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ON SOME PROPERTIES OF BERNOULLI AND EULER POLYNOMIALS

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Abstract Using elementary tools, we obtain some expressions for the Bernoulli and Euler polynomials and a formula linking the two polynomials. Moreover, we obtain super-congruences concerning sums of alternating powers.

1 Introduction and results

Since their initial identification in the post-Renaissance period, specifically by Bernoulli in his book "Ars Conjectandi" in 1713 [2], and Euler in his manuscript "Institutiones Calculi Differentialis" in 1755 [5], Bernoulli and Euler numbers and polynomials have been the subject of extensive research. These mathematical entities have captivated numerous mathematicians throughout history, who have been intrigued by the quest to discover explicit formulas, criteria, properties, and recurrent relationships associated with them. Even today, it remains a field of great interest for contemporary mathematicians [1, 3, 8].

Recall that the nth Bernoulli polynomial $B_n(x)$ $(n \in \mathbb{N} = \{0, 1, 2, \ldots\})$ and the nth Euler polynomial $E_n(x)$ can be defined by the generating function, respectively, as follows:

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} \quad (|z| < 2\pi) \tag{1.1}$$

$$\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} \quad (|z| < \pi). \tag{1.2}$$

Bernoulli numbers and Euler numbers can be defined by $B_n = B_n(0)$ and $E_n = 2^n E_n(\frac{1}{2})$, respectively. Setting $x = \frac{1}{2}$ in Relation (1.2), we obtain the nth Euler number E_n (see A122045 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [10]) which can also be defined by the generating series

$$\sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \frac{2}{e^z + e^{-z}} = \frac{1}{\cosh z}.$$

The expressions translations of Bernoulli and Euler polynomials are

$$B_n(y + z) = \sum_{k=0}^{n} \binom{n}{k} B_k(y) z^{n-k} \tag{1.3}$$

$$E_n(y + z) = \sum_{k=0}^{n} \binom{n}{k} E_k(y) z^{n-k}. \tag{1.4}$$

It is well-known that the alternating powers sum $T_n(k) = \sum_{r=1}^{n-1}(-1)^r r^k$ can be written via Euler polynomials as follows:

$$T_n(k) = \frac{E_k(0) - (-1)^n E_k(n)}{2}, \quad \text{for } n, k \geq 1 \tag{1.5}$$

with $E_k(0)$ as a function of B_k is $E_k(0) = \frac{2}{k+1}(1 - 2^{k+1})B_{k+1}$.

The previous properties of the polynomials of Bernoulli and Euler (1.1), (1.2), (1.3), (1.4) and (1.5) can be found in [1, p. 804].

Another important property of Bernoulli and Euler polynomials is that they are Appell polynomials. Recall that a sequence of polynomials $A_n(x)$ is said to be an Appell polynomial sequence if $A_n'(x) = nA_{n-1}(x)$, for $n \geq 1$ and $A_0(x)$ is a nonzero constant polynomial.

In 2003, Cheon [4] proved the following relation between the polynomials of Bernoulli and those of Euler

$$B_n(x) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k(0) E_{n-k}(x).$$

In the next theorem, we present some simple properties of Bernoulli and Euler polynomials.

Theorem 1.1.

(I) *There exists a family of polynomials with complex coefficients $(b_n(x))$ such that for any integer $n \geq 1$, we have*

$$\deg(b_n(x)) = n \text{ and } B_{2n}(x) - B_{2n} = 2nb_n\left(\frac{1}{2}x(x-1)\right). \tag{1.6}$$

(II) *There exists a family of polynomials with complex coefficients $(e_n(x))$ such that for any integer $n \geq 1$, we have*

$$\deg(e_n(x)) = n \text{ and } E_{2n}(x) = 2ne_n\left(\frac{1}{2}x(x-1)\right). \tag{1.7}$$

In the following theorem we establish an explicit formula for Euler polynomials $E_n(x)$ in terms of Stirling numbers of the second kind.

Theorem 1.2. *For $m \geq 0$, we have*

$$E_m(x) = \sum_{k=0}^{m} \sum_{j=1}^{k+1} \binom{m}{k} \frac{(-1)^{j+k-1}(j-1)!}{2^{j+1}} S(k+1, j) x^{m-k}, \tag{1.8}$$

where $S(k, j)$ (OEIS A008277) is the Stirling number of the second kind.

Theorem 1.3. *For any integer $n \geq 0$, we have*

$$E_n(x) = \frac{2}{n+1}\left(B_{n+1}(x) - \sum_{k=0}^{n+1} \binom{n+1}{k} B_{n+1-k} E_k(x)\right). \tag{1.9}$$

Recall that, if p is an odd prime. $\mathbb{Z}_{(p)}$ denotes the ring of rational p-integers (those rational numbers whose denominators are not divisible by p). If $x, y \in \mathbb{Z}_{(p)}$, then we say that x is congruent to y modulo p^n (where $n \geq 2$) if and only if $x - y \in p^n\mathbb{Z}_{(p)}$ and denote this relation by $x \equiv y \pmod{p^n}$.

The following result gives a congruence for the alternating sum $T_{\frac{p+1}{2}}(2k+1)$ in terms of Euler numbers.

Theorem 1.4. *For any prime number $p \geq 3$, and for any integers k and $m \geq 1$, we have*

$$\sum_{r=1}^{\frac{p-1}{2}} (-1)^r r^{2k+1} \equiv \frac{1}{2}E_{2k+1}(0) + \frac{(-1)^{\frac{p-1}{2}}}{2^{2k+2}} \sum_{j=0}^{m-1} \binom{2k+1}{2j+1} p^{2j+1} E_{2k-2j} \pmod{p^{2m+s}}, \tag{1.10}$$

with $s \in \{0, 1\}$.

2 Proofs

In this section, we will provide proofs of our theorems.

Proof of Theorem 1.1.

Case I. Applying (1.3) for $y = \frac{1}{2}$ and $z = \frac{2x-1}{2}$, we get

$$B_n(x) = B_n\left(\frac{1}{2} + \frac{2x-1}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} B_k\left(\frac{1}{2}\right)\left(\frac{2x-1}{2}\right)^{n-k}. \tag{2.1}$$

It is easy to show that $B_{2k}\left(\frac{1}{2}\right) = (2^{1-2k}-1)B_{2k}$ and $B_{2k+1}\left(\frac{1}{2}\right) = 0$ for any $k \geq 0$, which allows us to write (2.1) as follows:

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n} \binom{n}{2k} B_{2k}\left(\frac{1}{2}\right)\left(\frac{2x-1}{2}\right)^{n-2k} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} B_{2k}\left(\frac{1}{2}\right)\frac{(2x-1)^{n-2k}}{2^{n-2k}} \\ &= \frac{1}{2^n}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}(2-2^{2k})B_{2k}(2x-1)^{n-2k}. \end{aligned} \tag{2.2}$$

By noting that

$$(2x-1)^2 = 8\left(\frac{x(x-1)}{2}\right) + 1, \tag{2.3}$$

from Relations (2.2) and (2.3) we deduce that

$$B_{2n}(x) - B_{2n} = 2n \times b_n\left(\frac{1}{2}x(x-1)\right),$$

where

$$b_n(t) = \frac{1}{n2^{2n+1}}\sum_{k=0}^{n}\binom{2n}{2k}(2-2^{2k})B_{2k}((8t+1)^{n-k}-1).$$

Case II. According to the following relationship

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k},$$

then using Relation (2.3), we can easily see that we have

$$E_{2n}(x) = 2n \times e_n\left(\frac{1}{2}x(x-1)\right),$$

where

$$e_n(t) = \frac{1}{n2^{2n+1}}\sum_{k=0}^{n}\binom{2n}{2k}E_{2k}(8t+1)^{n-k}.$$

This achieves the proof. □

Remark 2.1. Since Bernoulli polynomials and Euler polynomials are Appell polynomials, by using $B'_{2n}(x) = 2nB_{2n-1}(x)$ and $E'_{2n}(x) = 2nE_{2n-1}(x)$, for any integer $n \geq 1$, respectively, according to (1.6) and (1.7), we have

$$B_{2n-1}(x) = \left(x-\frac{1}{2}\right)b'_n\left(\frac{1}{2}x(x-1)\right) \text{ and } E_{2n-1}(x) = \left(x-\frac{1}{2}\right)e'_n\left(\frac{1}{2}x(x-1)\right).$$

Remark 2.2. From Relations (1.6) and (1.7), we have $b_n(0) = 0$ and $e_n(0) = 0$ and following Relations (1.6) and (1.7), we have also $b'_n(0) = 0$ and $e'_n(0) \neq 0$, which allows to deduce that

$$x^2 \text{ divides } b_n(x) \text{ for } n \geq 2 \text{ and } x \text{ divides } e_n(x) \text{ for } n \geq 1.$$

Proof of Theorem 1.2. Differentiating both sides of Equation (1.2) with respect to z, m times, and applying Leibniz formula for derivation we get

$$\sum_{n=m}^{\infty} E_n(x) \frac{z^{n-m}}{(n-m)!} = 2 \sum_{k=0}^{m} \binom{m}{k} (e^{xz})^{(m-k)} \left(\frac{1}{e^z + 1} \right)^{(k)}.$$

Applying the following identity [11, Theorem 3.1]

$$\left(\frac{1}{1 - \lambda e^{\alpha z}} \right)^{(k)} = (-\alpha)^k \sum_{j=1}^{k+1} \frac{(-1)^{j-1}(j-1)!}{(1 - \lambda e^{\alpha z})^j} S(k+1, j), \tag{2.4}$$

for $\lambda = -1$ and $\alpha = 1$, we get

$$\left(\frac{1}{1 + e^z} \right)^{(k)} = (-1)^k \sum_{j=1}^{k+1} \frac{(-1)^{j-1}(j-1)!}{(1 + e^z)^j} S(k+1, j).$$

Consequently, it follows

$$\begin{aligned} E_m(x) &= \lim_{z \to 0} \sum_{n=m}^{\infty} E_n(x) \frac{z^{n-m}}{(n-m)!} \\ &= 2 \sum_{k=0}^{m} \binom{m}{k} x^{m-k} (-1)^k \sum_{j=1}^{k+1} (-1)^{j-1} (j-1)! S(k+1, j) \lim_{z \to 0} \frac{e^{xz}}{(1 + e^z)^j} \\ &= \sum_{k=0}^{m} \sum_{j=1}^{k+1} \binom{m}{k} \frac{(-1)^{j+k-1}(j-1)}{2^{j+1}} ! S(k+1, j) x^{m-k}. \end{aligned}$$

This completes the proof of the theorem. □

Remark 2.3. Note that Relation (1.8) and Relation (2.4) have been obtained respectively by Luo [9] using difference operator and by Xu [11] using Faà di Bruno formula. Note also that for $x = n$, $x = 0$ in Relation (1.8) and by Relation (1.5), we get Theorem 3 of [7].

Proof of Theorem 1.3. We note that

$$\frac{ze^{xz}}{e^z + 1} = \frac{ze^{xz}}{e^z - 1} - \frac{2ze^{xz}}{e^{2z} - 1},$$

and we have

$$\begin{aligned} \sum_{n=0}^{\infty} (nE_{n-1}(x)) \frac{z^n}{n!} &= \frac{2ze^{xz}}{e^z + 1} \\ &= 2 \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} - 2 \frac{2e^{xz}}{e^z + 1} \frac{z}{e^z + 1} \\ &= 2 \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} - 2 \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \end{aligned}$$

Thanks to the product

$$\left(\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} b_n \frac{z^n}{n!} \right) = \sum_{n=0}^{\infty} \left(\left(\sum_{k=0}^{n} \binom{n}{k} a_{n-k} b_k \right) \right) \frac{z^n}{n!}$$

we obtain the following expression

$$E_n(x) = \frac{2}{n+1}\left(B_{n+1}(x) - \sum_{k=0}^{n+1}\binom{n+1}{k}B_{n+1-k}E_k(x)\right).$$

This completes the proof of the theorem. □

The following two identities come immediately by combining Theorem 1.1, Theorem 1.3, and putting $x = \frac{1}{2}$ in Relation (1.9).

Corollary 2.4. *We have*

$$E_{2n-1}(x) = \frac{1}{n}\left(2nb_n\left(\frac{1}{2}x(x-1)\right) - \sum_{k=1}^{2n}\binom{2n}{k}B_{2n-k}E_k(x)\right),$$

and

$$E_n = \frac{2}{n+1}\left((1-2^n)B_{n+1} - \sum_{k=0}^{n+1}\binom{n+1}{k}2^{n-k}B_{n+1-k}E_k\right).$$

Proof of Theorem 1.4. Replacing $n = \frac{p+1}{2}$ in Identity (1.5), we get

$$\sum_{r=1}^{\frac{p-1}{2}}(-1)^r r^{2k+1} = \frac{E_{2k+1}(0) + (-1)^{\frac{p-1}{2}}E_{2k+1}(\frac{p}{2}+\frac{1}{2})}{2}.$$

By expanding $E_{2k+1}(\frac{p}{2}+\frac{1}{2})$ with the help of Relation (1.4), we obtain

$$\begin{aligned}\sum_{r=1}^{\frac{p-1}{2}}(-1)^r r^{2k+1} &= \frac{1}{2}\left(E_{2k+1}(0) + (-1)^{\frac{p-1}{2}}\sum_{j=0}^{2k+1}\binom{2k+1}{j}\left(\frac{p}{2}\right)^j E_{2k+1-j}\left(\frac{1}{2}\right)\right)\\ &= \frac{1}{2}\left(E_{2k+1}(0) + \frac{(-1)^{\frac{p-1}{2}}}{2^{2k+1}}\sum_{j=0}^{2k+1}\binom{2k+1}{j}p^j E_{2k+1-j}\right).\end{aligned}$$

As $E_{2k+1} = 0$, we deduce that

$$\begin{aligned}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k k^{2k+1} &= \frac{1}{2}\left(E_{2k+1}(0) + \frac{(-1)^{\frac{p-1}{2}}}{2^{2k+1}}\sum_{j=1}^{2k+1}\binom{2k+1}{j}p^j E_{2k+1-j}\right)\\ &= \frac{1}{2}\left(E_{2k+1}(0) + \frac{(-1)^{\frac{p-1}{2}}}{2^{2k+1}}\sum_{j=0}^{2k}\binom{2k+1}{j+1}p^{j+1} E_{2k-j}\right)\\ &= \frac{1}{2}E_{2k+1}(0) + \frac{(-1)^{\frac{p-1}{2}}}{2^{2k+2}}(C_m + p^{2m}D_m),\end{aligned}$$

where

$$C_m = \sum_{j=0}^{2m-1}\binom{2k+1}{j+1}p^{j+1}E_{2k-j}$$

and

$$D_m = \frac{1}{p^{2m}}\sum_{j=2m}^{2k}\binom{2k+1}{j+1}p^{j+1}E_{2k-j}.$$

By noting that Euler numbers E_{2k-j} are equal to zero for $0 \leq j \leq 2m-1$, we can write C_m as follows:

$$C_m = \sum_{j=0}^{m-1} \binom{2k+1}{2j+1} p^{2j+1} E_{2k-2j}$$
$$\equiv \sum_{j=0}^{m-1} \binom{2k+1}{2j+1} p^{2j+1} E_{2k-2j} \pmod{p^{2m+s}}.$$

We have

$$D_m = \sum_{j=2m}^{2k} \binom{2k+1}{j+1} p^{j+1-2m} E_{2k-j} \in \mathbb{Z}_{(p)}.$$

Then the result follows. □

Note that for $m = 1$ in (1.10), we get

$$\sum_{r=1}^{\frac{p-1}{2}} (-1)^r r^{2k+1} \equiv \frac{1}{2}\left(E_{2k+1}(0) + \frac{(-1)^{\frac{p-1}{2}}}{2} p(2k+1) E_{2k}\left(\frac{1}{2}\right)\right) \pmod{p^2}.$$

This last result is exactly congruence (31) given in [6].

The particular cases of Theorem 1.4 yield the following congruences.

Corollary 2.5. *For any prime number $p \geq 3$ and for any integer k, we have*

$$\sum_{r=1}^{\frac{p-1}{2}} (-1)^r r^{2k+1} \equiv \frac{1}{2} E_{2k+1}(0) + \frac{(-1)^{\frac{p-1}{2}}}{2^{2k+2}} (2k+1) p E_{2k} + \frac{(-1)^{\frac{p-1}{2}}}{3 \times 2^{2k+2}} (4k^2-1) p^3 E_{2k-2} \pmod{p^4}.$$

Moreover, for $s = 1$, we have

$$\sum_{r=1}^{\frac{p-1}{2}} (-1)^r r^{2k+1} \equiv \frac{1}{2} E_{2k+1}(0) + \frac{(-1)^{\frac{p-1}{2}}}{2^{2k+2}} (2k+1) p E_{2k} + \frac{(-1)^{\frac{p-1}{2}}}{3 \times 2^{2k+2}} (4k^2-1) p^3 E_{2k-2} \pmod{p^5}.$$

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