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$\nu-$ideals of Almost Distributive Lattice

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Abstract In an Almost Distributive Lattice(ADL), the notion of $\nu-$ideals is introduced and their properties are investigated. To characterize an $E-$complemented ADL, a set of equivalent conditions is established for every maximum ideal of an ADL to become into a $\nu-$ideal. In addition, an ADL's $\nu-$ideals are characterized using minimal prime $E-$ideals.

1 Introduction

The idea of an Almost Distributive Lattice(ADL) was presented by Swamy U.M., and Rao G.C., [11] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. The concept of dense complemented ideal was introduced by Ramesh Sirisetti and Jogarao in [9]. In [10], the notions of $\delta-$primary ideals and weakly $\delta-$primary ideals with the help of an ideal expansion were introduced. In [8], the concept of closure ideal was introduced in an $MS-$ADL and their properties were studied. In this paper, the concept of $\nu-$ideals are introduced and their properties are investigated in an ADL that is analogous to a distributive lattice. Every ADL's maximum ideal becomes into a $\nu-$ideal by a set of equivalent conditions, which leads to a characterization of $E-$complemented ADLs. Some necessary conditions are proved for proper $E-$ideal to become $\nu-$ideal. Finally, minimal prime $E-$ideals are used to characterize the $\nu-$ideals of an ADL.

2 Preliminaries

In this section, we go through some ideas as well as significant observations from [2] and [11], that are necessary for the paper's text.

Definition 2.1. [11] An algebra $R = (R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions:

(1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$;

(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

(3) $(x \vee y) \wedge y = y$;

(4) $(x \vee y) \wedge x = x$;

(5) $x \vee (x \wedge y) = x$;

(6) $0 \wedge x = 0$;

(7) $x \vee 0 = x$, for all $x, y, z \in R$.

Example 2.2. Each non-empty set A can be classified as an ADL in the following way:
Take $a_0 \in A$. Define the $\vee$and $\wedge$ binary operations on A by

$$a \vee b = \begin{cases} a & \text{if } a \neq a_0 \\ b & \text{if } a = a_0 \end{cases} \qquad a \wedge b = \begin{cases} b & \text{if } a \neq a_0 \\ a_0 & \text{if } a = a_0. \end{cases}$$

Then $(A, \vee, \wedge, a_0)$ is an ADL (where a_0 is the zero) and is said to be a discrete ADL.

For any $x, y \in R$, define $x \leq y$ if and only if $x = x \wedge y$ (or alternatively, $x \vee y = y$), then $\leq$ is a partial ordering on R.

Theorem 2.3. *[11] For any $x, y, z \in R$, we have the following:*

(1) $x \vee y = x \Leftrightarrow x \wedge y = y$;

(2) $x \vee y = y \Leftrightarrow x \wedge y = x$;

(3) $\wedge$ *is associative in* R;

(4) $x \wedge y \wedge z = y \wedge x \wedge z$;

(5) $(x \vee y) \wedge z = (y \vee x) \wedge z$;

(6) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

(7) $x \wedge (x \vee y) = x, \ \ (x \wedge y) \vee y = y$ *and* $x \vee (y \wedge x) = x$;

(8) $x \wedge x = x$ *and* $x \vee x = x$.

An ADL R exhibits nearly all the properties of a distributive lattice, except for the absence of the right distributivity of $\vee$ over $\wedge$, as well as the non-commutativity of both $\vee$ and $\wedge$. However, an ADL R can be considered a distributive lattice as long as any one of these characteristics holds. In the context of an ADL R, we define an element $m \in R$ as maximal if it stands as the utmost element within the partially ordered set $(R, \leq)$. In other words, for every $a \in R$, the condition $m \leq a$ implies that $m = a$.

In ADL structures, motivated by the notions of distributive lattices [1, 6], we define a non-empty subset I of R as an ideal if, for any elements $a, b \in I$ and $x \in R$, both conditions $a \vee b \in I$ and $a \wedge x \in I$ are satisfied. In a similar way, a non-empty subset F of R is termed a filter when, for elements $a, b \in F$ and $x \in R$, both $a \wedge b \in F$ and $x \vee a \in F$ are true.

The collection $\mathfrak{I}(R)$ of all ideals in R forms a bounded distributive lattice. It possesses a least element, denoted as $\{0\}$, and a greatest element, which is the entire set R, both ordered by set inclusion. Within this lattice, for any two ideals I and J in $\mathfrak{I}(R)$, their infimum is represented as $I \cap J$, while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. Furthermore, a proper ideal(filter) P of R said to be a prime if, for any elements x and y in R, the condition $x \wedge y \in P (x \vee y \in P)$ implies that either $x \in P$ or $y \in P$. Additionally, a proper ideal(filter) M of R is called maximal when there is no other proper ideal(filter) of R that contains it. It is observed that every maximal ideal(filter) of R is prime. Furthermore, any proper ideal(filter) of R contained in a maximal ideal(filter) in R. Given any subset S of an ADL R, the smallest ideal containing S is denoted as $(S]$ and is defined as $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$. When S consists of a single element, say $S = \{s\}$, we simplify the notation to $(s]$, and such an ideal is referred to as the principal ideal of R. Similarly, for any subset S of R, the smallest filter containing S is denoted as $[S)$ and is defined as $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$. When S contains just one element, i.e., $S = \{s\}$, we use the notation $[s)$, and such a filter is termed the principal filter of R. It can be demonstrated that, for any two elements a and b in R, we have that $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$. These relationships establish that the collection $(\mathfrak{I}^{PI}(R), \vee, \cap)$ of all principal ideals of R forms a sublattice of the distributive lattice $(\mathfrak{I}(R), \vee, \cap)$, which consists of all ideals of R. Furthermore, it should be noted that the set $(\mathfrak{F}(R), \vee, \cap)$ of all filters in R constitutes a bounded distributive lattice.

Theorem 2.4. *[5] Let R be an ADL with maximal elements. Then P is a prime ideal of R if and only if $R \setminus P$ is a prime filter of R.*

It is known that, for any $x, y \in R$ with $x \leq y$, the interval $[x, \ y]$ is a bounded distributive lattice. Now, an ADL R is said to be relatively complemented if, for any $x, y \in R$ with $x \leq y$, the interval $[x, \ y]$ is a complemented distributive lattice.

Theorem 2.5. *[12] An ADL R with maximal elements is relatively complemented if and only if B(R) = R, where $B(R) = \{x \in R \mid x \wedge y = 0, \ x \vee y$ is maximal, for some $y \in R\}$.*

Definition 2.6. [4] For any nonempty subset A of an ADL R, define $A^+ = \{ x \in R \mid a \vee x$ is maximal element, for all $a \in A \}$. Here A^+ is called the dual annihilator of A in R.

For any $a \in R$, we have $\{a\}^+ = [a)^+$, where $[a)$ is the principal filter generated by a. An element a of an ADL R is called dual dense element if $(a]^+ = \mathcal{M}$, where $\mathcal{M}$ is the set of all maximal elements of R and the set E of all dual dense elements in an ADL is an ideal if E is non-empty.

Definition 2.7. [7] An ideal G of R is said to be an $E-$ideal of R if $E \subseteq G$. An $E-$ideal Q is said to be proper if $Q \subsetneq R$. A proper $E-$ideal Q is said to be maximal if it is not properly contained in any proper $E-$ideal of R. A proper $E-$ideal Q of an ADL R is said to be a prime $E-$ideal if Q is prime ideal of R.

Definition 2.8. [7] A prime $E-$ideal M of an ADL R containing an $E-$ideal G is said to be a minimal prime $E-$ideal belonging to G if there exists no prime $E-$ideal N such that $G \subseteq N \subseteq M$.

Note that if we take $E = G$ in the above definition then we say that M is a minimal prime $E-$ideal.

Definition 2.9. [7] For any nonempty subset S of R, define $(S, E) = \{a \in R \mid s \wedge a \in E,\ for\ all\ s \in S\}$. We call this set as relative dual annihilator of S with respect to the ideal E.

For $S = \{s\}$, we denote $(\{s\}, E)$ by (s, E).

Theorem 2.10. *[7] For any $x, y \in R$ we have the following:*
(1) $((x], E) = (x, E)$;
(2) $x \leq y \Rightarrow (y, E) \subseteq (x, E)$;
(3) $(x \vee y, E) = (x, E) \cap (y, E)$;
(4) $((x \wedge y, E), E) = ((x, E), E) \cap ((y, E), E)$;
(5) $(x, E) = R \Leftrightarrow x \in E$.

3 $\nu-$ideals of an ADL

In this section, the concept of $\nu-$ideals is introduced in an ADL. The class of all $\nu-$ideals are characterized in terms of minimal prime $E-$ideal.

Definition 3.1. An element a of an ADL R with maximal elements is said to be $E-$complemented, if there exists an element $b \in R$ such that $a \wedge b \in E$ and $a \vee b$ is a maximal element of L. An ADL R with maximal elements is said to be an $E-$complemented ADL L, if every element of an ADL R is $E-$complemented.

Now, we have the following.

Proposition 3.2. *For any prime ideal M of an $E-$complemented ADL R with maximal elements, the following are equivalent:*
(1) $E \subseteq M$;
(2) for any $a \in R,\ a \in M$ *if and only if* $(a, E) \nsubseteq M$;
(3) for any $a, b \in R$ *with* $(a, E) = (b, E), a \in M$ *implies that* $b \in M$;
(4) $E \cap (R \setminus M) = \emptyset$.

Proof. $(1) \Rightarrow (2)$: Assume (1). Suppose $a \in M$. Since R is $E-$complemented, there exists $b \in R$ such that $a \wedge b \in E$ and $a \vee b$ is maximal. Then $b \in (a, E)$. Clearly, we have $b \notin M$ and hence $(a, E) \nsubseteq M$. Conversely, assume that $(a, E) \nsubseteq M$. Then there exists $b \in R$ such that $b \in (a, E)$ and $b \notin M$. Clearly, $a \wedge b \in E \subseteq M$. Since M is prime and $b \notin M$, we get $a \in M$.
$(2) \Rightarrow (3)$: Assume (2). Let $a, b \in R$ with $(a, E) = (b, E)$. Suppose $a \in M$ By our assumption, we get $(a, E) \nsubseteq M$ and hence $(b, E) \nsubseteq M$. Therefore $b \in M$.
$(3) \Rightarrow (4)$: Assume (3). Let $a \in R$. If $a \in E \cap (R \setminus M)$. Then $(a, E) = R$ and $a \notin M$. That implies $(a, E) = R = (0, E)$. Since $0 \in M$, by our assumption, we get $a \in M$, which is a contradiction. Hence $E \cap (R \setminus M) = \emptyset$.
$(4) \Rightarrow (1)$: Assume (4). Then we have that $E \subseteq M$. □

Theorem 3.3. *Let a' be an $E-$complement of a in an ADL R with maximal elements. Then every prime $E-$ideal contain exactly one of a or a'.*

Proof. Since a' be an $E-$complement of a, we have that $a \wedge a' \in E$ and $a \vee a'$ is maximal. Let M be a prime $E-$ideal of R. Clearly, $a \wedge a' \in E \subseteq M$. Since M is prime, we get $a \in M$ or $a' \in M$. Suppose $a \in M$ and $a' \in M$. Then $a \vee a' \in M$, which is a contradiction. Hence M must contain exactly one of a or a'. □

Proposition 3.4. *Let R be an $E-$complemented ADL. Then the following conditions are equivalent:*
(1) R is a relatively complemented ADL;
(2) every prime ideal contains exactly one of a or a', where a' is the $E-$complement of a in R;
(3) every prime ideal is an $E-$ideal;
(4) every minimal prime ideal is an $E-$ideal.

Proof. $(1) \Rightarrow (2)$: Assume (1). Let M be a prime ideal of R and $a \in M$. By our assumption, there exists an element $a' \in R$ such that $a \wedge a' = 0$ and $a \vee a'$ is a maximal element. Since $a \wedge a' = 0$, we get $a \wedge a' \in M$. Since M is prime, we get $a \in M$ or $a' \in M$. Since $a \vee a'$ is maximal, we get M contain exactly one of a or a'.
$(2) \Rightarrow (3)$: Assume the condition (2). Let M be a prime ideal of R. Let $a \in E$. Since R is $E-$complemented, we get that $a' \in (a)^+ = \mathcal{M}$. Hence $a' \notin M$. By the condition (2), we get $a \in M$. Thus $E \subseteq M$. Therefore M is an $E-$ideal of R.
$(3) \Rightarrow (4)$: It is clear.
$(4) \Rightarrow (1)$: Assume (4). Let $a \in R$. Suppose $a \wedge a' \neq 0$. Then there exists a maximal filter M of R such that $a \wedge a' \in M$. Clearly, $R \setminus M$ is a minimal prime ideal such that $a \wedge a' \notin R \setminus M$. Hence $a \notin R \setminus M$ and $a' \notin R \setminus M$. By the hypothesis, we get $E \subseteq R \setminus M$. By Theorem-3.3, $R \setminus M$ must contain exactly one of a or a', which is a contradiction. Therefore $a \wedge a' = 0$ and hence R is a relatively complemented ADL. □

Theorem 3.5. *For any proper deal M of an $E-$complemented ADL R, M is maximal; if and only if M is a prime $E-$ideal.*

Proof. Let M be any proper ideal of R. Assume that M is a maximal ideal of R. Clearly, M is prime. Let $a \in E$. Then $(a)^+ = \mathcal{M}$. Suppose $a \notin M$. Then $M \vee (a] = R$. There exist $s \in M$ and $t \in (a]$ such that $s \vee t$ is maximal. That implies $s \in (a)^+$. Since $(a)^+ = \mathcal{M}$, we get s is maximal. That implies $s \in M$, this leads $M = R$, which is a contradiction. Hence $a \in M$. Thus $E \subseteq M$. Therefore M is a prime $E-$ideal of R. Conversely, assume that M is a prime $E-$ideal of R. Suppose M is not maximal. Then there exists a proper ideal N of R such that $M \subsetneq N$. Choose $a \in N \setminus M$. Since R is $E-$complemented, there exists $a' \in R$ such that $a \wedge a' \in E \subseteq M$ and $a \vee a'$ is maximal. Since M is prime and $a \notin M$, we get $a' \in M \subset N$. Then $a \vee a' \in N$, which is a contradiction. Therefore M is maximal. □

In an $E-$complemented ADL, the class of all maximal ideals and the class of all prime $E-$ideals of R are the same. Since every prime $E-$ideal is maximal, we can conclude that every prime $E-$ideal is minimal in an $E-$complemented ADL. Hence maximal ideals, prime $E-$ideal, and minimal prime $E-$ideals are the same in an $E-$complemented ADL.

Definition 3.6. For any filter F of an ADL R, define $\nu(F) = \{a \in R \mid a \wedge s \in E, \text{ for some } s \in F\}$.

Clearly, we have that $\nu(F) = \bigcup_{a \in F} (a, E)$.

Proposition 3.7. *Let F be a filter of an ADL R. Then $\nu(F)$ is an $E-$ideal of R.*

Proof. Clearly, $E \subseteq \nu(F)$. Let $a, b \in \nu(F)$. Then there exist $s, t \in F$ such that $a \wedge s \in E$ and $b \wedge t \in E$. Since E is an ideal of R, we get $s \wedge t \wedge a \in E$ and $s \wedge t \wedge b \in E$. Then $(s \wedge t \wedge a) \vee (s \wedge t \wedge b) \in E$ and hence $((s \wedge t) \wedge (a \vee b) \in E$. That implies $(a \vee b) \wedge (s \wedge t) \in E$. Since $s, t \in F$, we get $s \wedge t \in F$ and hence $a \vee b \in \nu(F)$. Let $a \in \nu(F)$ Then there exists $s \in F$ such that $a \wedge s \in E$. Let $r \in R$. Since E is an ideal of R, we get $(a \wedge r) \wedge s \in E$ and hence $a \wedge s \in \nu(F)$. Therefore $\nu(F)$ is an $E-$ideal of R. □

Lemma 3.8. *Let G, H be two filters of an ADL R. Then we have the following:*
(1) $G \cap \nu(G) \neq \emptyset \Leftrightarrow \nu(G) = R$;
(2) $G \subseteq H \Rightarrow \nu(G) \subseteq \nu(H)$;
(3) $\nu(G) \cap \nu(G) = w(G \cap H)$.

Proof. (1). Assume that $G \cap \nu(G) \neq \emptyset$. Then choose an element $a \in G \cap \nu(G)$. Then $a \in G$ and $a \in \nu(G)$. Since $a \in \nu(G)$, there exists $s \in G$ such that $a \wedge s \in E$. By Theorem-2.10(5), we get $(a \wedge s, E) = R$. Since $a \in G$ and $s \in G$, we get $a \wedge s \in G$. Hence $\nu(G) = \bigcup\limits_{a \in G} (a, E) = R$. Conversely, assume that $\nu(G) = R$. Then for any $m \in \mathcal{M}$ such that $m \in \nu(G)$ and hence $m \in G \cap \nu(G)$. Thus $G \cap \nu(G) \neq \emptyset$.
(2). Assume $G \subseteq H$. Let $a \in \nu(G)$. Then there exists $s \in G$ such that $a \wedge s \in E$. Since $G \subseteq H$, we get $s \in H$ and hence $a \in \nu(H)$. Thus $\nu(G) \subseteq \nu(H)$.
(3). Clearly, $\nu(G \cap H) \subseteq \nu(G) \cap \nu(H)$. Let $a \in \nu(G) \cap \nu(H)$. Then there exist $s \in G$ and $t \in H$ such that $a \wedge s \in E$ and $a \wedge t \in E$. Since $s \in G$ and $t \in H$, we get $s \vee t \in G \cap H$ and hence $a \wedge (s \vee t) = (a \wedge s) \vee (a \wedge t) \in E$. Therefore $a \in \nu(G \cap H)$. Hence $\nu(G) \cap \nu(H) \subseteq \nu(G \cap H)$. □

Proposition 3.9. *If G, H are two filters of an ADL R with $\nu(G) \cap H = \emptyset$, then there exists a prime $E-$ideal M such that $\nu(G) \subseteq M$ and $M \cap H = \emptyset$.*

Proof. Let G and H be two filters of an ADL R such that $\nu(G) \cap H = \emptyset$. Then there exists a prime filter P such that $H \subseteq P$ and $\nu(G) \cap P = \emptyset$. Since $\nu(G) \cap P = \emptyset$, we get that $E \subseteq \nu(G) \subseteq R \setminus P$. Since $R \setminus P$ is a prime ideal of R, we get that $R \setminus P = M$ is a prime $E-$ideal of R containing $\nu(G)$. □

Now we have the following definition of $\nu-$ideal in an ADL.

Definition 3.10. An $E-$ideal G of an ADL R is said to be a $\nu-$ideal if $G = \nu(F)$, for some filter F of R such that $F \cap E = \emptyset$.

From the above definition, it is easy to verify that for any $m \in \mathcal{M}$, $\nu(m) = E$. Hence E is proper and the smallest $E-$ideal of R.

Example 3.11. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and define $\vee$, $\wedge$ on R as follows:

$\wedge$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	1	2	3	4	5	6	7
3	0	3	3	3	0	0	3	0
4	0	4	5	0	4	5	7	7
5	0	4	5	0	4	5	7	7
6	0	6	6	3	7	7	6	7
7	0	7	7	0	7	7	7	7

$\vee$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	1	2	3	1	2	6	6
4	4	1	1	1	4	4	1	4
5	5	2	2	2	5	5	2	5
6	6	1	2	6	1	2	6	6
7	7	1	2	6	4	5	6	7

Then $(R, \vee, \wedge)$ is an ADL. Clearly, we have that $E = \{0, 7\}$. Consider the $E-$ideals $I_1 = \{0, 3\}, I_2 = \{0, 4, 5, 7\}, I_3 = \{0, 3, 6, 7, \}, I_4 = \{0, 7\} = E$ and filters $F_1 = \{1, 2, 4\}, F_2 = \{1.2.6\}$. Now, $\nu(F_1) = \{0, 3, 6, 7\} = I_3$. Hence G is a $\nu-$ideal of R. But $\nu(F_2) = \{0, 4, 5, 6\} \neq I_i$, for $i = 1, 2, 3, 4$. Hence

Proposition 3.12. *For any $a \notin E$ in an ADL R. we have (a, E) is a $\nu-$ideal of R.*

Proof. Let $a \notin E$. Clearly, we have $[a) \cap E = \emptyset$. Let $s \in (a, E)$. Then $s \wedge a \in E$. Since $a \in [a)$, we get $s \in \nu([a))$ and hence $(a, E) \subseteq \nu([a))$. Let $s \in \nu([a))$. Then there exists $b \in [a)$ such that $s \wedge b \in E$. That implies $s \wedge a \in E$, which gives that $s \in (a, E)$. Therefore $\nu([a)) \subseteq (a, E)$ and hence $(a, E) = \nu([a))$. Thus (a, E) is a $\nu-$ideal of R. □

Theorem 3.13. *Let M be a prime $E-$ideal of R with $(M, E) \neq E$. Then M is a $\nu-$ideal.*

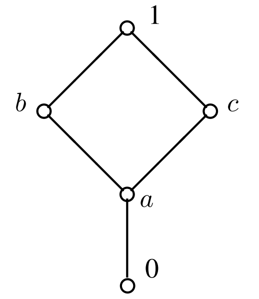
Proof. Assume that $(M,E) \neq E$. Since $E \subseteq (M,E)$, we get that $(M,E) \nsubseteq E$. Then there exists $a \in (M,E)$ such that $a \notin E$. Clearly, $[a) \cap E = \emptyset$ and $a \notin M$. Then $M \subseteq ((M,E),E) \subseteq (a,E)$. Therefore $M \subseteq (a,E)$. Let $s \in (a,E)$. Then $s \wedge a \in E \subseteq M$. Since $a \notin M$, we have $s \in M$. Then $(a,E) \subseteq M$. Therefore $M = (a,E) = \nu([a))$ and hence M is a $\nu-$ideal of R. □

Theorem 3.14. *Every minimal prime $E-$ideal of an ADL R is a $\nu-$ideal.*

Proof. Let M be a minimal prime $E-$ideal of R. Then $R \setminus M$ is a prime filter of R such that $E \cap (R \setminus M) = \emptyset$. Now prove that $M = \nu(R \setminus M)$. Let $a \in M$. Since M is minimal, there exists $b \in R \setminus M$ such that $a \wedge b \in E$. That implies $a \in \nu(R \setminus M)$. Therefore $M \subseteq \nu(R \setminus M)$. Let $a \in w(R \setminus M)$. Then there exists $s \in R \setminus M$ such that $a \wedge s \in E \subseteq M$. Since M is prime and $s \notin M$, we get $a \in M$. Therefore $\nu(R \setminus M) \subseteq M$ and hence $M = \nu(R \setminus M)$. Thus M is a $\nu-$ideal of R. □

We now turn our intension towards the converse of the above theorem. In general, every $\nu-$ideal of an ADL need not be a minimal prime $E-$ideal. In fact it need not even be a prime $E-$ideal. It can be observed in the following example:

Example 3.15. Consider a distributive lattice $L = \{0, a, b, c, 1\}$ and discrete ADL $A = \{0', a'\}$.



Clearly,
$R = A \times L = \{(0',0),(0',a),(0',b),(0',c),(0',1),(a',0),(a',a),(a',b),(a',c),(a',1)\}$ is an ADL with zero element $(0,0')$. Clearly, the dense set $E = \{(0',0),(0',a)\}$. Consider an $E-$ideal $I = \{(0',0),(0',a),(0',c)\}$ and a filter $F = \{(a',b),(a',1)\}$. Clearly, I is a $\nu-$ideal, but not prime.

Though every $\nu-$ideal need not be a prime $E-$ideal, we derive a necessary and sufficient condition for a $\nu-$ideal of an ADL to become a prime $E-$ideal.

Theorem 3.16. *A proper $\nu-$ideal G of an ADL R is a prime $E-$ideal if and only if G contains a prime $E-$ideal.*

Proof. Let G be a proper $\nu-$ideal of R. Assume that G is a prime $E-$ideal of R. Clearly, G contains a prime $E-$ideal G. Conversely, assume that G contains a prime $E-$ideal, say M. Since $E \subseteq M \subseteq G, G$ is an $E-$ideal of R. Since G is a $\nu-$ideal, we get $G = \nu(F)$, for some filter F of R with $F \cap E = \emptyset$. Let $s,t \in R$ such that $s \notin G$ and $t \notin G$. Since $M \subseteq G$, we get $s \notin M$ and $t \notin M$. Since M is prime, we get $s \wedge t \notin M$. That implies $(s \wedge t, E) \subseteq M \subseteq G = \nu(F)$. Suppose $s \wedge t \in G = \nu(F)$. Then there exists $x \in F$ such that $s \wedge t \wedge x \in E$. That implies $x \in (s \wedge t, E) \subseteq \nu(F)$. Therefore $x \in F \cap \nu(F)$ and hence $F \cap \nu(F) \neq \emptyset$. By Lemma-3.8(1), $G = \nu(F) = R$, which is a contradiction. Thus G is a prime $E-$ideal of R. □

In the above Theorem-3.16, It is observed that every minimal prime $E-$ideal is a prime $\nu-$ideal of R. Now we established the equivalency between prime $\nu-$ideals and minimal prime $E-$ideals of an ADL.

Theorem 3.17. *Every prime $\nu-$ideal of an ADL R is a minimal prime $E-$ideal.*

Proof. Let M be a prime $\nu-$ideal of R. Then $M = \nu(F)$, for some filter F of R with $F \cap E = \emptyset$. Let $a \in M = \nu(F)$. Then there exists $b \in F$ such that $a \wedge b \in E$. Suppose $b \in M$. Then $b \in F \cap \nu(F)$. That implies $F \cap \nu(F) \neq \emptyset$. By Lemma-3.8(1), $M = \nu(F) = R$ which is a contradiction. Therefore $b \notin M$ and hence M is a minimal prime $E-$ideal. □

Theorem 3.18. *In an ADL R, the following are equivalent:*
(1) R is $E-$complemented;
(2) every prime $E-$ideal is a $\nu-$ideal;
(3) every prime $E-$ideal is minimal;
(4) every maximal ideal is a minimal prime $E-$ideal;
(5) every maximal ideal is a $\nu-$ideal.

Proof. $(1) \Rightarrow (2)$: Assume (1). Let M be a prime $E-$ideal of R. Then $R \setminus M$ is a prime ideal of R such that $(R \setminus M) \cap E = \emptyset$. Now prove that $M = \nu(R \setminus M)$. Let $a \in M$. Since R is $E-$complemented, there exists $b \in R$ such that $a \wedge b \in E$ and $a \vee b$ is maximal. Clearly, $b \notin M$, which gives that $b \in R \setminus M$. Since $a \wedge b \in E$, we get $a \in \nu(R \setminus M)$. Therefore $M \subseteq \nu(R \setminus M)$. Let $a \in \nu(R \setminus M)$. Then there exists $b \in R \setminus M$ such that $a \wedge b \in E$. Since $a \wedge b \in E \subseteq M$ and $b \notin M$, we get $a \in M$. Therefore $\nu(R \setminus M) \subseteq M$. Hence M is a $\nu-$ideal of R.
$(2) \Rightarrow (3)$: Assume (2). Let M be a prime $E-$ideal of R. By our assumption, P is a prime $\nu-$ideal. By Theorem-3.17, P is minimal.
$(3) \Rightarrow (4)$: It is clear.
$(4) \Rightarrow (5)$: It is clear.
$(5) \Rightarrow (1)$: Assume (5). Let $a \in R$ and $m \in \mathcal{M}$. Suppose $m \notin (a] \vee (a, E)$. Then there exists a maximal ideal M such that $(a] \vee (a, E) \subseteq M$. That implies $a \in M$ and $(a, E) \subseteq M$. By the assumption, M is a $\nu-$ideal. Since M is prime, by Theorem-3.17, M is minimal prime $E-$ideal. Then $a \notin M$, which is a contradiction. That implies $m \in (a] \vee (a, E)$. There exists $s \in (a, E)$ such that $a \vee s = m$. Since $s \in (a, E)$, we get $s \wedge a \in E$. Thus R is $E-$complemented. □

We conclude this paper with a characterization theorem of $\nu-$ideals in terms of minimal prime $E-$ideals. For this, we first need the following results.

Lemma 3.19. *Let F be a filter of an ADL R such that $F \cap E = \emptyset$. If M is a minimal prime $E-$ideal containing $\nu(F)$, then $F \cap M = \emptyset$.*

Proof. Let M be a minimal prime $E-$ideal of R with $\nu(F) \subseteq M$. Suppose $a \in F \cap M$. Then $a \in M$ and $a \in F$. Since M is minimal and $a \in M$, there exists $b \notin M$ such that $a \wedge b \in \nu(F)$. Then there exists $x \in F$ such that $(a \wedge b) \wedge x \in E$. That implies $b \wedge (a \wedge x) \in E$ and $a \wedge x \in F$. Therefore $b \in \nu(F) \subseteq M$, which is a contradiction. Thus $F \cap M = \emptyset$. □

Lemma 3.20. *Every minimal prime $E-$ideal of an ADL R containing a $\nu-$ideal is a minimal prime $E-$ideal in R.*

Proof. Let G be a $\nu-$ideal of R. Then $G = \nu(F)$, for some filter F of R such that $F \cap E = \emptyset$. Let M be a minimal prime $E-$ideal containing $G = \nu(F)$. By the above lemma, $F \cap M = \emptyset$. Let $a \in M$. Then there exists $b \notin M$ such that $a \wedge b \in \nu(F)$. There exists $x \in F$ such that $(a \wedge b) \wedge x \in E$. Therefore $a \wedge (b \wedge x) \in E \subseteq M$ and $b \wedge x \notin M$. Thus M is a minimal prime $E-$ideal of R. □

Now, $\nu-$ideals are characterized in terms of minimal prime $E-$ideals.

Theorem 3.21. *Every $\nu-$ideal of an ADL R is the intersection of all minimal prime $E-$ideals containing it.*

Proof. Let G be a $\nu-$ideal of R. Then $G = \nu(F)$, for some filter F of R such that $F \cap E = \emptyset$. Let $H = \bigcap\{M | M \text{ is a minimal prime } E - \text{ideal containing } G\}$. Clearly, $G \subseteq H$. Let $x \notin G = \nu(F)$. Then $x \wedge s \notin E$, for all $s \in F$. Then there exists a minimal prime $E-$ideal M such that $x \wedge s \notin M$. That implies $x \notin M$ and $s \notin M$. Since M is prime, $(s, E) \subseteq M$, for all $s \in F$. Then $G = \nu(F) \subseteq M$. Hence M is minimal such that $G \subseteq M$ and $x \notin M$. Therefore $x \notin H$, which leads $H \subseteq G$. Thus $G = H$. □

Theorem 3.22. *Let $\{G_\alpha\}_{\alpha\in\triangle}$ be a class of $\nu-$ideals of an ADL R. Then $\bigcap\limits_{\alpha\in\triangle} G_\alpha$ is a $\nu-$ideal of R.*

Proof. For each $\alpha \in \triangle$, let $G_\alpha = \nu(F_\alpha)$ where F_α is a filter of R such that $F_\alpha \cap E = \emptyset$. Then $\{F_\alpha\}_{\alpha\in\triangle}$ will be an arbitrary family of filters in R such that $F_\alpha \cap E = \emptyset$ for each $\alpha \in \triangle$. Hence $\bigcap\limits_{\alpha\in\triangle} F_\alpha$ is a filter of R such that $\Big(\bigcap\limits_{\alpha\in\triangle} F_\alpha\Big) \cap E = \emptyset$. By Lemma-3.8(3), we get $\bigcap\limits_{\alpha\in\triangle} \nu(F_\alpha) = \nu\Big(\bigcap\limits_{\alpha\in\triangle} F_\alpha\Big)$. Therefore $\bigcap\limits_{\alpha\in\triangle} G_\alpha$ is a $\nu-$ideal of R. □

Note that the class of all $\nu-$ideals of an ADL is closed under set-intersection. In general, $\nu-$ideals need not be closed under finite joins. However, in the following, we prove that the class $\mathfrak{I}_\nu(R)$ of all $\nu-$ideals of an ADL R forms a complete lattice.

Theorem 3.23. *Let G, H be two filters of an ADL R such that $G \cap E = H \cap E = \emptyset$. Then $\nu(G \vee H)$ is the smallest $\nu-$ideal containing both $\nu(G)$ and $\nu(H)$.*

Proof. Let G, H be two filters of R such that $G \cap E = H \cap E = \emptyset$. Clearly, $(G \vee H) \cap E = \emptyset$. By Lemma-3.8(2), we get $\nu(G) \subseteq \nu(G \vee H)$ and $\nu(H) \subseteq \nu(G \vee H)$. Suppose $\nu(G) \subseteq \nu(K)$ and $\nu(H) \subseteq \nu(K)$, for some filter K of R with $K \cap E = \emptyset$. Let $a \in \nu(G \vee H)$. Then there exist $s \in G$ and $t \in H$ such that $a \wedge (s \wedge t) \in E$. That implies $a \wedge s \in \nu(H) \subseteq \nu(K)$. There exists $x \in K$ such that $a \wedge s \wedge x \in E$. Since $x \wedge y \in K$, we get $a \in \nu(K)$. Therefore $\nu(G \vee H)$ is the supremum of $\nu(G)$ and $\nu(H)$. Consider this supremum by $\nu(G) \sqcup \nu(H)$. Thus $(\mathfrak{I}_\nu(R), \cap, \sqcup)$ forms a lattice. □

Corollary 3.24. *Let $\{\nu(F_\alpha)\}_{\alpha\in\triangle}$ be a class of $\nu-$ideals of an ADL R where $F_\alpha \cap E = \emptyset$ for each $\alpha \in \triangle$. Then $\bigsqcup\limits_{\alpha\in\triangle} \nu(F_\alpha)$ is the smallest $\nu-$idealr containing each $\nu(F_\alpha)$.*

It can be easily observed that the class of all $\nu-$ideals of an ADL forms a complete lattice with respect to set inclusion $\subseteq$, in which for any $\{\nu(F_\alpha)\}_{\alpha\in\triangle}$ of $\nu-$ideals, $inf\{\nu(F_\alpha)\}_{\alpha\in\triangle} = \nu(\bigcap\limits_{\alpha\in\triangle} F_\alpha)$ and the $sup\{\nu(F_\alpha)\}_{\alpha\in\triangle} = \nu\Big(\bigvee\limits_{\alpha\in\triangle} F_\alpha\Big)$. Since the class of all filters of an ADL forms a complete distributive lattice, the class $\mathfrak{I}_\nu(R)$ of all $\nu-$ideals of an ADL R forms a complete distributive lattice. In general, the class $\mathfrak{I}_\nu(R)$ of all $\nu-$ideals of an ADL R is not a sublattice of the ideal lattice $\mathfrak{I}(R)$. However, in the following, we derive a set of equivalent conditions for $\mathfrak{I}_\nu(R)$ to become a sublattice of $\mathfrak{I}(R)$. For this, we first need the following result.

Lemma 3.25. *Every proper $\nu-$ideal is contained in a minimal prime $E-$ideal.*

Proof. Let G be a proper $\nu-$ideal of R. Then $G = \nu(F)$ for some filter F of R with $F \cap E = \emptyset$. Hence $E \subseteq \nu(F) = G$. Clearly, $G \cap F = \nu(F) \cap F = \emptyset$. Consider, the set $\mathfrak{I} = \{H \mid H \text{ is a filter of } R \text{ such that } F \subseteq H \text{ and } G \cap H = \emptyset\}$. Clearly $F \in \mathfrak{I}$ and $\mathfrak{I}$ satisfies the Zorn's lemma. Let N be a maximal element of $\mathfrak{I}$. Then N is an ideal of R such that $F \subseteq N$ and $G \cap N = \emptyset$. Since $E \subseteq G$, we get $E \cap N = \emptyset$. That implies N is an ideal which is maximal with respect to the property that $E \cap N = \emptyset$. Hence $R \setminus N$ is a minimal prime $E-$ideal such that $G \subseteq R \setminus N$. □

Theorem 3.26. *In an ADL R, the following are equivalent:*
(1) $\mathfrak{I}_\nu(R)$ is a sublattice of $\mathfrak{I}(R)$;
(2) for $x, y \in R, x \wedge y \in E$ implies $(x, E) \vee (y, E) = R$;
(3) for $x, y \in R, (x, E) \vee (y, E) = (x \wedge y, E)$;
(4) for $G, H \in \mathfrak{F}(R), G \vee H = R$ implies $\nu(G) \vee \nu(H) = R$;
(5) for $G, H \in \mathfrak{F}(R), \nu(G) \vee \nu(H) = \nu(G \vee H)$.

Proof. $(1) \Rightarrow (2)$: Assume (1). Let $x, y \in R$ with $x \wedge y \in E$. Suppose $(x, E) \vee (y, E) \neq R$. Since (x, E) and (y, E) are $\nu-$ideals of R, by hypothesis, we get that $(x, E) \vee (y, E)$ is a proper $\nu-$ideal of R. By Lemma-3.25, there exists a minimal prime $E-$ideal M such that $(x, E) \vee (y, E) \subseteq M$. Hence $(x, E) \subseteq M$ and $(y, E) \subseteq M$. Since M is a minimal prime $E-$ideal, we get that $x \notin M$ and $y \notin M$. Since M is a prime ideal, we get that $x \wedge y \notin M$, which is a contradiction to that

$x \wedge y \in E \subseteq M$. Therefore $(x, E) \vee (y, E) = R$.
(2) $\Rightarrow$ (3) : Assume (2). Let $x, y \in R$. Clearly $(x, E) \vee (y, E) \subseteq (x \wedge y, E)$. Let $s \in (x \wedge y, E)$. Then $s \wedge (x \wedge y) \in E$. That implies $(s \wedge x) \wedge (s \wedge y) \in E$. By our assumption, we have that $(s \wedge x, E) \vee (s \wedge y, E) = R$. Then $s \in (s \wedge x, E) \vee (s \wedge y, E)$. There exist $a \in (s \wedge x, E)$ and $t \in (s \wedge y, E)$ such that $s = a \vee t$. Since $a \in (s \wedge x, E)$, we get $a \wedge s \in (x, E)$. Similarly, we have that $t \wedge s \in (y, E)$. Clearly, $(s \wedge a) \vee (s \wedge t) \in (x, E) \vee (y, E)$, which leads $s \wedge (a \vee t) \in (x, E) \vee (y, E)$. Since $s = a \vee s$, we get that $s \in (x, E) \vee (y, E)$. Therefore $(x \wedge y, E) \subseteq (x, E) \vee (y, E)$ and hence $(x, E) \vee (y, E) = (x \wedge y, E)$.
(3) $\Rightarrow$ (4) : Assume (3). Let G, H be two filters of R with $G \vee H = R$. Let $x \in E$. Then there exist $s \in G$ and $t \in H$ such that $x = s \wedge t$. By our assumption, we get $R = (x, E) = (s \wedge t, E) = (s, E) \vee (t, E) \subseteq \nu(G) \vee \nu(H)$. Hence $\nu(G) \vee \nu(H) = R$.
(4) $\Rightarrow$ (5) : Let G, H be two filters of R. Clearly we have that $\nu(G) \vee \nu(H) \subseteq \nu(G \vee H)$. Let $a \in \nu(G \vee H)$. Then there exists $s \in G \vee H$ such that $a \wedge s \in E$. Since $s \in G \vee H$, there exist $x \in G$ and $y \in H$ such that $s = x \wedge y$. Since $a \wedge s \in E$, we get $a \wedge (x \wedge y) \in E$. That implies $[(a \vee x) \vee (a \vee y)) = [E)$, which gives $[a \wedge x) \cap [a \wedge y) = R$. Therefore $\nu([a \wedge x)) \vee \nu([a \wedge y)) = R$ and hence $(a \wedge x, E) \vee (a \wedge y, E) = R$. Since $a \in R$, we have $a \in (a \wedge x, E) \vee (a \wedge y, E)$. Then there exist $s \in (a \wedge x, E)$ and $t \in (a \wedge y, E)$ such that $a = s \vee t$. Since $s \in (a \wedge x, E)$ and $t \in (a \wedge y, E)$, we get $a \wedge s \in (x, E)$ and $a \wedge t \in (y, E)$. Then $(a \wedge s) \vee (a \wedge t) \in (x, E) \vee (y, E)$, which leads $a \wedge (s \vee t) \in (x, E) \vee (y, E)$. Since $s \vee t = a$, we get $a \in (x, E) \vee (y, E)$. Since $(x, E) \vee (y, E) \subseteq \nu(G) \vee \nu(H)$, we get $a \in \nu(G) \vee \nu(H)$. Therefore we get $\nu(G \vee H) \subseteq \nu(G) \vee \nu(H)$. Hence $\nu(G \vee H) = \nu(G) \vee \nu(H)$.
(5) $\Rightarrow$ (1) : It is clear. □

Theorem 3.27. *Let $\Im_\nu(R)$ be a sublattice of $\Im(R)$. If $\{G_\alpha\}_{\alpha \in \triangle}$ be any class of $\nu-$ideals of R, then $\bigvee\limits_{\alpha \in \triangle} G_\alpha$ is again a $\nu-$ideal of R.*

Proof. For each $\alpha \in \triangle$, let $G_\alpha = \nu(F_\alpha)$ where F_α is a filter of R such that $F_\alpha \cap E = \emptyset$. Then $\{F_\alpha\}_{\alpha \in \triangle}$ will be any class family of filters of R with $F_\alpha \cap E = \emptyset$, for all $\alpha \in \triangle$. Clearly, $(\vee F_\alpha) \cap E = \emptyset$. Since $G_\alpha = \nu(F_\alpha) \subseteq \nu(\vee F_\alpha)$ for each $\alpha \in \triangle$, we get $\vee G_\alpha \subseteq \nu(\vee F_\alpha)$. Let $a \in \nu(\vee F_\alpha)$. Then there exists $s \in \vee F_\alpha$ such that $a \wedge s \in E$. Then there exists a positive integer n such that $s = s_1 \wedge s_2 \wedge \cdots \wedge s_n$ where $s_i \in F_{\alpha_i}$. By condition (4) of Theorem-3.26, we get $a \wedge s \in E \Rightarrow a \wedge (s_1 \wedge s_2 \wedge \cdots \wedge s_n) \in E \Rightarrow (a \wedge s_1) \wedge (a \wedge s_2) \wedge \cdots \wedge (a \wedge s_n) \in E \Rightarrow [a \wedge s_1) \cap [a \wedge s_2) \cap \cdots \cap [a \wedge s_n) = R \Rightarrow \nu([a \wedge s_1)) \vee \nu([a \wedge s_2)) \vee \cdots \vee \nu([a \wedge s_n)) = R \Rightarrow (a \wedge s_1, E) \vee (a \wedge s_2, E) \vee \cdots \vee (a \wedge s_n, E) = R$. Since $a \in R$ we get $a \in (a \wedge s_1, E) \vee (a \wedge s_2, E) \vee \cdots \vee (a \wedge s_n, E)$. Then there exists $t_i \in (a \wedge s_i, E)$ for $i = 1, 2, \cdots, n$ such that $a = t_1 \vee t_2 \vee \cdots \vee t_n$. Now, $a = a \wedge a = a \wedge (t_1 \vee t_2 \vee \cdots \vee t_n) = (a \wedge t_1) \vee (a \wedge t_2) \vee \cdots \vee (a \wedge t_n) \in (s_1, E) \vee (s_2, E) \vee \cdots \vee (s_n, E) \subseteq \nu(F_1) \vee \nu(F_2) \vee \cdots \vee \nu(F_n) = G_1 \vee G_2 \vee \cdots \vee G_n \subseteq \vee G_\alpha$. That implies $\nu(\vee F_\alpha) \subseteq \vee G_\alpha$. Thus $\vee G_\alpha$ is a $\nu-$ideal of R. □

Theorem 3.28. *Let $\Im_\nu(R)$ be a sublattice of $\Im(R)$. For any $E-$ideal G, there exists a unique $\nu-$ideal contained in G.*

Proof. Let G be any $E-$ideal of R. Consider $\mathfrak{M} = \{H \in \Im_\nu(R) \mid H \subseteq G\}$. Since E is the $\nu-$ideal and $E \subseteq G$, we get $E \in \mathfrak{M}$. Clearly, $\mathfrak{M}$ satisfies the hypothesis of Zorn's Lemma. Then $\mathfrak{M}$ has a maximal element let it be N. It is enough to show that N is unique. Let Q be any maximal element of $\mathfrak{M}$ such that $N \subseteq Q$. Clearly, $N \vee Q \subseteq G$. By Theorem-3.26, $N \vee Q \in \mathfrak{M}$. Therefore $N = N \vee Q = Q$. Thus $\mathfrak{M}$ has a unique maximal element, which is the required $\nu-$ideal contained in G. □

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