$$\begin{aligned}
\|\tilde{\mathbf{B}}(\theta)\|_Y &= \Big\|\theta_1 - \mathcal{R}_{(\alpha,v)}(b,0)\ \theta_0 + \mathcal{R}_{(\alpha,v)}(b,0)\ h(v) - \int_0^b \mathcal{R}_{(\alpha,v)}(b,s)\Big(f(s,v(s)) \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} g(s,\tau,v(\tau))\ d\tau\Big)\ ds - \sum_{i=1}^{m} \mathcal{R}_{(\alpha,v)}(b,t_i) I_i(v(t_i))\Big\| \\
&\le \|\theta_1\|_Y\ +\ \|\ \mathcal{R}_{(\alpha,v)}(b,0)\|_Y\ \|\theta_0\|_Y\ +\ \|\mathcal{R}_{(\alpha,v)}(b,0)\|_Y\|\ h(v)\|_Y \\
&+ \int_0^b \Big\|\mathcal{R}_{(\alpha,v)}(b,v)\Big\|\Big(f(s,v(s)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau))\ d\tau\Big)\ ds \\
&+ \sum_{i=1}^{m} \Big\|\mathcal{R}_{(\alpha,v)}(b,t_i)\Big\|\Big(\|I_i(v(t_i)) - I_i(0)\| + \|I_i(0)\|\Big) \\
&\le \|\theta_1\|_Y + \mathbf{M}_0\|\theta_0\|_Y + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\left\{\sum_{i=1}^{m} I_i\delta + \|I_i(0)\|\right\} \\
&\le \|\theta_1\|_Y + \mathbf{M}_0\|\theta_0\|_Y + \mathbf{M}_0\mathbf{K}_1 + \mathbf{M}_0\tilde{\tau} + \mathbf{M}_0\xi, \qquad (3.5)
\end{aligned}$$

where $\xi = \sum_{i=1}^{m}\left(I_i\delta + \|I_i(0)\|\right)$.

Substituting the inequalities (3.4) and (3.5) into the inequality (3.3) gives

$$\begin{aligned}
I_2 &\le \mathbf{M}_1\mathbf{M}_2\mathbf{K}2\max\left\{\|\tilde{\mathbf{A}}(\theta), \tilde{\mathbf{B}}(v)\|_Y\right\} a^2 \max_{\tau\in J}\|\theta(\tau) - v(\tau)\| \\
&\le \mathbf{M}_1\mathbf{M}_2 2\mathbf{K}a^2\left\{\|\theta_1\|_Y + \mathbf{M}_0\left(\|\theta_0\|_Y + \mathbf{K}_1 + \tilde{\tau} + \xi\right)\right\}\ \max_{\tau\in J}\|\theta(\tau) - v(\tau)\|. \qquad (3.6)
\end{aligned}$$

Again, Lemma 3.2 and Lemma 3.3, ($\mathbf{H}_3$), ($\mathbf{H}_4$) and ($\mathbf{H}_6$), yields

$$\begin{aligned}
I_3 &\leq \int_0^t \Bigg\{ \Bigg\| \mathcal{R}_{(\alpha,\theta)}(t,s)\Bigg(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau))\, d\tau \Bigg) \\
&- \mathcal{R}_{(\alpha,v)}(t,s)\Big(f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau))\, d\tau \Big) \Bigg\| \\
&+ \Bigg\| \mathcal{R}_{(\alpha,v)}(t,s)\Big(f(s,v(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,v(\tau))\, d\tau \Big) \\
&- \mathcal{R}_{(\alpha,v)}(t,s)\Big(f(s,v(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,v(\tau))\, d\tau \Big) \Bigg\| \Bigg\}\, ds \\
&\leq \mathbf{K} \int_0^t \|f(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} g(s,\tau,\theta(\tau))\, d\tau\|_Y \int_s^t \|\theta(\tau)-v(\tau)\|\, d\tau\, ds \\
&+ \mathbf{M}_0 \int_0^t \Big(\|f(s,\theta(s)) - f(s,v(s))\| \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \|g(s,\tau,\theta(\tau))\, d\tau - g(s,\tau,v(\tau))\|\, d\tau \Big)\, ds \\
&\leq \mathbf{K}\, \tilde{\tau} a \max_{\tau\in J} \|\theta(\tau)-v(\tau)\| \\
&+ \mathbf{M}_0\mathbf{K}_1 \int_0^t \|\theta(s)-v(s)\| ds + \mathbf{M}_0\mathbf{K}_3 \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| \\
&\leq \mathbf{K}\tilde{\tau} a \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| + \mathbf{M}_0\mathbf{K}_1 a \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| + \mathbf{M}_0\mathbf{K}_3 a \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| \\
&= \Big(\mathbf{K}\tilde{\tau} a + \mathbf{M}_0\mathbf{K}_1 a + \mathbf{M}_0\mathbf{K}_3 a \Big) \max_{\tau\in J}\|\theta(\tau)-v(\tau)\|. \qquad (3.7)
\end{aligned}$$

Finally, using the Lemma 3.2, ($\mathbf{H}_5$) and ($\mathbf{H}_6$), yields

$$\begin{aligned}
I_4 &\leq \sum_{i=1}^m \Big(\|\mathcal{R}_{(\alpha,\theta)}(t,t_i) I_i(\theta(t_i)) - \mathcal{R}_{(\alpha,v)}(t,t_i) I_i(v(t_i))\| + \|\mathcal{R}_{(\alpha,v)}(t,t_i) I_i(\theta(t_i)) \\
&- \mathcal{R}_{(\alpha,v)}(t,t_i) I_i(v(t_i))\| \Big) \\
&\leq \sum_{i=1}^m \Big[\mathbf{K}\Big(\|I_i(\theta(t_i)) - I_i(0)\| + \|I_i(0)\| \Big) \int_{t_i}^t \|\theta(\tau)-v(\tau)\| d\tau \\
&+ \mathbf{M}_0 \|I_i(\theta(t_i)) - I_i(v(t_i))\| \Big] \\
&\leq \sum_{i=1}^m \Big[\mathbf{K}\Big(l_i\delta + \|I_i(0)\| \Big) \int_{t_i}^t \|\theta(\tau)-v(\tau)\| d\tau + \mathbf{M}_0 l_i \|\theta(\tau)-v(\tau)\| \Big] \\
&\leq \mathbf{K} \sum_{i=1}^m \Big(l_i\, \delta + \|I_i(0)\| \Big) a \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| + \mathbf{M}_0 \sum_{i=1}^m l_i \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| \\
&= \mathbf{K}\xi a \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| + \mathbf{M}_0 \sum_{i=1}^m l_i \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| \\
&= \Big(\mathbf{K}\xi a + \mathbf{M}_0 \sum_{i=1}^m l_i \Big) \max_{\tau\in J}\|\theta(\tau)-v(\tau)\|. \qquad (3.8)
\end{aligned}$$

Substituting the inequalities (3.2), (3.6), (3.7) and (3.8) in the inequality (3.1), we obtain

$$\begin{aligned}
\| (\mathbf{Q}\theta_\mu)(t)\| &\leq I_1 + I_2 + I_3 + I_4 \\
&= \Big(\mathbf{K}\,\|\theta_0\| a + \mathbf{K}\mathbf{K}_1 + \mathbf{M}_0\mathbf{K}_2\Big) \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| \\
&+ \mathbf{M}_1\mathbf{M}_2 2a^2\mathbf{K}\Big(\|\theta_1\|_Y + \mathbf{M}_0(\,\|\theta_0\|_Y + \mathbf{K}_1 + \tilde{\tau} + \xi)\Big) \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| \\
&+ \Big(\mathbf{K}\xi a + \mathbf{M}_0\sum_{i=1}^{m} l_i\Big) \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| + \Big(\mathbf{K}\xi a \\
&+ \mathbf{M}_0\sum_{i=1}^{m} l_i\Big) \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| \\
&= \Big\{\mathbf{K}\|\theta_0\| a + \mathbf{K}a\mathbf{K}_1 + \mathbf{M}_0\mathbf{K}_2 \\
&+ \mathbf{M}_1\mathbf{M}_2 2a\mathbf{K}\Big(\|\theta_1\|_Y \\
&+ \mathbf{M}_0(\|\theta_0\|_Y + \mathbf{K}_1 + \tilde{\tau} + \xi)\Big) + \mathbf{K}\tilde{\tau}a + \mathbf{M}_0\mathbf{K}_1 a + \mathbf{M}_0\mathbf{K}_3 + \mathbf{K}\xi a \\
&+ \mathbf{M}_0\sum_{i=1}^{m} l_i\Big\} \max_{\tau\in J}\|\theta(\tau)-v(\tau)\| \\
&= \lambda \max_{\tau\in J}\|\theta(\tau)-v(\tau)\|,
\end{aligned}$$

where $\lambda = \mathbf{K}\|\theta_0\| a + \mathbf{K}a\mathbf{K}_1 + \mathbf{M}_0\mathbf{K}_2 + \mathbf{M}_1\mathbf{M}_2 2a\mathbf{K}\Big(\|\theta_1\|_Y + \mathbf{M}_0(\|\theta_0\|_Y + \mathbf{K}_1 + \tilde{\tau} + \xi)\Big)$ $+\mathbf{K}\tilde{\tau}a + \mathbf{M}_0\mathbf{K}_1 a + \mathbf{M}_0\mathbf{K}_3 + \mathbf{K}\xi a + \mathbf{M}_0\sum_{i=1}^{m} l_i$.

Therefore, $\mathbf{Q}$ is a contraction mapping and hence there exists a unique fixed point $\theta \in \Lambda$, such that $\mathbf{Q}\theta(t) = \theta(t)$. Every fixed point of $\mathbf{Q}$ is a mild solution of (1.1)-(1.3) on J satisfying s $\theta(a) = \theta_1$. Thus the system (1.1)-(1.3) is controllable on J.

□

4 Conclusion

The results on the controllability of solutions of a new class of fractional impulsive integro-differential control systems in Banach spaces contribute significantly to the theory of fractional PDE, in particular to the area of fractional operators. In addition to the above, we can highlight that the result obtained here in the context of Caputo-type fractional operators allowed us to address other open questions in the area, in particular, a problem on mild-type solutions for the ψ-Hilfer fractional derivative. In this sense, we believe that further results in this context will be published in the near future.

Author Contributions

Formal analysis, Priscila. S. Ramos, J. Vanterler da C. Sousa, E. Capelas de Oliveira; Investigation, Priscila. S. Ramos, J. Vanterler da C. Sousa, E. Capelas de Oliveira Methodology, J. Vanterler da C. Sousa; Supervision, E. Capelas de Oliveira; Validation, Priscila. S. Ramos.; Writing—original draft, Priscila. S. Ramos; Writing—review and editing, E. Capelas de Oliveira All authors have read and agreed to the published version of the manuscript.

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