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C-Recurrent operators

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Abstract In this paper, we extend the concepts of cyclic transitivity, recurrence, and super-recurrence by introducing and examining a novel notion called C-recurrence. We then present a C-recurrence Criterion, inspired by the Hypercyclicity Criterion and the Cyclicity Criterion. Finally, we characterize C-recurrence for weighted shifts.

1 Introduction and preliminaries

Let X be a Banach space over the field $\mathbb{C}$ of complex numbers. The set of linear continuous operators on X is denoted by $\mathcal{B}(X)$.

An operator $T \in \mathcal{B}(X)$ is considered hypercyclic if there exists a vector $x \in X$ such that its orbit under the action of T,

$$\text{Orb}(x, T) := \{T^n(x) : n \in \mathbb{N}\},$$

is dense in the entire space X. The vector x itself is referred to as a hypercyclic vector for the operator T. The collection of all hypercyclic vectors for T is denoted by $HC(T)$. In the context of separable Banach spaces, an equivalent notion of hypercyclicity, called topological transitivity, was introduced by Birkhoff in [12]. According to this notion, an operator T acting on a separable Banach space X is hypercyclic if and only if it is topologically transitive; that is, for every pair of non-empty open sets (U, V) in X, there exists an $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset.$$

In 1974, Hilden and Wallen introduced the concept of supercyclicity in [22]. An operator $T \in \mathcal{B}(X)$ is termed *supercyclic* if there exists a vector $x \in X$ such that the set

$$\mathbb{C} \cdot \text{Orb}(x, T) := \{\lambda T^n(x) \mid n \in \mathbb{N}, \lambda \in \mathbb{C}\}$$

is dense in X. This vector x is referred to as a supercyclic vector for T. The collection of all supercyclic vectors for T is denoted by $SC(T)$. Furthermore, an operator T acting on a separable Banach space X is supercyclic if and only if, for each pair (U, V) of nonempty open subsets of X, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that

$$\lambda T^n(U) \cap V \neq \emptyset.$$

An operator $T \in \mathcal{B}(X)$ is referred to as cyclic if there exists a vector $x \in X$ such that the set

$$\mathbb{C}[T]x := \text{span}\{\text{Orb}(x, T)\} = \{p(T)(x) \mid p \text{ polynomial}\}$$

is dense in X. The set of all cyclic vectors for T is denoted by $C(T)$. The phenomenon of cyclic transitivity was introduced in [11]. An operator $T \in \mathcal{B}(X)$ is said to be cyclic transitive if, for any pair (U, V) of non-empty open subsets of X, there exists a complex polynomial p such that

$$p(T)(U) \cap V \neq \emptyset.$$

It is evident that cyclic transitivity implies cyclicity. However, the converse does not hold in general, as shown in [17].

For further information about hypercyclic, supercyclic, and cyclic operators and their properties, refer to the book by K.G. Grosse-Erdmann and A. Peris [20], and the book by F. Bayart and E. Matheron [8], as well as the survey article by K.G. Grosse-Erdmann [21]. See also [1, 2, 3, 4, 5, 9].

Another crucial concept in linear dynamics is that of recurrence, originally introduced by Poincaré in [23]. Later, it was studied by Gottschalk and Hedlund [18], as well as by Furstenberg [16]. Recently, recurrent operators have been the subject of study in [13].

An operator $T \in \mathcal{B}(X)$ is considered recurrent if, for each open subset U of X, there exists a positive integer n such that

$$T^n(U) \cap U \neq \emptyset.$$

A vector $x \in X$ is called a recurrent vector for T if there exists an increasing sequence (n_k) of positive integers such that $T^{n_k}x$ converges to x as k approaches infinity. The set of all recurrent vectors for T is denoted by $Rec(T)$. Additionally, T is recurrent if and only if $Rec(T)$ is dense in X.

Recently, a new class of operators, known as the class of super-recurrent operators, has been introduced. An operator $T \in \mathcal{B}(X)$ is considered super-recurrent if, for each open subset U of X, there exists $\lambda \in \mathbb{C}$ and a positive integer n such that

$$\lambda T^n(U) \cap U \neq \emptyset.$$

A vector $x \in X$ is called a recurrent vector for T if there exists an increasing sequence (n_k) of positive integers and a sequence (λ_{n_k}) of complex numbers such that $\lambda_{n_k} T^{n_k}x$ converges to x as k approaches infinity. The set of all recurrent vectors for T is denoted by $SRec(T)$. Furthermore, T is super-recurrent if and only if $SRec(T)$ is dense in X. For further information about this class of operators, see [6, 10].

In this paper, we introduce a novel class of operators called C-recurrent operators, which bridge the relationship between cyclic transitivity and recurrence. Specifically, this class of operators combines the properties of being cyclic and recurrent, encompassing the phenomena of cyclic transitivity and recurrence simultaneously.

In Section 2, we present the concept of C-recurrent operators. We demonstrate that every super-recurrent operator and every cyclic transitivity operator belong to the class of C-recurrent operators. Additionally, we establish the existence of an operator that is classified as C-recurrent but is not categorized as super-recurrent or cyclic transitive. This finding illustrates that the class of C-recurrent operators provides a broader scope that encompasses both super-recurrent and cyclic transitivity operators while also introducing new examples that do not fall under either of these categories.

In Section 3, we establish several properties for C-recurrent operators. In particular, we prove that an operator T is C-recurrent if and only if it admits a dense set of C-recurrent vectors.

In Section 4, we introduce a C-recurrent Criterion, providing guidelines to determine whether an operator belongs to this class. Furthermore, we establish some sufficient conditions for $T \oplus T$ to be C-recurrent when T is C-recurrent. Additionally, we conduct an in-depth investigation of the C-recurrence properties of weighted shift operators on classical sequence spaces.

2 C-recurrent operators

In this section, we commence by introducing our primary definition. Let us recall that given $p \in \mathbb{C}[X]$ with $p(z) = \sum_{i=0}^{n} \lambda_i z^i$ and $T \in \mathcal{B}(X)$, we have the notation

$$p(T) = \sum_{i=0}^{n} \lambda_i T^i.$$

Definition 2.1. An operator T is considered to be C-recurrent if, for every nonempty open subset U of the vector space X, there exists a polynomial $p \in \mathbb{C}[X]$ such that the following condition is met:

$$p(T)(U) \cap U \neq \emptyset.$$

Furthermore, a vector $x \in X \setminus \{0\}$ is termed a C-recurrent vector for the operator T if there exists a sequence (p_k) of complex polynomials such that the following convergence occurs:

$$p_k(T)x \longrightarrow x.$$

The set containing all C-recurrent vectors for the operator T is denoted by $CRec(T)$.

Our initial contribution is to establish the relationship between our class of operators and the other class mentioned previously. We have the following diagram showing the relationships among supercyclic, super-recurrent, cyclic transitive, and C-recurrent operators.

$$\begin{array}{ccc}
\text{Supercyclicity} & \Longrightarrow & \text{Cyclic Transitivity} \\
\Downarrow & & \Downarrow \\
\text{Super-recurrent} & \Longrightarrow & \text{C-recurrent}
\end{array}$$

Note that the converse of all these implications does not hold in general, as demonstrated by the following examples.

Counterexamples:

• Cyclic Transitivity $\not\Rightarrow$ Supercyclicity: There exist operators that are cyclic transitive but not supercyclic; see [15, Proposition 7.2].

• Super-recurrence $\not\Rightarrow$ Supercyclicity: There exist operators that are super-recurrent but not supercyclic; see [6, Remarks 2.2].

• C-recurrence $\not\Rightarrow$ Super-recurrence: There exist operators that are C-recurrent but not super-recurrent, as shown by the following example:

Example 2.2. Let $\mathcal{H}(\mathbb{C})$ be the space of entire functions. We define an operator T on $\mathcal{H}(\mathbb{C})$ by

$$\begin{aligned} T : \mathcal{H}(\mathbb{C}) &\longrightarrow \mathcal{H}(\mathbb{C}) \\ f &\longmapsto T(f) = f'. \end{aligned}$$

The restriction of the operator T to the space of constant functions in $\mathcal{H}(\mathbb{C})$ is C-recurrent. To see this, let f be a non-zero constant function. Consider the polynomials $p_k(z) = z^k + \alpha_k$, where $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence of complex numbers such that $\alpha_k \to 1$. Then,

$$p_k(T)(f) = T^k(f) + \alpha_k f = \alpha_k f \longrightarrow f,$$

as $k \to \infty$. However, T cannot be super-recurrent in the space of constant functions in $\mathcal{H}(\mathbb{C})$. This is because, for every non-zero constant function f in $\mathcal{H}(\mathbb{C})$, we have

$$\lambda_k T^{n_k}(f) = 0 \nrightarrow f,$$

for every increasing sequence (n_k) of positive integers and a sequence (λ_k) of complex numbers.

• C-recurrence $\not\Rightarrow$ Cyclic Transitivity: There exist operators that are C-recurrent but not cyclic transitive, as shown by the following example:

Example 2.3. A cyclic transitive operator is C-recurrent, but the converse does not hold in general. Indeed, let

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

be a matrix on $\mathbb{C}^2$. T is a C-recurrent operator. To see this, let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and let $p_k(X) = 2^{-k}\beta_k X^k$, where $(\beta_k)_{k\in\mathbb{N}}$ is a sequence of complex numbers such that $\beta_k \to 1$. Then

$$p_k(T)u = \begin{bmatrix} p_k(2)u_1 \\ p_k(2)u_2 \end{bmatrix} \to u.$$

However, T cannot be cyclic; see [15]. Therefore, T is not cyclic transitive.

3 Some properties of C-recurrent operators.

In the following, we present certain properties that are satisfied by C-recurrent operators.

Proposition 3.1. *Suppose that $q \in \mathbb{C}[X]$ and $S \in \mathcal{B}(X)$ satisfy the condition $ST = TS$. Then it follows that $CRec(T)$ remains invariant under $q(S)$.*

Proof. Let $x \in CRec(T)$. Then, there exists a sequence (p_k) of polynomials such that

$$p_k(T)x \longrightarrow x$$

as $k \longrightarrow \infty$. Since $ST = TS$, it follows that

$$p_k(T)q(S) = q(S)p_k(T)$$

for all $q \in \mathbb{C}[X]$. Given that S is continuous, we can conclude that

$$p_k(T)q(S)x = q(S)p_k(T)x \longrightarrow q(S)x$$

as $k \longrightarrow \infty$. This implies that

$$q(S)x \in CRec(T).$$

□

Corollary 3.2. *If x is a C-recurrent vector for T, then the set of nonzero vectors generated by applying polynomials to T acting on x, i.e.,*

$$\{p(T)x \mid p \textit{ is a polynomial}\} \setminus \{0\},$$

is a subset of $CRec(T)$.

In particular, if T has a C-recurrent vector, then it possesses an invariant subspace consisting of C-recurrent vectors, except for the zero vector.

Proof. For a nonzero polynomial p, let $S = p(T)$. Since $x \in CRec(T)$, it follows from Proposition 3.1 that $p(T)x \in CRec(T)$. Therefore, for any nonzero polynomial p, the vector $p(T)x$ is also C-recurrent with respect to T. □

In the following proposition, we establish that if $q(T)$ is C-recurrent for a complex polynomial q, then T is also C-recurrent.

Proposition 3.3. *Suppose that $T \in \mathcal{B}(X)$ and $q \in \mathbb{C}[X]$. Then:*

(i) If $q(T)$ is a C-recurrent operator, then T is also a C-recurrent operator.

(ii) $CRec(q(T)) \subseteq CRec(T)$.

Proof. Proof of (i) is evident by definition.
To establish (ii), let $x \in CRec(q(T))$. Thus, there exists a sequence $(p_k)_{k\in\mathbb{N}}$ of complex polynomials such that

$$p_k(q(T))(x) \longrightarrow x.$$

As a result, $p_k(q)(T)(x) \longrightarrow x$ since $p_k(q)$ is a sequence of complex polynomials. Consequently, $x \in CRec(T)$.

□

Consider two Banach spaces, X and Y, and let T and S be operators acting on X and Y, respectively. The operators T and S are referred to as quasi-conjugate or quasi-similar if there exists an operator $\phi : X \to Y$ with a dense range such that

$$S \circ \phi = \phi \circ T.$$

If it is possible to choose ϕ as a homeomorphism (a bijective continuous linear operator with a continuous inverse), then T and S are called conjugate or similar.

Proposition 3.4. *If $T \in \mathcal{B}(X)$ is quasi-similar to $S \in \mathcal{B}(Y)$, then the property of T being C-recurrent in X implies that S is C-recurrent in Y.*

Proof. Suppose that T is C-recurrent. Let U be a nonempty open subset of Y. Since ϕ is a continuous linear operator with a dense range, $\phi^{-1}(U)$ is a nonempty open subset of X. As T is C-recurrent, there exist $p \in \mathbb{C}[X]$ and $x \in X$ such that

$$x \in \phi^{-1}(U) \quad \text{and} \quad p(T)x \in \phi^{-1}(U).$$

This implies that $\phi(x) \in U$ and $\phi \circ p(T)x \in U$. Since T and S are quasi-similar, it follows that $\phi(x) \in U$ and $p(S) \circ \phi(x) \in U$. Therefore, S is C-recurrent in Y.

□

Corollary 3.5. *Assume that $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$ are similar. Then, T is C-recurrent in X if and only if S is C-recurrent in Y.*

The following theorem provides both necessary and sufficient conditions for the C-recurrence of operators.

Theorem 3.6. *The following assertions are equivalent:*

(i) T is C-recurrent;

(ii) For each $x \in X$, there exists a sequence (x_k) of elements of X and a sequence (p_k) of complex polynomials such that

$$x_k \longrightarrow x \quad \textit{and} \quad p_k(T)(x_k) \longrightarrow x,$$

as $k \longrightarrow \infty$;

(iii) For each $x \in X$ and each neighborhood W of zero, there exist $z \in X$ and $p \in \mathbb{C}[X]$ such that

$$p(T)(z) - x \in W \quad \textit{and} \quad z - x \in W.$$

Proof. $(i) \Rightarrow (ii)$: Let $x \in X$. For all $k \geq 1$, consider $U_k = B(x, \frac{1}{k})$, which is a nonempty open subset of X. Since T is C-recurrent, there exists a polynomial $p_k \in \mathbb{C}[X]$ such that

$$p_k(T)(U_k) \cap U_k \neq \emptyset.$$

For each $k \geq 1$, choose $x_k \in U_k$ such that $p_k(T)(x_k) \in U_k$. It follows that

$$\|x_k - x\| < \frac{1}{k} \quad \text{and} \quad \|p_k(T)(x_k) - x\| < \frac{1}{k},$$

implying that

$$x_k \longrightarrow x \quad \text{and} \quad p_k(T)(x_k) \longrightarrow x$$

as $k \longrightarrow \infty$.

$(ii) \Rightarrow (iii)$: It is clear;

$(iii) \Rightarrow (i)$: Let U be a nonempty open subset of X, and let $x \in U$. Since for all $k \geq 1$, $W_k = B(0, \frac{1}{k})$ is a neighborhood of zero, there exist $z_k \in X$ and $p_k \in \mathbb{C}[X]$ such that

$$\|p_k(T)(z_k) - x\| < \frac{1}{k} \quad \text{and} \quad \|z_k - x\| < \frac{1}{k}.$$

This implies that

$$z_k \longrightarrow x \quad \text{and} \quad p_k(T)(z_k) \longrightarrow x$$

as $k \longrightarrow \infty$, which shows that T is C-recurrent.

□

Let X and Y be two Banach spaces, and consider $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$. The following proposition establishes the relationship between the cyclic recurrence of $T \oplus S$ on $X \oplus Y$ and the cyclic recurrence of T and S on X and Y, respectively.

Proposition 3.7. *If $T \oplus S$ is C-recurrent on $X \oplus Y$, then both T and S are C-recurrent on X and Y, respectively.*

Proof. Let U_1 and U_2 be nonempty open sets in X and Y, respectively. Then, $U_1 \oplus U_2$ is a nonempty open set in $X \oplus Y$. Since $T \oplus S$ is C-recurrent, there exists $p \in \mathbb{C}[X]$ such that

$$(p(T) \oplus p(S))(U_1 \oplus U_2) \cap (U_1 \oplus U_2) \neq \emptyset,$$

which implies that

$$p(T)(U_1) \cap U_1 \neq \emptyset \quad \text{and} \quad p(S)(U_2) \cap U_2 \neq \emptyset.$$

Therefore, T and S are C-recurrent. □

The following theorem establishes the relationship between the cyclic recurrence of an operator and the set of its C-recurrent vectors. In fact, it shows that T is C-recurrent if and only if it has a dense set of C-recurrent vectors.

Theorem 3.8. *Let T be an operator acting on X. The following assertions are equivalent:*

(i) The operator T is C-recurrent;

(ii) $\overline{CRec(T)} = X$.

Furthermore, the set of C-recurrent vectors for T is a G_δ subset of X.

Proof. $(ii) \Rightarrow (i)$: Assume that T has a dense set of C-recurrent vectors. Let U be an open set in X. Take a C-recurrent vector $y \in U$ and choose $\varepsilon > 0$ such that $B(y, \varepsilon) \subset U$. Since T has a dense set of C-recurrent vectors, there exists a polynomial $p \in \mathbb{C}[X]$ such that

$$\|p(T)y - y\| < \varepsilon.$$

Therefore,

$$y \in p(T)(U) \cap U \neq \emptyset,$$

and we conclude that T is C-recurrent.

$(i) \Rightarrow (ii)$: Let $U = B(x_0, \varepsilon_0)$ with $x_0 \in X$ and $0 < \varepsilon_0 < 1$. We assume that T is a C-recurrent operator. Thus, there exists a polynomial $p_1 \in \mathbb{C}[X]$ such that the set $p_1(T)(U) \cap U$ is both nonempty and open in X. Consequently, we can find $x_1 \in X$ and $\varepsilon_1 < \frac{1}{2}$ satisfying the condition:

$$U_1 := B(x_1, \varepsilon_1) \subset p_1(T)^{-1}(U) \cap U.$$

Once more, note that $U_1 = B(x_1, \varepsilon_1)$ is an open set in X. Therefore, we can find another polynomial $p_2 \in \mathbb{C}[X]$ such that the set $p_2(T)^{-1}(U_1) \cap U_1$ is nonempty and open in X. Consequently, we can choose $x_2 \in X$ and $\varepsilon_2 < \frac{1}{2^2}$ such that:

$$U_2 := B(x_2, \varepsilon_2) \subset p_2(T)^{-1}(U_1) \cap U_1.$$

Using induction, we construct a sequence $(x_k)_{k\in\mathbb{N}}$ of elements in X, a sequence $(p_k)_{k\in\mathbb{N}}$ of complex polynomials, and a sequence of positive real numbers $(\varepsilon_k)_{k\in\mathbb{N}}$ satisfying the following conditions:

$$B(x_k, \varepsilon_k) \subset B(x_{k-1}, \varepsilon_{k-1}) \quad \text{and} \quad p_k(T)(B(x_k, \varepsilon_k)) \subset B(x_{k-1}, \varepsilon_{k-1}).$$

Since X is complete, we can conclude by applying Cantor's theorem that:

$$\bigcap_k B(x_k, \varepsilon_k) = \{y\}$$

for some $y \in X$. Therefore, $p_k(T)y \to y$, implying that y is a C-recurrent vector for T, and $y \in U$. Finally, let's observe that:

$$CRec(T) = \bigcap_{s=1}^{\infty} \bigcup_{p\in\mathbb{C}[X]} \left\{ x \in X : \|p(T)x - x\| < \frac{1}{s} \right\}$$

which demonstrates that the set of C-recurrent vectors for T is a G_δ-set. □

Remark 3.9. Contrary to what we have proven in Theorem 3.8, the set of C-recurrent vectors is necessarily dense for every C-recurrent operator. In fact, this condition is not required for cyclic vectors of a cyclic operator, as shown in [17].

4 C-recurrence Criterion, the problem of $T \oplus T$ and weighted shifts operators

In this section, we present several conditions for $T \oplus T$ to be C-recurrent. We introduce a C-recurrence Criterion and analyze the C-recurrence properties of weighted shift operators on classical sequence spaces.

We start by presenting several equivalent conditions for $T \oplus T$ to be C-recurrent.

Proposition 4.1. *The following statements are equivalent:*

(i) $T \oplus T$ *is C-recurrent;*

(ii) For every pair of nonempty open sets $U, V \subset X$*, there exists a polynomial* $p \in \mathbb{C}[X]$ *such that*

$$p(T)(U) \cap U \neq \emptyset \quad \textit{and} \quad p(T)(V) \cap V \neq \emptyset;$$

(iii) For every nonempty open set $U \subset X$ *and every neighborhood* W *of* 0*, there exists a polynomial* $p \in \mathbb{C}[X]$ *such that*

$$p(T)(U) \cap U \neq \emptyset \quad \textit{and} \quad p(T)(W) \cap W \neq \emptyset.$$

Proof. Since the implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are straightforward, we only need to prove the implication $(iii) \Rightarrow (i)$.

$(iii) \Rightarrow (i)$: Let U and V be nonempty open subsets of X. Then, there exist a nonempty open set V_1 and a neighborhood W of 0 such that $W + V_1 \subset V$. By our hypothesis, there exists a polynomial $p \in \mathbb{C}[X]$ such that

$$p(T)(U) \cap U \neq \emptyset \quad \text{and} \quad p(T)(W) \cap W \neq \emptyset.$$

Consequently, we have

$$p(T)(W + V_1) \cap (W + V_1) \neq \emptyset,$$

which implies that

$$(p(T) \oplus p(T))(U \oplus V) \cap (U \oplus V) \neq \emptyset.$$

Hence, we obtain the desired result. □

To establish a C-recurrence Criterion, let us begin by revisiting the hypercyclicity and cyclicity criteria.

Theorem 4.2. *(Hypercyclicity Criterion [20]) Let* X *be a separable Banach space, and let* T *be an operator acting on* X*. Assume that there exist dense subsets* $X_0 \subset X$ *and* $Y_0 \subset X$*, an increasing sequence* $(n_k)_{k\geq 1}$ *of positive integers, and maps* $S_{n_k} : Y_0 \longrightarrow X$ *such that, for any* $x \in X_0$ *and* $y \in Y_0$*, the following conditions are satisfied:*

(i) $T^{n_k}(x) \longrightarrow 0$*,*

(ii) $S_{n_k}(y) \longrightarrow 0$*,*

(iii) $T^{n_k} S_{n_k}(y) \longrightarrow y$*.*

Then T *is hypercyclic.*

Theorem 4.3. *(Cyclicity Criterion [19]) Let* T *be an operator on a separable Banach space* X*. Suppose that there exist two dense subsets* V *and* W *of* X*, a sequence* (p_k) *of polynomials, and a sequence of maps* $S_k : W \longrightarrow X$ *such that:*

(i) For every $x \in V$*,* $p_k(T)x \longrightarrow 0$*,*

(ii) For every $x \in W$*,* $S_k(x) \longrightarrow 0$*,*

(iii) For every $x \in W$*,* $p_k(T)S_k(x) \longrightarrow x$*.*

Then $T \oplus T$ *is cyclic.*

We modify the hypercyclicity and cyclicity criteria to establish a C-recurrence criterion.

Theorem 4.4. *(C-recurrence Criterion) Let T be an operator on X. Suppose that there exists a dense subspace $Z \subset X$ and a sequence (p_k) of complex polynomials such that:*

(i) $p_k(T)x \longrightarrow 0$ for every $x \in Z$,

(ii) For every $x \in Z$, there exists a sequence (x_k) of elements of X such that $x_k \longrightarrow 0$ and $p_k(T)x_k \longrightarrow x$.

Then $T \oplus T$ is C-recurrent.

Proof. Let U and V be non-empty open subsets of X. We will demonstrate that there exists $k \geq 0$ such that $p_k(U) \cap U \neq \emptyset$ and $p_k(V) \cap V \neq \emptyset$. Since Z is dense in X, we can find

$$x \in U \cap Z \quad \text{and} \quad y \in V \cap Z.$$

Let ε_1 and ε_2 be chosen as strictly positive real numbers such that:

$$B(x, \varepsilon_1) \subseteq U \quad \text{and} \quad B(y, \varepsilon_2) \subseteq V.$$

By condition (i), we have $p_k(T)x \longrightarrow 0$ and $p_k(T)y \longrightarrow 0$ as $k \longrightarrow \infty$. Moreover, based on condition (ii), we know that there exist two sequences (x_k) and (y_k) of elements in X such that

$$x_k \longrightarrow 0, \quad p_k(T)x_k \longrightarrow x, \quad y_k \longrightarrow 0, \quad \text{and} \quad p_k(T)y_k \longrightarrow y$$

as $k \longrightarrow \infty$. Thus, we can choose a value of k large enough such that the following inequalities hold:

$$\|p_k(T)(x)\| < \frac{\varepsilon_1}{2}, \quad \|x_k\| < \varepsilon_1, \quad \|p_k(T)(x_k) - x\| < \frac{\varepsilon_1}{2},$$

and

$$\|p_k(T)(y)\| < \frac{\varepsilon_2}{2}, \quad \|y_k\| < \varepsilon_2, \quad \|p_k(T)(y_k) - y\| < \frac{\varepsilon_2}{2}.$$

Additionally, we obtain:

$$\|(x_k + x) - x\| = \|x_k\| < \varepsilon_1 \quad \text{and} \quad \|(y_k + y) - y\| = \|y_k\| < \varepsilon_2.$$

Hence, we have $x + x_k \in B(x, \varepsilon_1) \subset U$ and $y + y_k \in B(y, \varepsilon_2) \subset V$. Consequently, we obtain:

$$\begin{aligned} \|p_k(T)(x + x_k) - x\| &= \|p_k(T)(x) + p_k(T)(x_k) - x\| \\ &\leq \|p_k(T)(x)\| + \|p_k(T)(x_k) - x\| \\ &< \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1, \end{aligned}$$

and

$$\begin{aligned} \|p_k(T)(y + y_k) - y\| &= \|p_k(T)(y) + p_k(T)(y_k) - y\| \\ &\leq \|p_k(T)(y)\| + \|p_k(T)(y_k) - y\| \\ &< \frac{\varepsilon_2}{2} + \frac{\varepsilon_2}{2} = \varepsilon_2. \end{aligned}$$

Thus, we have established the existence of $k \in \mathbb{N}$ such that $p_k(T)(x+x_k) \in U$ and $p_k(T)(y+y_k) \in V$. Therefore, we can deduce that

$$x + x_k \in p_k(T)^{-1}(U) \cap U \quad \text{and} \quad y + y_k \in p_k(T)^{-1}(V) \cap V.$$

As a result, we can conclude that $T \oplus T$ is C-recurrent. □

Let's illustrate the C-recurrence Criterion with an example where we will prove that $B \oplus B$ is C-recurrent on $X = \ell^p(\mathbb{N})$, where $1 \leq p < \infty$, and B is the backward shift operator defined as follows:

$$B(x_0, x_1, \dots) = (x_1, x_2, \dots)$$

for all $(x_0, x_1, \dots) \in \ell^p(\mathbb{N})$. Note that the shift operator plays a big role not only in the dynamics but in operator theory in general, see for instance [14, 24].

Example 4.5. Let $X = \ell^p(\mathbb{N})$, where $1 \leq p < \infty$, or $X = c_0(\mathbb{N})$, and let $B : X \to X$ be the backward shift operator. We aim to prove that $B \oplus B$ is C-recurrent in $X \oplus X$.

Consider $Z := c_0(\mathbb{N})$, which denotes the space of finitely supported sequences. Let (p_k) be a sequence of polynomials defined by $p_k(B) = \lambda^k B^k$, where B is the backward shift operator. This is for each $k \in \mathbb{N}$ and each $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$. For any $x \in Z \subset X$, we have

$$p_k(B)(x) \longrightarrow 0,$$

since $p_k(B)(x) = \lambda^k B^k(x) = 0$ for sufficiently large k, as $B^k(x)$ becomes zero for large enough k due to the finitely supported nature of x in Z. Due to the density of Z in X, we can find a sequence (y_k) of elements in Z such that

$$y_k \longrightarrow x \quad \text{as} \quad k \to \infty.$$

Consider the forward shift operator S defined on X by

$$S(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots),$$

for all $(x_0, x_1, \ldots) \in \ell^p(\mathbb{N})$. Let $(x_k) := (\lambda^{-k} S^k(y_k))$. Considering that $\left\|\lambda^{-k} S^k\right\| \leq |\lambda|^{-k}$, we can deduce that

$$x_k = \lambda^{-k} S^k(y_k) \longrightarrow 0 \quad \text{and} \quad p_k(B)(x_k) = y_k \longrightarrow x.$$

This confirms that the backward shift operator B fulfills the C-recurrence Criterion. Thus, we can conclude that $B \oplus B$ is a C-recurrent operator on $X \oplus X$.

In the following proposition, we present an alternative formulation of the C-recurrence criterion. It will become evident later that this formulation is easier to work with compared to the initial one.

Proposition 4.6. *An operator T on a Banach space X satisfies the C-recurrence Criterion if and only if there exists a dense subspace $Z \subset X$ and a sequence (p_k) of polynomials such that for each $x \in Z$, there exists a sequence (x_k) of elements in X satisfying the following conditions:*

(i) $\|p_k(T)(x)\| \, \|x_k\| \longrightarrow 0$;

(ii) $p_k(T)x_k \longrightarrow x$.

Proof. It is evident that an operator satisfying the C-recurrence Criterion also satisfies the hypothesis of Proposition 4.6, so we need only to prove the converse.

For any $x \in Z$, there exists a sequence (x_k) consisting of elements from the set X that satisfies both properties (i) and (ii) described in Proposition 4.6. Assume that

$$\alpha_k := \|p_k(T)(x)\| \quad \text{and} \quad \beta_k := \|x_k\|$$

are not both zero. In the case where $\alpha_k \beta_k \neq 0$, we define $\lambda_k := \alpha_k^{-1/2} \beta_k^{1/2}$. In situations where $\alpha_k = 0$, we take $\lambda_k := 2^k \beta_k$, and if $\beta_k = 0$, we set $\lambda_k := 2^{-k} \alpha_k^{-1}$.

Now, we define

$$q_k = \lambda_k p_k \quad \text{and} \quad y_k = \lambda_k^{-1} x_k.$$

Then,

$$q_k(T)(x) \longrightarrow 0, \quad y_k \longrightarrow 0, \quad \text{and} \quad q_k(T)(y_k) = p_k(T)x_k \longrightarrow x.$$

Therefore, we have shown that T satisfies the C-recurrence criterion. □

Obviously, the criterion obtained in Proposition 4.6 is easier to apply than the C-recurrence Criterion (Theorem 4.4). This will become evident in the following result.

Let us first recall the definition of weighted shift operators.

Suppose X is either $\ell^p(\mathbb{N})$ with $1 \leq p < \infty$ or $X = c_0(\mathbb{N})$, and let $w = (w_n)_{n \in \mathbb{N}}$ be a bounded sequence of non-zero positive numbers in $\mathbb{C}$. Consider the unilateral weighted shift operator B_w on X. It is defined as follows:

$$B_w(e_n) = \begin{cases} w_n e_{n-1} & \text{for } n \geq 1, \\ 0 & \text{for } n = 0, \end{cases}$$

where $(e_n)_{n\in\mathbb{N}}$ represents the canonical basis of X.

The objective of the next proposition is to demonstrate how we can utilize the C-recurrence Criterion to establish that $B_{\mathbf{w}} \oplus B_{\mathbf{w}}$ is C-recurrent, and consequently, deduce that $B_{\mathbf{w}}$ is also C-recurrent.

Proposition 4.7. *Let $B_{\mathbf{w}} \in \mathcal{B}(X)$ be the unilateral weighted backward shift operator acting on $X = \ell^p(\mathbb{N})$ for $1 \leq p < \infty$ or $X = c_0(\mathbb{N})$. Then, $B_{\mathbf{w}} \oplus B_{\mathbf{w}}$ is a C-recurrent operator on $X \oplus X$. Consequently, $B_{\mathbf{w}}$ is also a C-recurrent operator on X.*

Proof. We will prove the result for the case of $X = \ell^p(\mathbb{N})$ for $1 \leq p < \infty$, since the proof for $X = c_0(\mathbb{N})$ is identical.

Let $Z := c_0(\mathbb{N})$ be the set of all finitely supported sequences, and let (p_k) be a sequence of polynomials defined by $p_k(t) = t^k$, for all $k \in \mathbb{N}$. Consider x belonging to Z. The density of Z in X guarantees the existence of a sequence (y_k) in Z such that y_k converges to x.

Let S_w be the linear map defined on X by

$$S_w(e_n) = w_{n+1}^{-1} e_{n+1},$$

and for each $k \in \mathbb{N}$, we set $x_k := S_w^k(y_k)$. Since $\|p_k(B_w)(x)\| = \left\|B_w^k(x)\right\| = 0$ for large enough k, we have

$$\|p_k(B_w)(x)\| \, \|x_k\| \longrightarrow 0$$

and

$$p_k(B_w)(x_k) = y_k \longrightarrow x.$$

As a consequence, we deduce that B_w satisfies the C-recurrence Criterion (Proposition 4.6). Hence, $B_{\mathbf{w}} \oplus B_{\mathbf{w}}$ is a C-recurrent operator on $X \oplus X$. □

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