Palestine Journal of Mathematics

Vol 13(4)(2024) , 1065–1072

On β-CHANGE OF SQUARE ROOT FINSLER METRIC

M. Ramesha, M. S. Siddesha, K. Chandru and S. K. Narasimhamurthy

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53B40; Secondary 53C60.

Keywords and phrases: Finsler space, Square root Finsler metric, Fundamental Metric Tensor, Cartan Tensor, Projective flatness, Locally dually flatness.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: M. Ramesha

Abstract In this paper, we mainly characterize the Fundamental Metric Tensor and Cartan Tensor of square root Finlser metric which is obtained by β-change. Later on, we investigate the necessary and sufficient conditions for the square root Finlser metirc which is obtained by β-change to be projective flat and locally dually flat.

1 Introduction

Let (M, F) be an n-dimensional Finler space with the metric $F = \sqrt[m]{A}$, where A is given by $a_{i_1}, a_{i_2}, ...a_{i_m}(x)y^{i_1}y^{i_2}...y^{i_m}$ with $a_{i_1}, a_{i_2}, ...a_{i_m}$ symmetric in all its indices. Then F is called an m^{th}-root Finsler metric.The m^{th}-root metric $F = \sqrt[m]{A}$ is regarded as a direct generalization of the Riemannian metric if $m = 2$. If $F = \sqrt{A}$, where $A = a_{i_1 i_2}(x)y^{i_1}y^{i_2}$ with $a_{i_1 i_2}(x)$ symmetric in both the indices, then F is called the square root Finsler metric. The concept of m^{th}-root metric was introduced by H. Shimada [16] in 1979 and applied to ecology by Antonelli[1]. Later on, many more authors worked on m^{th}-root Finsler metrics in different views [7], [8],[13], [18], [19], [20].

In 1941, Randers [10], has introduced the Finsler change $\bar{L}(x, y) = L(x, y) + \beta(x, y)$, where L is a Riemannian metric and β is a differential 1-form on M. Such a change in Finsler space is termed as Randers change. Randers well-known method for giving examples of Finsler spaces has the form $L(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$, where a_{ij} is a Riemannian metric and $\beta = b_i(x)y^i$ is one form with the condition $||b|| = \sqrt{a^{ij}b_i b_j} < 1$. If we change $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ to a given Finsler metric, this will lead to another Finsler metric. In 1984, C. Shibata [15] also used the notation of β-change in Finsler geometry. There are two important β-changes namely Randers change defined as $\bar{F} \to F + \beta$ and Kropina change defined by $\bar{F} \to \frac{F^2}{\beta}$. If $F = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, then the obtained metrics are the Randers metric and the Kropina metric respectively.

In Finsler geometry, the regular case of Hilbert's fourth problem is to study the projective flat Finsler metrics. Finsler metrics on an open domain in R^n are said to be projectively flat if their geodesics are straight lines. Therefore, it is important to study and characterize the projective flat Finsler metrics. The theory of dually flatness in Riemannian geometry was given by Amari and Nagaoka [2] while studying information geometry. Information geometry provides mathematical science with a new framework for analysis. Information geometry is an investigation of different geometric structures in probability distribution. It is also applicable in statistical physics, statistical inferences, etc. Z. Shen [14] extended the notation of dually flatness in Finsler spaces. After Shen's work, many authors characterize the conditions for dually flatness [8],[11],[12], [21].

With the results discussed in above articles, we carried out our research work by applying β-change to the special Finsler square root metric, which is defined as $\bar{F} = F + \frac{\beta^2}{F}$, where

$F = \sqrt{A}$. In particular, if F is a Riemannian metric, then $\bar{F} = \alpha + \frac{\beta^2}{\alpha}$. In the first part, we find the Fundamental Metric Tensor and Cartan Tensor for the special Finsler square root metric, which is obtained by β-change. In the continuation part, we find the necessary and sufficient conditions for the square root Finsler metric obtained by β-change to be projective flat and locally dually flat.

2 Preliminaries

A Finsler structure of a manifold M is a function $F : TM \to [0, \infty)$ with the following properties:

(i) Regularity: F is c^∞ on the entire slit tangent bundle $TM|_{\{0\}}$,

(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$,

(iii) Strong convexity: The $n \times n$ Hessian matrix $g_{ij} = \left(\left[\frac{1}{2}F^2\right]_{y^i y^j} \right)$ is positive definite at every point of $TM|_{\{0\}}$,

where $TM|_{\{0\}}$ denotes the tangent vector 'y' is non-zero in the tangent bundle TM. The pair $(M, F) = F^n$ is called a Finsler space. F is called the fundamental function, and g_{ij} is the fundamental tensor of the Finsler space.

The normalized supporting element l_i, angular metric tensor h_{ij}, and metric tensor g_{ij} are defined as follows:

$$l_i = \frac{\partial F}{\partial y^i}, \ h_{ij} = F \frac{\partial^2 F}{\partial y^i \partial y^j}, \ g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{2.1}$$

A Finsler metric $F(x, y)$ is called an (α, β)-metric [3], [4], [9], [17], if the metric function $F(x, y)$ is a positively homogeneous function $F(\alpha, \beta)$ of the first degree in two variables $\alpha(x, y) = \sqrt{a_{ij}(x) y^i y^j}$ and 1-form $\beta = b_i(x) y^i$.

The n^3 quantities

$$C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F^2, \tag{2.2}$$

constitute a tensor of (0,3)-type which is positive homogeneous and symmetric in all its indices. This tensor is called hv-torsion tensor or the Cartan tensor.

For a non-zero 1-form $\beta(x, y) = b_i(x) y^i$ on a Finsler manifold (M, F), a Finsler change is defined by $F(x, y) \to \bar{F}(x, y) = f(F, \beta)$, where $f = f(F, \beta)$ is a positively homogeneous function of F and β. This change in Finsler metric is called a β-change.

It is the Hillbert Fourth problem in the regular case to study and characterize Finsler metrics on an open domain D in R such that geodesics are straight lines. Finsler metrics on D with this property are said to be projectively flat. According to G. Hamel [6], a Finsler metric $F = F(x, y)$ on D is projectively flat, if it satisfies the partial differential equation $F_{x^k y^i} y^k = F_{x^i}$.

Lemma 2.1. *A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is projectively flat if and only if it satisfies the following equation:*

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{2.3}$$

In this case, the local function $P = P(x, y)$ is given by $P = \frac{F_{x^m} y^m}{2F}$. It is one of the important problem in Finsler geometry to study the solution to the above equation.

A Finsler metric on a smooth manifold M^n is called locally dually flat if, at any point, there is a standard coordinate system (x^i, y^i) in TM, (x^i) is called an adapted local coordinate system such that

$$L_{x^i y^j} y^i - 2L_{x^j} = 0, \ where \ L = F^2. \tag{2.4}$$

Consider a square root Finsler metric $F = \sqrt{A}$, where $A = a_{i_1 i_2}(x) y^{i_1 i_2}$ with $a_{i_1 i_2}(x)$ symmetric in indices. Let us define the following notations:

$$\begin{aligned} A_i &= \frac{\partial A}{\partial y^i}, \ A_{x^i} = \frac{\partial A}{\partial x^i}, \ A_0 = A_{x^i} y^i, \ A_{0l} = A_{x^i y^l} y^i, \ A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \\ B_i &= \frac{\partial B}{\partial y^i}, \ B_{x^i} = \frac{\partial B}{\partial x^i}, \ B_0 = B_{x^i} y^i, \ B_{0l} = B_{x^i y^l} y^i. \end{aligned}$$

3 Fundamental Metric Tensor of Special Finsler Space

Let $(M, \bar{F})$ be an n-dimensional Finsler space with square root Finsler metric which is obtained by β-change

$$\bar{F} = F + \frac{\beta^2}{F}. \tag{3.1}$$

The fundamental metric tensor of an n-dimensional Finsler space is defined as

$$g_{ij} = \left[\frac{F^2}{2}\right]_{y^i y^j} = h_{ij} + l_i l_j, \tag{3.2}$$

where $l_i = F_{y^i}$ and $h_{ij} = F F_{y^i y^j}$.
Differentiating (3.1) concerning (y^i) which yields

$$\bar{F}_{y^i} = \left(1 - \frac{\beta^2}{F^2}\right) F_{y^i} + \frac{2\beta}{F} b_i. \tag{3.3}$$

Differentiating (3.3) concerning (y^j) which yields

$$\bar{F}_{y^i y^j} = \left(1 - \frac{\beta^2}{F^2}\right) F_{y^i y^j} + \frac{2\beta^2}{F^3} F_{y^i} F_{y^j} - \frac{2\beta}{F^2}(F_{y^i} b_j + F_{y^j} b_i) + \frac{2}{F} b_i b_j. \tag{3.4}$$

Now the fundamental metric tensor of Finsler space $(M, \bar{F})$ with square root Finsler metric which is obtained by β-change is

$$\begin{aligned} \bar{g}_{ij} &= \bar{F}\bar{F}_{y^i y^j} + \bar{F}_{y^i}\bar{F}_{y^j}. \\ &= \left(F + \frac{\beta^2}{F}\right)\left[\left(1 - \frac{\beta^2}{F^2}\right) F_{y^i y^j} + \frac{2\beta^2}{F^3} F_{y^i} F_{y^j} - \frac{2\beta}{F^2}(F_{y^i} b_j + F_{y^j} b_i) + \frac{2}{F} b_i b_j\right] \\ &+ \left[\left(1 - \frac{\beta^2}{F^2}\right) F_{y^i} + \frac{2\beta}{F} b_i\right]\left[\left(1 - \frac{\beta^2}{F^2}\right) F_{y^j} + \frac{2\beta}{F} b_j\right]. \end{aligned}$$

$$\bar{g}_{ij} = \left(1 - \frac{\beta^4}{F^4}\right) g_{ij} + \left(\frac{4\beta^4}{F^4}\right) F_{y^i} F_{y^j} - \left(\frac{4\beta^3}{F^3}\right)(F_{y^i} b_j + F_{y^j} b_i) + \left(2 + \frac{6\beta^2}{F^2}\right) b_i b_j. \tag{3.5}$$

Proposition 3.1. *Let $(M, \bar{F})$ be an n-dimensional Finlser space with square root Finsler metric which is obtained by β-change $\bar{F} = F + \frac{\beta^2}{F}$. Then its covariant metric tensor is given as (3.5)*

4 Cartan Tensor of Special Finsler Space

Let $(M, \bar{F})$ be n-dimensional Finsler space. The Cartan torsion of n-dimensional Finsler space is defined as

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}[F^2]_{y^i y^j y^k}, \tag{4.1}$$

which is symmetric in all three indices i, j, and k.

Consider an n-dimensional Finsler space $(M, \bar{F})$ with square root Finsler metric which is obtained by β-change

$$\bar{F} = F + \frac{\beta^2}{F}. \tag{4.2}$$

According to the definition, the Cartan torsion of Finsler Space $(M, \bar{F})$ is

$$2\bar{C}_{ijk} = \frac{\partial \bar{g}_{ij}}{\partial y^k}. \tag{4.3}$$

By using (3.5) in (4.3), we have

$$\begin{aligned}
2\bar{C}_{ijk} &= \left(1-\frac{\beta^4}{F^4}\right)2c_{ijk}+\left[\frac{4\beta^4}{F^6}y_k-\frac{4\beta^3}{F^4}\right]\left(h_{ij}+\frac{y_iy_j}{F^2}\right)-\frac{4\beta^3}{F^4}(h_{ik}b_j+h_{jk}b_i)\\
&+ \frac{4\beta^4}{F^6}(h_{ik}y_j+h_{jk}y_i)+\frac{16\beta^3}{F^4}\left(\frac{b_kF^2-\beta y_k}{F^2}\right)\frac{y_iy_j}{F^2}-\frac{12\beta^2}{F^3}\left(\frac{F^2b_k-\beta y_k}{F^2}\right)\\
&\times \left(\frac{y_ib_j+y_jb_i}{F}\right)+\frac{12\beta}{F^4}(F^2b_k-\beta y_k)b_ib_j.\\
2\bar{C}_{ijk} &= 2\left(1-\frac{\beta^4}{F^4}\right)c_{ijk}\\
&+ \frac{4\beta^3}{F^4}\left[\left(\frac{\beta}{F^2}y_k-b_k\right)h_{ij}+\left(\frac{\beta}{F^2}y_i-b_i\right)h_{jk}+\left(\frac{\beta}{F^2}y_j-b_j\right)h_{ki}\right]\\
&+ \frac{12\beta}{F^2}\left[b_ib_jb_k-\frac{\beta}{F^2}\left(b_ib_jy_k+b_jb_ky_i+b_kb_iy_j\right)-\frac{\beta^3}{F^6}y_iy_jy_k\right]\\
&+ \frac{12\beta}{F^2}\left[\frac{\beta^2}{F^4}\left(y_iy_jb_k+y_jy_kb_i+y_ky_ib_j\right)\right].\\
2\bar{C}_{ijk} &= 2\frac{(F^4-\beta^4)}{F^4}C_{ijk}+\frac{4\beta^3}{F^4}\sum_{cyclicsum}\left(\frac{\beta}{F^2}y_k-b_k\right)h_{ij}+\prod_{cyclicproduct}\left(b_i-\frac{\beta}{F^2}y_i\right).
\end{aligned}$$

$$\bar{C}_{ijk}=\frac{(F^4-\beta^4)}{F^4}C_{ijk}+\frac{2\beta^3}{F^4}(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j)+\frac{6\beta}{F^2}m_im_jm_k. \tag{4.4}$$

Proposition 4.1. *Let $(M,\bar{F})$ be an n-dimensional Finsler space with square root Finsler metric which is obtained by β-change $\bar{F}=F+\frac{\beta^2}{F}$. Then its Cartan torsion is given as (4.4).*

5 Projective Flatness on β-Change of Special Finsler Space

Let $(M,\bar{F})$ be an n-dimensional Finsler space with square root Finsler metric which is obtained by β-change

$$\bar{F}=F+\frac{\beta^2}{F}, \tag{5.1}$$

where $F^2=A$.

In this section, we find the necessary and sufficient conditions for an n-dimensional Finsler space $(M,\bar{F})$ to be projective flat.

Consider a square root Finsler metric defined in (5.1) can be rewritten as

$$\bar{F}=\sqrt{A}+\frac{\beta^2}{\sqrt{A}}. \tag{5.2}$$

Differentiating (5.2) concerning (x^k), we have

$$\bar{F}_{x^k}=\left(\frac{A-\beta^2}{2A\sqrt{A}}\right)A_{x^k}+\frac{2\beta}{\sqrt{A}}\beta_{x^k}. \tag{5.3}$$

Differentiating (5.3) concerning (y^l), which yields

$$\begin{aligned}
\bar{F}_{x^ky^l} &= \left(\frac{A-\beta^2}{2A\sqrt{A}}\right)A_{x^ky^l}+\frac{3\beta^2-A}{4A^2\sqrt{A}}A_lA_{x^k}-\frac{\beta}{A\sqrt{A}}\left(A_{x^k}\beta_l+\beta_{x^k}A_l\right)\\
&+ \frac{2}{\sqrt{A}}\left(\beta_l\beta_{x^k}+\beta\beta_{x^ky^l}\right).
\end{aligned} \tag{5.4}$$

Transvecting (5.4) with (y^k), we have

$$\begin{aligned}\bar{F}_{x^k y^l} y^k &= \left(\frac{A-\beta^2}{2A\sqrt{A}}\right) A_{0l} + \frac{3\beta^2 - A}{4A^2\sqrt{A}} A_l A_0 - \frac{\beta}{A\sqrt{A}} (A_0\beta_l + \beta_0 A_l) \\ &+ \frac{2}{\sqrt{A}} (\beta_l\beta_0 + \beta\beta_{0l}). \end{aligned} \tag{5.5}$$

Replacing (x^k) by (x^l) in (5.2), we have

$$\bar{F}_{x^l} = \left(\frac{A-\beta^2}{2A\sqrt{A}}\right) A_{x^l} + \frac{2\beta}{\sqrt{A}}\beta_{x^l}. \tag{5.6}$$

We know that a Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is projective flat if and only if it satisfies the following PDE [15]

$$F_{x^k y^l} y^k - F x^l = 0. \tag{5.7}$$

From (5.5) and (5.6), equation (5.7) becomes

$$\begin{aligned} &\frac{1}{4} A^{-\frac{5}{2}} \left[2A^2 A_{0l} - A A_l A_0 - 2A A_{0l}\beta^2 + 3A_l A_0 \beta^2 - 4A A_0 \beta\beta_l - 4A_l \beta\beta_0\right] \\ + &\frac{1}{4} A^{-\frac{5}{2}} \left[8A^2(\beta_l\beta_0 + \beta\beta_{0l}) - 2A^2 A_{x^l} + 2A A_{x^l}\beta^2 - 8A^2\beta\beta_{x^l}\right] = 0. \end{aligned}$$

In simplified form, the above equation can be rewritten as

$$\begin{aligned} 3\beta^2 A_l A_0 &+ 2A\left[\beta^2(A_{x^l} - A_{0l}) - 2\beta(A_l\beta_0 + A_0\beta_l) - \frac{A_l A_0}{2}\right] \\ &+ 2A^2\left[(A_{0l} - A_{x^l}) + 4\beta_l\beta_0 + 4\beta(\beta_{0l} - \beta_{x^l})\right] = 0. \end{aligned} \tag{5.8}$$

Based on the above discussion and to prove the projective flatness, we use the following lemma given by [12].

Lemma 5.1. *Suppose that the equation $\phi A^2 + \varphi A + \theta = 0$ holds, where ϕ, φ, θ are polynomials in y. Then $\phi = \varphi = \theta = 0$.*

Above lemma(5.1) and equation (5.8), we conclude that $\bar{F}$ is projective flat if and only if the following conditions hold:

$$A_l A_0 = 0. \tag{5.9}$$

$$2\beta^2(A_{x^l} - A_{0l}) - 4\beta(A_l\beta_0 + A_0\beta_l) - A_l A_0 = 0. \tag{5.10}$$

$$(A_{0l} - A_{x^l}) + 4\beta_l\beta_0 + 4\beta(\beta_{0l} - \beta_{x^l}) = 0. \tag{5.11}$$

Theorem 5.2. *Let $\bar{F} = F + \frac{\beta^2}{F}$ be an n-dimensional special Finsler space which is obtained by β-change of square root Finsler metric. Then $\bar{F}$ is projective flat if and only if (5.9), (5.10) and (5.11) hold.*

6 Locally Dually Flatness on β-Change of Special Finsler Space

Let us consider an n-dimensional Finsler space $(M, \bar{F})$ with square root Finsler metric, which is obtained by β-change

$$\bar{F} = F + \frac{\beta^2}{F}, \tag{6.1}$$

where $F^2 = A$.

In this section, we find the necessary and sufficient conditions for an n-dimensional Finsler space $(M, \bar{F})$ with square root metric Finsler metric which is obtained by β-change defined in (6.1) to be locally dually flat.

Consider (6.1) as

$$\bar{F}^2 = A + \frac{\beta^4}{A} + 2\beta^2. \tag{6.2}$$

Differentiate (6.2) concerning (x^i) which yields

$$\bar{F}^2_{x^i} = \left(\frac{A^2 - \beta^4}{A^2}\right) A_{x^i} + \frac{4}{A}(\beta^3 + A\beta)\beta_{x^i}. \tag{6.3}$$

Differentiate (6.3) concerning (y^j) we have

$$\begin{aligned} \bar{F}^2_{x^i y^j} &= \left(\frac{A^2 - \beta^4}{A^2}\right) A_{x^i y^j} + \frac{4}{A}(\beta^3 + A\beta)\beta_{x^i y^j} - \frac{4\beta^3}{A^2}(\beta_{x^i} A_j + \beta_j A_{x^i}) \\ &+ \frac{4}{A}(3\beta^2 + A)\beta_{x^i}\beta_j + \frac{2\beta^4}{A^3} A_{x^i} A_j. \end{aligned} \tag{6.4}$$

Transvecting (6.4)by (y^i) and by treating $\bar{F}^2 = L$, we have

$$\begin{aligned} L_{x^i y^j} y^i &= \left(\frac{A^2 - \beta^4}{A^2}\right) A_{0j} + \frac{4}{A}(\beta^3 + A\beta)\beta_{0j} - \frac{4\beta^3}{A^2}(\beta_0 A_j + \beta_j A_0) \\ &+ \frac{4}{A}(3\beta^2 + A)\beta_0\beta_j + \frac{2\beta^4}{A^3} A_0 A_j. \\ L_{x^i y^j} y^i &= \frac{1}{A^3}\left(2\beta^4 A_0 A_j\right) - \frac{1}{A^2}\left[4\beta^3(\beta_0 A_j + A_0\beta_j) + \beta^4 A_{0j}\right] \\ &+ \frac{1}{A}\left[12\beta^2\beta_0\beta_j + 4\beta^3\beta 0j\right] + A_{0j} + 4(\beta\beta_{0j} + \beta_0\beta_j). \end{aligned} \tag{6.5}$$

Replace i by j in (6.3) which yields

$$L_{x^j} = A_{x^j} + \frac{4\beta^3}{A}\beta_{x^j} - \frac{\beta^4}{A^2} A_{x^j} + 4\beta\beta_{x^j}.$$

Then

$$2L_{x^j} = -\frac{1}{A^2}\left(2\beta^4 A_{x^j}\right) + \frac{1}{A}(8\beta\beta_{x^j}) + 2A_{x^j} + 8\beta\beta_{x^j}. \tag{6.6}$$

Using (6.5) and (6.6) in (2.4) we have,

$$\begin{aligned} &\frac{1}{A^3}[2\beta^4 A_0 A_j] - \frac{1}{A^2}[4\beta^3(\beta_0 A_j + \beta_j A_0) + \beta^4 A_{0j} + 2\beta^4 A_{x^j}] \\ &+\frac{1}{A}[12\beta^2 + \beta_0\beta_j + 4\beta^3\beta_{0j} - 8\beta^3\beta_{x^j}] + A_{0j} - 2A_{x^j} + 4\beta\beta_{0j} - 8\beta\beta_{x^j} = 0. \end{aligned} \tag{6.7}$$

With the above discussion, we conclude that $\bar{F}$ is locally dually flat if and only if the following conditions must hold.

$$2\beta^4 A_0 A_j = 0. \tag{6.8}$$

$$4\beta^3(\beta_0 A_j + A_0\beta_j) + \beta^4(A_{0j} - 2A_{x^j}) = 0. \tag{6.9}$$

$$12\beta^2\beta_0\beta_j + 4\beta^3\beta_{0j} - 8\beta^3\beta_{x^j} = 0. \tag{6.10}$$

$$A_{0j} - 2A_{x^j} = 0. \tag{6.11}$$

$$\beta(\beta_{0j} - 2\beta_{x^j}) + \beta_0\beta_j = 0.. \tag{6.12}$$

From equation (6.8), we have

$$A_0 A_j = 0. \tag{6.13}$$

Using (6.11) in (6.9), we have

$$A_0\beta_j + \beta_0 A_j = 0. \tag{6.14}$$

By using (6.10) in (6.12), we have

$$\beta_0\beta_j = 0. \tag{6.15}$$

with all the above observations leads to the main result,

Theorem 6.1. *Let $\bar{F} = F + \frac{\beta^2}{F}$ be an n-dimensional Finsler space obtained by β-change of square root Finsler metric. Then $\bar{F}$ is locally dually flat if and only if*

$$A_0 A_j = 0; A_{0j} = 2A_j; \beta_0\beta_j = 0; A_0\beta_j + \beta_0 A_j = 0.$$

7 Conclusion remarks

This paper aims to characterize the properties of an n-dimensional Finsler space $(M, \bar{F})$ which is obtained by β-change of square root Finsler metric. In the first part, we investigate the Fundamental Metric Tensor and Cartan Tensor of an n-dimensional Finsler space $(M, \bar{F})$, which is obtained by β-change of square root Finsler metric, which are as follows:
The Fundamental metric tensor and Cartan Tensor of $\bar{F}$ is as followed by

$$\bar{g}_{ij} = \left(1 - \frac{\beta^4}{F^4}\right) g_{ij} + \left(\frac{4\beta^4}{F^4}\right) F_{y^i} F y^j - \left(\frac{4\beta^3}{F^3}\right)(F_{y^i} b_j + F_{y^j} b_i) + \left(2 + \frac{6\beta^2}{F^2}\right) b_i b_j.$$

$$\bar{c}_{ijk} = \frac{(F^4 - \beta^4)}{F^4} C_{ijk} + \frac{2\beta^3}{F^4}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{6\beta}{F^2} m_i m_j m_k.$$

In the next part, we investigate the necessary and sufficient conditions for an n-dimensional Finsler space $(M, \bar{F})$ which is obtained by β-change of square root Finsler metric, to be projective flat and locally dually flat. In this regard, the results will be summarized as follows:
Let $(M, \bar{F})$ be an n-dimensional Finsler space with square root Finsler metric which is obtained by β-change defined as $\bar{F} = F + \frac{\beta^2}{F}$. The Finsler metric $\bar{F}$ is projectively flat if and only if the following conditions are satisfied:

$$A_l A_0 = 0.$$
$$2\beta^2(A_{x^l} - A_{0l}) - 4\beta(A_l \beta_0 + A_0 \beta_l) - A_l A_0 = 0.$$
$$(A_{0l} - A_{x^l}) + 4\beta_l \beta_0 + 4\beta(\beta_{0l} - \beta_{x^l}) = 0.$$

Finally, we investigate the necessary and sufficient conditions for $\bar{F}$ to be locally dually flat if and only if

$$A_0 A_j = 0; A_{0j} = 2A_j; \beta_0 \beta_j = 0; A_0 \beta_j + \beta_0 A_j = 0.$$

References

[1] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Application in Physics and Biology*, Fundam. Theor. Phys., **58**, kluwer acad. publ., Netherlands.(1993).

[2] S. I. Amari and H. Nagaoka, *Methods of Information Geometry,Translation of Mathematical Monographs*, AMS, 191, Oxford Univ. Press.(2000).

[3] AjayKumar AR and Pradeep Kumar, *On projective relation of generalized metric with certain (α, β)-metric*, Palestine Journal of Mathematics, **12(3)**, 199-–210.(2023).

[4] Brijesh Kumar Tripathi, *On Finslerian hypersurface of Finsler space with special generalized (α, β)-metric*, Palestine Journal of Mathematics, **12(3)**, 332-–340.(2023).

[5] S. S. Chern and Z. Shen, *Riemannian Finsler Geometry*, Nankai Tracts in Mathematics, World Scientific, Singapore.(2005).

[6] G. Hamel, *Uber Die Geometrieen in Denen Die Geraden Die Kurzesten Sind*, Math. Ann, **57**, 231–264,(1903).

[7] Manoj Kumar, *On randers change of generalized m^{th}-root metric*, Khayyam J. Math, **5 (1)**, 69–78, (2019).

[8] Manoj Kumar, *Locally dually flatness and locally projectively flatness of Matsumoto change with m^{th}-root Finsler metrics*, Journal of Finsler geometry and its applications, **2(1)**, 31–39, (2021).

[9] Makoto Matsumoto, *Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys, **31 (1)**, 43–83, (1992).

[10] G. Randers, *On the asymmetrical metric in the four-space of general relativity*, Phys. Rev, **59(2)** 195–199, (1941).

[11] Sarita Rani, Gauree Shankar and K. Kaur, *Dually flat Finsler spaces with transformed metrics*, Journal of Finsler geometry and its applications, **2(2)**, 122–133.

[12] G. Shankar, S. Rani, and K. Kaur, *Some classes of projectively and dually flat Finsler spaces with Randers change*, https://doi.org/10.48550/arXiv.1712.07865, (2017).

[13] G. Shankar, R. K. Sharma and R. D. S. Kushwaha, *On matsmoto change of Finsler space with m^{th}-root metric*, Acta Math. Acad. Paedagog. Nyiregyhaziensis, **33**, 255–264, (2017).

[14] Z. Shen, *Riemannian-Finsler geometry with applications to information geometry*, Chinese Ann. Math. Ser. B, **27(1)**, 73–94, (2006).

[15] C. Shibata, *On invariant tensors of β-change of Finsler metrics*, Kyoto J. Math, 24, 163–188, (1984).

[16] H. Shimada, *On Finsler space with metric* $L = \sqrt{m}a_{1_1}a_{1_2}...a_{1_m}y^{y_1}y^{y_2}...y^{y_m}$,Tensors N. S, **33**, 365–372, (1979).

[17] Surekha Desai, S. K. Narasimhamurthy, and R. S. Raghavendra R S, *Flag curvature and homogeneous geodesics of homogeneous Finsler space with* (α, β)*-metric*, Palestine Journal of Mathematics, **12(2)**, 254-–262.(2023).

[18] A. Tayebi, T. Tabatabaeifar and E. Peyghan, *On Kropina change for* m^{th}*-root Finsler metrics*, Ukr. Math. J, **66 (1)**, 160–164, (2014).

[19] A. Tayebi, M. Shahbazi Nia and E. Peyghan, *On randers change for* m^{th}*-root Finsler metrics*, Int. Electron. J. Geom, **8 (1)**, 14–20, (2015).

[20] B. Tiwari, Manoj Kumar and A. Tayebi, *On Genralized Kropina change of Generalized* m^{th}*-root Finsler metrics*, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. (2020).

[21] B. Tiwari, and Ghanashyam KR Prajapati, *On Locally DUally flat Special Finsler* (α, β)*-metrics*, Acta Math. Acad. Paedagog. Nyiregyhaziensis, *34*, 21–33 (2023).

Author information

M. Ramesha, Department of Mathematics, Jain Global Campus, Jain (Deemed-to-be University), Jakkasandra, Kanakpura, Karnataka, India.
E-mail: `ramfins@gmail.com`

M. S. Siddesha, Department of Mathematics, Jain Global Campus, Jain (Deemed-to-be University), Jakkasandra, Kanakpura, Karnataka, India.
E-mail: `mssiddesha@gmail.com`

K. Chandru, Department of Mathematics, DVS college of Arts and Science, Shivamogga, Karnataka, India.
E-mail: `kchandru2510@gmail.com`

S. K. Narasimhamurthy, Department of P.G. Studies and Research in Mathematics, Kuvempu University, Shankaraghatta, Shivamogga, Karnataka, India.
E-mail: `nmurthysk@gmail.com`

Received: 2023-10-21
Accepted: 2024-02-15