Palestine Journal of Mathematics

Vol 13(4)(2024) , 1044–1054

OSCILLATION OF FRACTIONAL DIFFERENCE EQUATIONS WITH DELAY TERMS

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Communicated by Martin Bohner

MSC 2010 Classifications: Primary 39A10.

Keywords and phrases: Delay terms, fractional difference equation, oscillation, Riccati transformation, solutions.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract The aim of this paper is to discuss the oscillatory behavior of following equation.

$$\Delta\left(a(i)\Delta\left(b(i)\Delta\left(c(i)\Delta^{\beta}x(i-u)\right)^{\alpha}\right)\right)+\alpha(i)x^{\kappa}(i-v)=0,$$

where $i \in N_{i_0+1-\beta}$. Here Δ^{β} denotes the Riemann fractional difference operator of order β. We obtain some oscillation results by using the general Riccati transformation technique. Some examples provide the potency of the main results.

1 Introduction

Fractional calculus's relevance in many scientific and engineering domains has led to its rise in prominence over the last decades. While the concept of fractional difference is relatively new, the ideas of fractional calculus may be found in Euler's writings. The definition of the fractional difference was established by Diaz and Osler, who took the seemingly obvious step of permitting any real or complex integer to be used as an index of differencing in the conventional formula for the nth difference. Later on, Hirota used Taylor's series to define the fractional order difference operator.

A recent discovery has shown the existence and uniqueness of solutions for several forms of fractional differential equations, heralding the beginning of the study of fractional differential equations. While the theory of integro-differential equations has largely developed in conjunction with the theory of differential equations, a significant amount of material on fractional integro-differential equations is also accessible. In the theory of fractional order difference equations, very little has been established.

In this research, we discuss oscillatory solution of fourth order fractional difference equation with delay terms

$$\Delta\left(a(i)\Delta\left(b(i)\Delta\left(c(i)\Delta^{\beta}x(i-u)\right)^{\alpha}\right)\right)+\alpha(i)x^{\kappa}(i-v)=0, \tag{1.1}$$

where $i \in N_{i_0+1-\beta}$, Δ^{β} denotes the Riemann fractional difference operator of order β , $N_{i_0} = \{i_0, i_0+1, i_0+2, ...\}$ and $a(i), b(i), c(i)$ are positive sequences. Throughout this paper, we assume that α, κ are the odd non-negative integers, $\kappa \le \alpha$, $\alpha(i)$ is a positive sequence. The delay term in (1.1) helps incorporate the influence of past states on the current state with more flexibility, accommodating the fractional order to model complex dynamics and memory effects more effectively .

These are the conditions that we employ to demonstrate the oscillatory results:

A1) Assume that v and u are non-negative integers with $\lim_{i\to\infty}(i-v) = \lim_{i\to\infty}(i-u) = \infty$;

A2) $\frac{x^{\kappa}(i-v)}{b(i)\Delta(c(i)\Delta^{\beta}x(i-u))^{\alpha}} \geq A > 0$ and $\frac{\rho(i)\Delta\Big(b(i)\Delta\big(c(i)\Delta^{\beta}x(i-u)\big)^{\alpha}\Big)}{\Delta\Big(b^{\frac{1}{\alpha}}(i)\Delta^{\frac{1}{\alpha}}(c(i)\Delta^{\beta}x(i-u))^{\alpha}\Big)\cdot\Big(b^{\frac{1}{\alpha}}(i)\Delta^{\frac{1}{\alpha}}(c(i)\Delta^{\beta}x(i-u))\Big)^{\alpha-1}}$ $\geq B > 0$, for $b(i)\Delta\left(c(i)\Delta^{\beta}x(i-u)\right)^{\alpha} \neq 0$;

A3) $\sum_{s=l_0}^{i-\alpha}\frac{1}{c(s)} < \infty$ as $i \to \infty$;

A4) Let F be a function such that $F(w) = Pw - Qw^{1+\frac{1}{\alpha}}$ where $P, Q > 0$. The function F has the maximum value at $w = \left(\frac{\alpha M}{(\alpha+1)N}\right)^{\alpha}$ such that $F_{max}(w) = \left(\frac{\alpha}{N}\right)^{\alpha}\left(\frac{M}{(\alpha+1)}\right)^{\alpha+1}$.

The nontrivial sequence $x(i)$ is a solution of (1.1) defined for all $i \leq min\{-u, -v\}$ and satisfies equation (1.1) for all large i. A solution $x(i)$ of (1.1) is oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called non-oscillatory.

A fractional difference equation is a type of difference equation where the difference operator is raised to a fractional power. This concept generalizes the idea of discrete dynamical systems by incorporating fractional calculus into discrete method. Fractional difference and differential equation have been proved in various fields including applied mathematics, physics, and engineering (see [14, 15]) as they provide a way to model processes with memory and hereditary properties in discrete systems. In recent years, many researchers involved with the research of oscillatory behavior of fractional order derivatives (see [2, 3, 5, 6, 8, 9, 10, 11, 12, 13, 16, 18, 21]). The oscillatory solutions to difference equations serve several important purposes across various fields: System Analysis and stability, Signal processing, Numerical methods and simulations, Population dynamics, and Engineering applications. Identifying the oscillatory solutions is crucial for understanding and controlling systems with periodic behavior. Especially finding oscillatory solutions in fractional difference equations aids in understanding, predicting, and controlling systems with complex dynamics and memory effects. Motivated by their research, this paper aims to obtain the oscillatory solution of (1.1).

2 Preliminaries

In this section, we remind the basic lemmas of fractional equations. These preliminaries are used to prove the main results.

Lemma 2.1 defined the fractional sum, it is an extension of the summation. It can be represented using the Gamma function and involves fractional orders. Additional lemmas deal with some properties of fractional difference equations.

Lemma 2.1 (See [4]). *Let $\beta > 0$. The β th fractional sum is defined by*

$$\Delta^{-\beta}f(i) = \frac{1}{\Gamma(\beta)}\sum_{s=a}^{i-\beta}(i-s-1)^{(\beta-1)}f(s),$$

for all $i \in N_{a+\beta}$ where f is defined for $s \equiv a\ mod(1)$ and $\Delta^{-\beta}(f)$ is defined for $i = (a+\beta)\ mod(1)$.

The falling factorial power function is

$$i^{\beta} = \frac{\Gamma(i+1)}{\Gamma(i+1-\beta)}.$$

The fractional sum $\Delta^{-\beta}f(i)$ maps function defined in N_a to functions defined in $N_{a+\beta}$.

Lemma 2.2 (See [4]). *Let $\kappa > 0$ and $m-1 < \kappa < n$ where m is a positive integer $m = (\lceil \kappa \rceil)$. Set $\beta = m - \kappa$. The κth factorial difference is defined as*

$$\Delta^{\kappa} f(i) = \Delta^{m-\beta} f(i) = \Delta^{m} \Delta^{-\beta} f(i),$$

where $\Delta^{-\beta} f(i)$ is the βth fractional sum.

Lemma 2.3 (See [7]). *The fundamental properties of Δ as follows:*

$$\Delta [U(i)V(i)] = \Delta U(i).V(i+1) + U(i)\Delta V(i).$$

$$\Delta \left[\frac{U(i)}{V(i)}\right] = \frac{\Delta U(i).V(i+1) - U(i+1).\Delta V(i)}{V(i).V(i+1)}.$$

A fundamental property of difference equations is their ability to model discrete systems and processes. Unlike differential equations, which deal with continuous changes, difference equations are used to describe the system's evolution in discrete steps or time intervals.

Lemma 2.4 (See [4]). *Let f be a real valued function defined on N_a and $\kappa, \alpha > 0$, then the following equalities hold:*

(i) $\Delta^{-\alpha}[\Delta^{-\kappa} f(i)] = \Delta^{-(\kappa+\alpha)} f(i) = \Delta^{-\kappa}[\Delta^{-\alpha} f(i)]$;

(ii) $\Delta^{-\alpha}\Delta f(i) = \Delta\Delta^{-\alpha} f(i) - \frac{(i-a)^{(\alpha-1)}}{\rho(\alpha)} f(a)$.

3 Main Results

In this section, by employing the Riccati difference transformation to convert a nonlinear difference equation into a linear one, we demonstrate the oscillation outcomes of fractional differece equation. This strategy is typically used to solve particular kinds of nonlinear difference equations by leveraging the linear structure that results from the transformation.

Theorem 3.1. *Suppose that conditions A1-A4 is true. If there is a positive sequence $\rho(i)$ such that*

$$\limsup_{i\to\infty} \sum_{s=i_3}^{i-\beta} \frac{1}{c(s)} = \infty \tag{3.1}$$

and

$$\limsup_{i\to\infty} \sum_{s=i_3}^{i-1} \left(A\alpha(s)\rho(s) - \left[\left(\frac{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)\alpha}{B\rho(s)} \right)^{\alpha} \left(\frac{\Delta\rho(s)}{(\alpha+1)\rho(s+1)} \right)^{\alpha+1} \right] \right) = \infty, \tag{3.2}$$

then equation (1.1) is oscillatory.

Proof. Without loss of generality, we may assume that $x(i)$ is non-oscillatory $\ni x(i)$ is eventually positive solution. Then $\exists\, i_1 \geq i_0 \ni x(i) > 0, x(i-u) > 0$ and $x^{\kappa}(i-v) > 0$, $\forall\, i \geq i_1 \geq i_0$. Therefore by equation (1.1), we have

$$\Delta \left(a(i)\Delta \left(b(i)\Delta \left(c(i)\Delta^{\beta} x(i-u) \right)^{\alpha} \right) \right) = -\alpha(i) x^{\kappa}(i-v) < 0. \tag{3.3}$$

Then $a(i)\Delta \left(b(i)\Delta \left(c(i)\Delta^{\beta} x(i-u) \right)^{\alpha} \right)$ is an eventually decreasing sequence on $[i_1, \infty)$ and $\Delta^{\beta} x(i-u)^{\alpha}$, $\Delta \left(c(i)\Delta^{\beta} x(i-u) \right)^{\alpha}$, $\Delta \left(b(i)\Delta \left(c(i)\Delta^{\beta} x(i-v) \right)^{\alpha} \right)$ are eventually of one sign. For $i_2 > i_1$ is large, we will claim that $\Delta \left(b(i)\Delta \left(c(i)\Delta^{\beta} x(i-v) \right)^{\alpha} \right) > 0$ on $[i_2, \infty)$. Suppose that there is an integer $i_3 > i_2$ such that $\Delta \left(b(i)\Delta \left(c(i)\Delta^{\beta} x(i-v) \right)^{\alpha} \right) < 0$ on $[i_3, \infty)$. For $[i_3, \infty)$ and there is a constant $c_1 > 0$, we have

$$\left(a(i)\Delta \left(b(i)\Delta \left(c(i)\Delta^{\beta} x(i-u) \right)^{\alpha} \right) \right) \leq -c_1 < 0,$$

$$\Rightarrow \Delta \left(b(i)\Delta \left(c(i)\Delta^{\beta} x(i-u) \right)^{\alpha} \right) \leq -\frac{c_1}{a(i)} < 0, \tag{3.4}$$

Summing inequality (3.4), $\exists\, c_2 > 0 \ni$

$$b(i)\Delta\left(c(i)\Delta^{\beta}x(i-u)\right) \leq -c_2 < 0,$$

$$\Rightarrow \Delta\left(c(i)\Delta^{\beta}x(i-u)\right)^{\alpha} \leq -\left[\frac{c_2}{b(i)}\right] < 0. \tag{3.5}$$

Again summing inequality (3.5), $\exists\, c_3 > 0 \ni$

$$\Delta^{\beta}x(i-u) \leq -c_3^{\frac{1}{\alpha}}\left[\frac{1}{c(i)}\right] < 0. \tag{3.6}$$

We can write inequality (3.6) as

$$\Delta(\Delta^{-(1-\beta)}x(i-u)) \leq -c_3^{\frac{1}{\alpha}}\left[\frac{1}{c(i)}\right]. \tag{3.7}$$

Summing aforementioned inequality from i_3 to $i-1$, we get

$$\Delta^{-(1-\beta)}x(i-u) \leq \Delta^{-(1-\beta)}x(i_3-u) - \sum_{s=i_3}^{i-1} c_3^{\frac{1}{\alpha}}\left[\frac{1}{c(s)}\right]. \tag{3.8}$$

Operating $\Delta^{(1-\beta)}$ on both sides of inequality (3.8), we carry

$$x(i-u) \leq -c_3^{\frac{1}{\alpha}}\Delta^{(1-\beta)}\sum_{s=i_3}^{i-1}\left[\frac{1}{c(s)}\right]. \tag{3.9}$$

Using Lemma 2.1 and Lemma 2.4 in (3.9), we can write

$$\begin{aligned} x(i-u) &\leq \frac{(i-a)^{(\beta-1)}}{\Gamma(\beta)}c_4 - \frac{c_3^{\frac{1}{\alpha}}}{\Gamma(\beta)}\sum_{s=i_3}^{i-\beta}(i-s-1)^{\beta-1}\frac{1}{c(s)} \\ &\leq \frac{1}{\Gamma(\beta)}\left((i-a)^{\beta-1}c_4 - c_3^{\frac{1}{\alpha}}(i-l_3-1)^{\beta-1}\sum_{s=i_3}^{i-\beta}\frac{1}{c(s)}\right), \end{aligned}$$

where $c_4 = \Delta^{-\beta}x(0)$. Taking limit supremum as $i \to \infty$, the above inequality becomes

$$\begin{aligned} \limsup_{i\to\infty} &\left(x(i-u) \leq \frac{(i-a)^{(\beta-1)}}{\Gamma\beta)}c_4 - \frac{c_3^{\frac{1}{\alpha}}}{\Gamma(\beta)}\sum_{s=i_3}^{i-\beta}(i-s-1)^{\beta-1}\frac{1}{c(s)}\right) \\ &\leq \limsup_{i\to\infty}\left(\frac{1}{\Gamma(\beta)}\left((i-a)^{\beta-1}c_4 - c_3^{\frac{1}{\alpha}}(i-i_3-1)^{\beta-1}\sum_{s=i_3}^{i-\beta}\frac{1}{c(s)}\right)\right) = -\infty, \end{aligned}$$

by equation (3.1) which is a contradiction. Hence $\Delta\left(b(i)\Delta\left(c(i)\Delta^{\beta}x(i-v)\right)^{\alpha}\right) > 0$. Define the function

$$z(i) = \rho(i)\frac{a(i)\Delta\left(b(i)\Delta\left(c(i)\Delta^{\beta}x(i-u)\right)^{\alpha}\right)}{b(i)\Delta\left(c(i)\Delta^{\beta}x(i-u)\right)^{\alpha}}, \tag{3.10}$$

for $i \in [i_1, \infty)$. Then we have $z(i) > 0$. Using Lemma 2.3, we write

$$\begin{aligned} \Delta z(i) = \quad & \Delta\rho(i)\left[\frac{a(i+1)\Delta\left(b(i+1)\Delta\left(c(i+1)\Delta^{\beta}x(i+1-u)\right)^{\alpha}\right)}{b(i+1)\Delta\left(c(i+1)\Delta^{\beta}x(i+1-u)\right)^{\alpha}}\right] \\ + \quad & \rho(i)\Delta\left[\frac{a(i)\Delta\left(b(i)\Delta\left(c(i)\Delta^{\beta}x(i-u)\right)^{\alpha}\right)}{b(i)\Delta\left(c(i)\Delta^{\beta}x(i-u)\right)^{\alpha}}\right]. \end{aligned}$$

$$\Delta z(i) = \Delta\rho(i)\frac{z(i+1)}{\rho(i+1)} + \rho(i)\left[\frac{b(i+1)\Delta\left(c(i+1)\Delta^\beta x(i+1-u)\right)^\alpha .\Delta\left(a(i)\Delta\left(b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha\right)\right)}{b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha .b(i+1)\Delta\left(c(i+1)\Delta^\beta x(i+1-u)\right)^\alpha}\right] - \rho(i)\left[\frac{a(i+1)\Delta\left(b(i+1)\Delta\left(c(i+1)\Delta^\beta x(i+1-u)\right)^\alpha\right).\Delta b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha}{b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha .b(i+1)\Delta\left(c(i+1)\Delta^\beta x(i+1-u)\right)^\alpha}\right]. \tag{3.11}$$

By (3.3), equation (3.11) becomes

$$\begin{aligned}\Delta z(i) \quad \leq \quad & \Delta\rho(i)\frac{z(i+1)}{\rho(i+1)} - \alpha(i)\rho(i)\frac{x^\kappa(i-v)}{b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha} \\ & -\frac{z(i+1)}{\rho(i+1)}\rho(i)\frac{\Delta b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha}{b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha}\end{aligned}$$

$$\begin{aligned}\Rightarrow \Delta z(i) \leq \Delta\rho(i)\frac{z(i+1)}{\rho(i+1)} &- \alpha(i)\rho(i)\frac{x^\kappa(i-v)}{b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha} \\ &- \frac{z(i+1)}{\rho(i+1)}\rho^{\frac{1}{\alpha}+1}(i)\left(\frac{a^{\frac{1}{\alpha}}(i)\left(\Delta b^{\frac{1}{\alpha}}(i)\Delta^{\frac{1}{\alpha}}\left(c(i)\Delta^\beta x(i-u)\right)\right)}{b^{\frac{1}{\alpha}}(i)\Delta^{\frac{1}{\alpha}}\left(c(i)\Delta^\beta x(i-u)\right)}\right) \\ &\left(\frac{\Delta\left(b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha\right)}{\rho^{\frac{1}{\alpha}}(i)a^{\frac{1}{\alpha}}(i)\left(\Delta b^{\frac{1}{\alpha}}(i)\Delta^{\frac{1}{\alpha}}\left(c(i)\Delta^\beta x(i-u)\right)\right)\left(b^{\frac{1}{\alpha}}(i)\Delta^{\frac{1}{\alpha}}\left(c(i)\Delta^\beta x(i-u)\right)\right)^{\alpha-1}}\right).\end{aligned} \tag{3.12}$$

$$\begin{aligned}\Rightarrow \Delta z(i) \leq \Delta\rho(i)\frac{z(i+1)}{\rho(i+1)} - \alpha(i)\rho(i)\frac{x^\kappa(i-v)}{b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha} - \frac{z(i+1)}{\rho(i+1)}z^{\frac{1}{\alpha}}(i) \\ \left(\frac{\rho(i)\Delta\left(b(i)\Delta\left(c(i)\Delta^\beta x(i-u)\right)^\alpha\right)}{\rho^{\frac{1}{\alpha}}(i)a^{\frac{1}{\alpha}}(i)\Delta b^{\frac{1}{\alpha}}(i)\Delta^{\frac{1}{\alpha}}\left(c(i)\Delta^\beta x(i-u)\right)\left(b^{\frac{1}{\alpha}}(i)\Delta^{\frac{1}{\alpha}}\left(c(i)\Delta^\beta x(i-u)\right)\right)^{\alpha-1}}\right).\end{aligned} \tag{3.13}$$

From (3.13) and condition (A2), we obtain

$$\Delta z(i) \leq \Delta\rho(i)\frac{z(i+1)}{\rho(i+1)} - A\alpha(i)\rho(i) - \frac{z(i+1)}{\rho(i+1)}z^{\frac{1}{\alpha}}(i)\frac{B}{\rho^{\frac{1}{\alpha}}(i)a^{\frac{1}{\alpha}}(i)}. \tag{3.14}$$

Since $z(i)$ is decreasing, $z(i+1) \leq z(i)$ and

$$\Delta z(i) \leq \Delta\rho(i)\frac{z(i+1)}{\rho(i+1)} - A\alpha(i)\rho(i) - z^{1+\frac{1}{\alpha}}(i+1)\frac{B}{\rho(i+1)\rho^{\frac{1}{\alpha}}(i)a^{\frac{1}{\alpha}}(i)}. \tag{3.15}$$

Summing up inequality (3.15) from i_3 to $i-1$ becomes

$$\begin{aligned}\sum_{s=i_3}^{i-1}\Delta z(i) \quad \leq \quad & \sum_{s=i_3}^{i-1}\left[\Delta\rho(s)\frac{z(i+1)}{\rho(i+1)}\right] - \sum_{s=i_3}^{i-1}A\alpha(i)\rho(i) \\ & -\sum_{s=i_3}^{i-1}z^{1+\frac{1}{\alpha}}(s+1)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(i)},\end{aligned} \tag{3.16}$$

$$\begin{aligned}\Rightarrow [z(s)]_{i_3}^{i} &\le \sum_{s=i_3}^{i-1}\Delta\rho(s)\frac{z(s+1)}{\rho(s+1)}-\sum_{s=i_3}^{i-1}A\alpha(s)\rho(s)\\ &\quad -\sum_{s=i_3}^{i-1}z^{1+\frac{1}{\alpha}}(s+1)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)},\end{aligned}$$

$$\begin{aligned}\Rightarrow z(i) &\le z(i_3)+\sum_{s=i_3}^{i-1}\Delta\rho(s)\frac{z(s+1)}{\rho(s+1)}-\sum_{s=i_3}^{i-1}A\alpha(s)\rho(s)\\ &\quad -\sum_{s=i_3}^{i-1}z^{1+\frac{1}{\alpha}}(s+1)\rho(s)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)},\end{aligned}$$

$$\begin{aligned}\Rightarrow z(i) &\le z(i_3)-\sum_{s=i_3}^{i-1}A\alpha(s)\rho(s)\\ &\quad +\sum_{s=i_3}^{i-1}\left[\Delta\rho(s)\frac{z(s+1)}{\rho(s+1)}-z^{1+\frac{1}{\alpha}}(s+1)\rho(s)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}\right], \quad (3.17)\end{aligned}$$

Using the condition (A4), then inequality (3.17) can be written as

$$\Rightarrow z(i)\le z(i_3)-\sum_{s=i_3}^{i-1}A\alpha(s)\rho(s)+\sum_{s=i_3}^{i-1}\left(\frac{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)\alpha}{B\rho(s)}\right)^{\alpha}\left(\frac{\Delta\rho(s)}{(\alpha+1)\rho(s+1)}\right)^{\alpha+1}.$$

$$\sum_{s=i_3}^{i-1}\left(A\alpha(s)\rho(s)-\left(\frac{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)\alpha}{B\rho(s)}\right)^{\alpha}\left(\frac{\Delta\rho(s)}{(\alpha+1)\rho(s+1)}\right)^{\alpha+1}\right)\le z(i_3)-z(i)<z(i_3).$$

Thus

$$\limsup_{i\to\infty}\sum_{s=i_3}^{i-1}\left(A\alpha(s)\rho(s)-\left(\frac{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)\alpha}{B\rho(s)}\right)^{\alpha}\left(\frac{\Delta\rho(s)}{(\alpha+1)\rho(s+1)}\right)^{\alpha+1}\right)<z(i_3)<\infty, \quad (3.18)$$

which runs counter to (3.2). Our proof of Theorem 1 is completed. □

Under these circumstances, the ensuring theorem is demonstrated.
Let $H(i,s)$ be a positive sequence such that $H(i,i)=0$, for $i\ge i_0$, $H(i,s)>0$ and $\Delta_2H(i,s)=H(i,s+1)-H(i,s)<0$, for $i\ge s\ge i_0$. Here $H(i,s)$ is a double sequence that refers to a sequence defined over two indices. It can be used to model more complex systems than single sequences. They are frequently encountered in applications such as signal processing, control theory, and numerical analysis.

Theorem 3.2. *Assume that the conditions A1-A4 hold. If there exists a positive sequence $\rho(i)\ni$*

$$\limsup_{i\to\infty}\left[\frac{1}{H(i,i_0)}\sum_{s=i_0}^{i-1}A\alpha(s)\rho(s)H(i,s)-\left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right]=\infty, \quad (3.19)$$

where

$$h(i,s)=\Delta_2H(i,s)+H(i,s)\Delta\rho(s)\frac{z(s+1)}{\rho(s+1)}.$$

Then every solution of (1.1) is oscillatory.

Proof. Suppose that $x(i)$ is a non-oscillatory solution of (1.1). Without loss of generality assume that $x(i)$ is an eventually non-negative solution.
Then $\exists\, i_1 \in [i_0, \infty] \ni x(i) > 0, x(i-u) > 0, x^{\kappa}(i-v) > 0, \forall\, i \geq i_1$.
Proceeding as in the proof of Theorem 3.1, inequality (3.13) holds.
Then multiplying both sides of inequality (3.15) by $H(i,s)$ and then summing from i_3 to $i-1$, we get

$$\sum_{s=i_3}^{i-1} \Delta z(s) H(i,s) \leq \sum_{s=i_3}^{i-1} H(i,s)\Delta\rho(s)\frac{z(s+1)}{\rho(s+1)} - \sum_{s=i_3}^{i-1} A\alpha(s)\rho(s)H(i,s) - \sum_{s=i_3}^{i-1} H(i,s) z^{1+\frac{1}{\alpha}}(s+1)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}, \tag{3.20}$$

$$\Rightarrow \sum_{s=i_3}^{i-1} A\alpha(s)\rho(s)H(i,s) \leq -\sum_{s=i_3}^{i-1} \Delta z(s) H(i,s) + \sum_{s=i_3}^{i-1} H(i,s)\Delta\rho(s)\frac{z(s+1)}{\rho(s+1)} - \sum_{s=i_3}^{i-1} H(i,s) z^{1+\frac{1}{\alpha}}(s+1)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}. \tag{3.21}$$

Now

$$-\sum_{s=i_3}^{i-1} \Delta z(s) H(i,s) = [-H(i,s)z(s)]_{i_3}^{i} + \sum_{s=i_3}^{i-1} z(s+1)\Delta_2 H(i,s),$$

$$\Rightarrow -\sum_{s=i_3}^{i-1} \Delta z(s) H(i,s) = [H(i,i_3)z(i_3)] + \sum_{s=i_3}^{i-1} z(s+1)\Delta_2 H(i,s).$$

Therefore (3.21) becomes,

$$\sum_{s=i_3}^{i-1} A\alpha(s)\rho(s)H(i,s) \leq [H(i,i_3)z(i_3)] + \sum_{s=i_3}^{i-1} z(s+1)\Delta_2 H(i,s) + \sum_{s=i_3}^{i-1} H(i,s)\Delta\rho(s)\frac{z(s+1)}{\rho(s+1)} - \sum_{s=i_3}^{i-1} H(i,s) z^{1+\frac{1}{\alpha}}(s+1)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)},$$

$$\Rightarrow \sum_{s=i_3}^{i-1} A\alpha(s)\rho(s)H(i,s) \leq H(i,i_3)z(i_3) + \sum_{s=i_3}^{i-1} h(i,s)z(s+1) - \sum_{s=i_3}^{i-1} H(i,s) z^{1+\frac{1}{\alpha}}(s+1)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)},$$

where $h(i,s) = \Delta_2 H(i,s) + H(i,s)\Delta\rho(s)\frac{z(s+1)}{\rho(s+1)}$.

$$\Rightarrow \sum_{s=i_3}^{i-1} A\alpha(s)\rho(s)H(i,s) \leq H(i,i_3)z(i_3) + \sum_{s=i_3}^{i-1}\left(h(i,s)z(s+1) - H(i,s)z^{1+\frac{1}{\alpha}}(s+1)\frac{B}{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}\right).$$

Using the condition (A4), the aforementioned inequality can be expressed as

$$\sum_{s=i_3}^{i-1} A\alpha(s)\rho(s)H(i,s) \le H(i,i_3)z(i_3) + \sum_{s=i_3}^{i-1}\left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}.$$

or

$$\sum_{s=i_3}^{i-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right) \le H(i,i_0)z(i_3), \tag{3.22}$$

for $i > i_3 > i_0$.

Now

$$\begin{aligned}
&\sum_{s=i_0}^{i-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\\
&\quad= \sum_{s=i_0}^{i_3-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\\
&\quad+ \sum_{s=i_3}^{i-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right).
\end{aligned} \tag{3.23}$$

Substituting inequality (3.22) in (3.23), we have

$$\begin{aligned}
&\sum_{s=i_0}^{i-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\\
&\qquad\le \sum_{s=i_0}^{i_3-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\\
&\qquad+ H(i,i_0)z(i_3),
\end{aligned}$$

$$\begin{aligned}
\Rightarrow &\sum_{s=i_0}^{i-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\\
&\qquad\le \sum_{s=i_0}^{i_3-1} A\alpha(s)\rho(s)H(i,s) + H(i,i_0)z(i_3),
\end{aligned}$$

$$\begin{aligned}
\Rightarrow &\sum_{s=i_0}^{i-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\\
&\qquad\le H(i.i_0)\sum_{s=i_0}^{i_3-1} A\alpha(s)\rho(s) + z(i_3),
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{1}{H(i,i_0)}&\sum_{s=i_0}^{i-1}\left(A\alpha(s)\rho(s)H(i,s) - \left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\\
&\qquad\le \sum_{s=i_0}^{i_3-1} A\alpha(s)\rho(s) + z(i_3),
\end{aligned}$$

Taking limit supremum as $i \to \infty$, we get

$$\limsup_{i\to\infty}\left[\frac{1}{H(i,i_0)}\sum_{s=i_0}^{i-1}\left(A\alpha(s)\rho(s)H(i,s)-\left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\right]$$
$$\leq \sum_{s=i_0}^{i_3-1}A\alpha(s)\rho(s)+z(i_3)<\infty,$$

which contradicts (3.19). This completes the proof. □

4 Application

Example 4.1. Consider

$$\Delta\left(i^3\Delta\left(i^2\Delta\left(i\Delta^{\frac{1}{4}}x(i-1)\right)^3\right)\right)+i^{-3}x^5(i-2)=0. \tag{4.1}$$

Here, $i_0=2, a(i)=i^3, b(i)=i^2, c(i)=i$, $\beta=\frac{1}{4}$,$u=1, v=2, \alpha=3, \kappa=5$ and choosing $\rho(i)=i^2$.

Now

$$\lim_{i\to\infty}\sum_{s=i_0}^{i-1}\left[\frac{1}{c(s)}\right]=\lim_{i\to\infty}\sum_{s=2}^{i-1}\left[\frac{1}{s}\right]=\infty.$$

and

$$\limsup_{i\to\infty}\sum_{s=i_3}^{i-1}\left(A\alpha(s)\rho(s)-\left(\frac{\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)\alpha}{B\rho(s)}\right)^{\alpha}\left(\frac{\Delta\rho(s)}{(\alpha+1)\rho(s+1)}\right)^{\alpha+1}\right)$$
$$=\limsup_{i\to\infty}\sum_{s=i_3}^{i-1}\left(As^{-3}s^2-\left(\frac{3(s+1)^2s^{\frac{2}{3}}s}{Bs^2}\right)^3\left(\frac{\Delta s^2}{4(s+1)^2}\right)^4\right)$$
$$=\limsup_{i\to\infty}\sum_{s=i_3}^{i-1}\left(A-\left(\frac{3(s+1)^2s^{\frac{2}{3}}}{Bs}\right)^3\left(\frac{\Delta s^2}{4(s+1)^2}\right)^4\right)=\infty$$

Thus all the conditions of Theorem 3.1 are satisfied. Hence we conclude that every solution of (4.1) is oscillatory.

Example 4.2. Consider

$$\Delta\left(i^{-2}\Delta\left(i^3\Delta\left(i^2\Delta^{\frac{1}{3}}x(i-1)\right)^5\right)\right)+i^{-1}x(i-2)=0. \tag{4.2}$$

Here, $i_0=2, a(i)=i^{-2}, b(i)=i^3, c(i)=i^2$, $\beta=\frac{1}{3}$,$u=1, v=2, \alpha=5, \kappa=1$ and choosing $\rho(i)=i^2$.

Now

$$\limsup_{i\to\infty}\left[\frac{1}{H(i,i_0)}\sum_{s=i_0}^{i-1}\left(A\alpha(s)\rho(s)H(i,s)-\left(\frac{\alpha\rho(s+1)\rho^{\frac{1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)}{H(i,s)B}\right)^{\alpha}\left(\frac{h(i,s)}{\alpha+1}\right)^{\alpha+1}\right)\right]$$
$$=\limsup_{i\to\infty}\left[\frac{1}{H(i,i_0)}\sum_{s=i_0}^{i-1}\left(sAH(i,s)-\left(\frac{5(s+1)^2s^{\frac{1}{5}}}{H(i,s)B}\right)^5\left(\frac{h(i,s)}{6}\right)^6\right)\right]=\infty.$$

Thus (3.19) is satisfied. All solutions to (4.2) are oscillatory, according to Theorem 3.2

5 Conclusion remarks

In this paper, by using Riccati transformation and summing techniques, the oscillatory solutions of equation (1.1) are established in Theorem 3.1 and Theorem 3.2. These techniques can transform an equation into a form that is easier to solve. Here, some sufficient conditions are proved. These conditions are new and authentic. Also the examples are demonstrated with the effect of main results.

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Received: 2023-10-16
Accepted: 2024-09-17