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Existence of Classical Solutions for a Class of Impulsive Hamilton-Jacobi Equations

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Abstract In this paper we investigate a class of impulsive Hamilton-Jacobi equations for existence of global solutions in spaces of continuous functions. We give conditions under which the considered equations have at least one and at least two classical solutions. To prove our main results we propose a new approach based upon recent theoretical results.

1 Introduction

Impulsive differential equations or impulsive partial differential equations are natural frameworks for mathematical simulation of phenomena which are abruptly changed in their states at short time perturbations whose duration is negligible in comparison with the duration of the phenomena. The theory of impulsive differential equations is richer than the corresponding theory of impulsive partial differential equations, see for example the books [8, 16] and the references therein. Some references on impulsive partial differential equations are: [2, 3, 7, 10, 13, 15]. For some other related studies, see [6, 17].

In this paper, we investigate the following class of impulsive Hamilton-Jacobi equations

$$\begin{aligned} u_t + H(t,x,u(t,x),Du(t,x)) &= 0, \quad t \in J\backslash\{t_1,\ldots,t_k\},\ J=[0,T], \quad x\in\mathbb{R}^n, \\ u(t_j^+,x) - u(t_j^-,x) &= I_j(x,u(t_j,x)), \quad j\in\{1,\ldots,k\},\ x\in\mathbb{R}^n, \\ u(0,x) &= u_0(x), \quad x\in\mathbb{R}^n, \end{aligned} \tag{1.1}$$

where $Du = (u_{x_1},\ldots,u_{x_n})$ and

$$u(t_j^+,x) = \lim_{t\to t_j^+} u(t,x), \quad u(t_j^-,x) = \lim_{t\to t_j^-} u(t,x) \quad x\in\mathbb{R}^n, \quad j\in\{1,\ldots,k\}.$$

Assume that

(A1) $n \geq 1$, $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = T$, $u_0 \in \mathcal{C}^1(\mathbb{R}^n)$, $0 \leq u_0 \leq B$ on $\mathbb{R}^n$ for some constant $B > 0$.

(A2) $I_j \in \mathcal{C}(\mathbb{R}^{n+1})$, $|I_j(x,v)| \leq a_j(x)|v|^{p_j}$, $x\in\mathbb{R}^n$, $v\in\mathbb{R}$, $a_j\in\mathcal{C}(\mathbb{R}^n)$, $0\leq a_j(x)\leq B$, $x\in\mathbb{R}^n$, $p_j\geq 0$, $j\in\{1,2,\ldots,k\}$,

(A3) $f \in \mathcal{C}(J\times\mathbb{R}^{2n+1})$,

$$|H(t,x,u(t,x),Du(t,x))| \leq b_1(t,x) + b_2(t,x)|u(t,x)|^l + \sum_{j=1}^n b_{3j}(t,x)|u_{x_j}(t,x)|^{l_j},$$

$(t, x) \in J \times \mathbb{R}^n$, $b_1, b_2, b_{3j} \in \mathcal{C}(J \times \mathbb{R}^n)$, $0 \le b_1, b_2, b_{3j} \le B$ on $J \times \mathbb{R}^n$, $l, l_j \ge 0$, $j \in \{1, \dots, n\}$.

The problem (1.1) is investigated in [11] for existence and uniqueness of solutions using the method of characteristics and a Krasnosel'skii type fixed point theorem. The main assumption in [11] is that H is Lipschitzian in the t-variable. Note that the Hamiltonian H can satisfies $(A3)$ and at the same time to be not Lipschitzian in the t-variable. For instance,

$$H(t, x, u(t, x), Du(t, x)) = \sqrt{t} + |u(t, x)|^3 + \sum_{j=1}^{n} |u_{x_j}(t, x)|^3, \quad (t, x) \in J \times \mathbb{R}^n,$$

satisfies $(A3)$, but it is not Lipschitzian in the t-variable. Therefore, the results in this paper one can consider as complimentary results to the results in [11]. In [12], the authors show that the solution of impulsive Hamilton-Jacobi equation is applied to traffic flow problem.

In this paper, we will investigate the problem (1.1) for existence of at least one classical solution and existence of at least two nonnegative classical solutions. The problem of existence of solutions of Cauchy-type problems for some partial differential equations in spaces of continuous functions was studied in [4].

This paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we prove existence of at least one classical solution for the problem (1.1). In Section 4, we prove existence of at least two nonnegative classical solutions. In Section 5, we give an example to illustrate our main results.

2 Preliminary Results

Below, assume that X is a real Banach space. Now, we will recall the definitions of compact and completely continuous mappings in Banach spaces.

Definition 2.1. Let $K : M \subset X \to X$ be a map. We say that K is compact if $K(M)$ is contained in a compact subset of X. K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.

Proposition 2.2. *(Leray-Schauder nonlinear alternative [1]) Let C be a convex, closed subset of a Banach space E, $0 \in U \subset C$ where U is an open set. Let $f : \overline{U} \to C$ be a continuous, compact map. Then*

(a) *either f has a fixed point in $\overline{U}$,*

(b) *or there exist $x \in \partial U$, and $\lambda \in (0, 1)$ such that $x = \lambda f(x)$.*

To prove our existence result we will use the following fixed point theorem which is a consequence of Proposition 2.2.

Theorem 2.3. *Let E be a Banach space, Y a closed, convex subset of E, U be any open subset of Y with $0 \in U$. Consider two operators T and S, where*

$$Tx = \varepsilon x, \ x \in \overline{U},$$

for $\varepsilon > 1$ and $S : \overline{U} \to E$ be such that

(i) *$I - S : \overline{U} \to Y$ continuous, compact and*

(ii) $\{x \in \overline{U} : x = \lambda(I - S)x, \quad x \in \partial U\} = \emptyset$, *for any* $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.

Then there exists $x^* \in \overline{U}$ such that

$$Tx^* + Sx^* = x^*.$$

Proof. We have that the operator $\frac{1}{\varepsilon}(I - S) : \overline{U} \to Y$ is continuous and compact. Suppose that there exist $x_0 \in \partial U$ and $\mu_0 \in (0, 1)$ such that

$$x_0 = \mu_0 \frac{1}{\varepsilon}(I - S)x_0,$$

that is

$$x_0 = \lambda_0 (I - S)x_0$$

where $\lambda_0 = \mu_0 \frac{1}{\varepsilon} \in \left(0, \frac{1}{\varepsilon}\right)$. This contradicts the condition (ii). From Leray-Schauder nonlinear alternative, it follows that there exists $x^* \in \overline{U}$ so that

$$x^* = \frac{1}{\varepsilon}(I - S)x^*$$

or

$$\varepsilon x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

□

Definition 2.4. Let X and Y be real Banach spaces. A map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which one has the following inequality

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Now, we will recall the definition for a cone in a Banach space.

Definition 2.5. A closed, convex set $\mathcal{P}$ in X is said to be cone if

(i) $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,

(ii) $x, -x \in \mathcal{P}$ implies $x = 0$.

Denote $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$. The next result is a fixed point theorem which we will use to prove existence of at least two nonnegative global classical solutions of the IVP (1.1). For its proof, we refer the reader to [5, 9].

Theorem 2.6. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous map and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) $Sx \neq (I - T)(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,

(ii) *there exists* $\epsilon \geq 0$ *such that* $Sx \neq (I - T)(\lambda x)$, *for all* $\lambda \geq 1 + \epsilon$, $x \in \partial U_2$ *and* $\lambda x \in \Omega$,

(iii) $Sx \neq (I - T)(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

3 Existence of at Least One Solution

Let $J_0 = J \backslash \{t_j\}_{j=1}^k$ and define the spaces $PC(J)$, $PC^1(J)$ and $PC^1(J, \mathcal{C}^1(\mathbb{R}^n))$ by

$$PC(J) = \{g : g \in \mathcal{C}(J_0), \quad \exists g(t_j^+),\ g(t_j^-) \text{ and } g(t_j^-) = g(t_j),\ j \in \{1, \dots, k\}\},$$

$$PC^1(J) = \{g : g \in PC(J) \cap \mathcal{C}^1(J_0),\ \exists g'(t_j^-),\ g'(t_j^+) \text{ and } g'(t_j^-) = g'(t_j),\ j \in \{1, \dots, k\}\}$$

and

$$PC^1(J,\mathcal{C}^1(\mathbb{R}^n)) = \{u : J\times\mathbb{R}^n \to \mathbb{R} : u(\cdot,x) \in PC^1(J),\ x \in \mathbb{R}^n \text{ and } u(t,\cdot) \in \mathcal{C}^1(\mathbb{R}^n),\ t \in J\}. \tag{3.1}$$

Suppose that $X := PC^1(J,\mathcal{C}^1(\mathbb{R}^n))$ is endowed with the norm

$$\begin{aligned} \|u\| &= \sup\Bigg\{ \sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n} |u(t,x)|, \quad \sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n} |u_{x_i}(t,x)|, \\ &\qquad \sup_{(t,x)\in[t_j,t_{j+1}]\times\mathbb{R}^n} |u_t(t,x)|, \quad j \in \{1,\ldots,k\}, \quad i \in \{1,\ldots,n\}\Bigg\}, \end{aligned}$$

provided it exists.

Lemma 3.1. *Suppose* $(A2)$ *and* $(A3)$. *Let* $u \in X$ *and* $\|u\| \le B$. *Then*

$$\begin{aligned} |H(t,x,u(t,x),Du(t,x))| &\le B\left(1 + B^l + \sum_{j=1}^n B^{l_j}\right), \\ |I_j(x,u(t,x))| &\le B^{p_j+1}, \quad j \in \{1,\ldots,k\}, \\ \left|\sum_{j=1}^k I_j(x,u(t,x))\right| &\le \sum_{j=1}^k B^{p_j+1}, \quad (t,x) \in J\times\mathbb{R}^n. \end{aligned}$$

Proof. We have

$$\begin{aligned} |H(t,x,u(t,x),Du(t,x))| &\le b_1(t,x) + b_2(t,x)|u(t,x)|^l + \sum_{j=1}^n b_{3j}(t,x)|u_{x_j}(t,x)|^{l_j} \\ &\le B + B^{l+1} + B\sum_{j=1}^n B^{l_j} \\ &= B\left(1 + B^l + \sum_{j=1}^n B^{l_j}\right), \quad (t,x) \in J\times\mathbb{R}^n, \end{aligned}$$

and

$$\begin{aligned} |I_j(x,u(t,x))| &\le a_j(x)|u(t,x)|^{p_j} \\ &\le B^{p_j+1}, \quad (t,x) \in J\times\mathbb{R}^n, \quad j \in \{1,\ldots,k\}, \end{aligned}$$

and

$$\begin{aligned} \left|\sum_{j=1}^k I_j(x,u(t,x))\right| &\le \sum_{j=1}^k |I_j(x,u(t,x))| \\ &\le \sum_{j=1}^k B^{p_j+1}, \quad (t,x) \in J\times\mathbb{R}^n. \end{aligned}$$

This completes the proof. □

For $u \in X = PC^1(J, \mathcal{C}^1(\mathbb{R}^n))$, define the operator

$$\begin{aligned} S_1 u(t,x) &= u(t,x) + \int_0^t H(s,x,u(s,x),Du(s,x))ds \\ &\quad - u_0(x) - \sum_{0<t_k<t} I_k(x,u(t_k,x)), \quad (t,x) \in J \times \mathbb{R}^n. \end{aligned}$$

Lemma 3.2. *Suppose* $(A1)$-$(A3)$. *If* $u \in X$ *satisfies the equation*

$$S_1 u(t,x) = 0, \quad (t,x) \in J \times \mathbb{R}^n, \tag{3.2}$$

then it is a solution to the IVP (1.1).

Proof. We have

$$\begin{aligned} 0 &= S_1 u(t,x) \\ &= u(t,x) + \int_0^t H(s,x,u(s,x),Du(s,x))ds \\ &\quad - u_0(x) - \sum_{0<t_k<t} I_k(x,u(t_k,x)), \quad (t,x) \in J \times \mathbb{R}^n. \end{aligned}$$

Hence,

$$\begin{aligned} u(t,x) &= -\int_0^t H(s,x,u(s,x),Du(s,x))ds \\ &\quad + u_0(x) + \sum_{0<t_k<t} I_k(x,u(t_k,x)), \quad (t,x) \in J \times \mathbb{R}^n. \end{aligned} \tag{3.3}$$

We differentiate (3.3) with respect to t and we find

$$u_t(t,x) = -H(t,x,u(t,x),Du(t,x)), \quad (t,x) \in J \times \mathbb{R}^n.$$

We put $t = 0$ in (3.3) and we get

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n.$$

Now, by (3.3), we obtain

$$\begin{aligned} u(t_j^+,x) &= -\int_0^{t_j} H(s,x,u(s,x),Du(s,x))ds \\ &\quad + u_0(x) + \sum_{0<t_k<t_j^+} I_k(x,u(t_k,x)), \quad x \in \mathbb{R}^n, \end{aligned}$$

$j \in \{1,\dots,k\}$, and

$$\begin{aligned} u(t_j^-,x) &= -\int_0^{t_j} H(s,x,u(s,x),Du(s,x))ds \\ &\quad + u_0(x) + \sum_{0<t_k<t_j^-} I_k(x,u(t_k,x)), \quad x \in \mathbb{R}^n, \end{aligned}$$

$j \in \{1,\dots,k\}$, whereupon

$$u(t_j^+,x) - u(t_j^-,x) = I_j(x,u(t_j,x)), \quad x \in \mathbb{R}^n, \quad j \in \{1,\dots,k\}.$$

This completes the proof. □

Let

$$B_1 = 2B + T\left(B + B^{1+l} + \sum_{j=1}^{n} B^{1+l_j}\right) + \sum_{j=1}^{k} B^{1+p_j}.$$

Lemma 3.3. *Suppose* $(A1)$-$(A3)$. *If* $u \in X$, $\|u\| \le B$, *then*

$$|S_1u(t,x)| \quad \le \quad B_1, \quad (t,x) \in J \times \mathbb{R}^n.$$

Proof. We apply Lemma 3.1 and we get

$$\begin{aligned}
|S_1u(t,x)| &= \Bigg| u(t,x) + \int_0^t H(s,x,u(s,x),Du(s,x))ds \\
&\quad - u_0(x) - \sum_{0<t_k<t} I_k(x,u(t_k,x)) \Bigg| \\
&\le |u(t,x)| + \int_0^t \left(b_1(s,x) + b_2(s,x)|u(s,x)|^l + \sum_{j=1}^{n} b_{3j}(s,x)|u_{x_j}(s,x)|^{l_j} \right) ds \\
&\quad + |u_0(x)| + \sum_{0<t_k<t} |I_k(x,u(t_k,x))| \\
&\le 2B + T\left(B + B^{1+l} + \sum_{j=1}^{n} B^{1+l_j}\right) + \sum_{j=1}^{k} B^{1+p_j} \\
&= B_1, \quad (t,x) \in J \times \mathbb{R}^n.
\end{aligned}$$

This completes the proof. □

In addition, we suppose

(A4) there exist a function $g \in \mathcal{C}(J \times \mathbb{R}^n)$ such that $g > 0$ on $(0,T] \times \left(\mathbb{R}^n \backslash \left(\cup_{j=1}^{k}\{x_j = 0\}\right)\right)$ and

$$\begin{aligned}
g(0,x) &= g(t,0,x_2,\dots,x_n) \\
&= g(t,x_1,0,x_3,\dots,x_n) \\
&\dots \\
&= g(t,x_1,\dots,x_{n-1},0) \\
&= 0, \quad t \in [0,T], \quad x = (x_1,\dots,x_n) \in \mathbb{R}^n,
\end{aligned}$$

and a positive constant A such that

$$2^n(1+t)\prod_{j=1}^{n}(1+|x_j|)\int_0^t \left|\int_0^x g(t_1,s)ds\right| dt_1 \le A, \quad (t,x) \in J \times \mathbb{R}^n,$$

where $\int_0^x = \int_0^{x_1} \dots \int_0^{x_n}$, $ds = ds_n \dots ds_1$.

For $u \in X = PC^1(J, \mathcal{C}^1(\mathbb{R}^n))$, define the operator

$$S_2u(t,x) = \int_0^t \int_0^x (t-t_1) \prod_{j=1}^n (x_j - s_j) g(t_1, s) S_1 u(t_1, s) ds dt_1, (t,x) \in J \times \mathbb{R}^n \tag{3.4}$$

Lemma 3.4. *Suppose* $(A1)$-$(A4)$*. If* $u \in X$ *and* $\|u\| \leq B$*, then*

$$\|S_2 u\| \leq AB_1.$$

Proof. We have

$$\begin{aligned}
|S_2u(t,x)| &= \left| \int_0^t \int_0^x (t-t_1) \prod_{j=1}^n (x_j - s_j) g(t_1, s) S_1 u(t_1, s) ds dt_1 \right| \\
&\leq \int_0^t \left| \int_0^x (t-t_1) \prod_{j=1}^n |x_j - s_j| g(t_1, s) |S_1 u(t_1, s)| ds \right| dt_1 \\
&\leq B_1 t 2^n \prod_{j=1}^n |x_j| \int_0^t \left| \int_0^x g(t_1, s) ds \right| dt_1 \\
&\leq B_1 2^n (1+t) \prod_{j=1}^n (1+|x_j|) \int_0^t \left| \int_0^x g(t_1, s) ds \right| dt_1 \\
&\leq AB_1, \quad (t,x) \in J \times \mathbb{R}^n,
\end{aligned}$$

and

$$\begin{aligned}
\left| \frac{\partial}{\partial t} S_2u(t,x) \right| &= \left| \int_0^t \int_0^x \prod_{j=1}^n (x_j - s_j) g(t_1, s) S_1 u(t_1, s) ds dt_1 \right| \\
&\leq \int_0^t \left| \int_0^x \prod_{j=1}^n |x_j - s_j| g(t_1, s) |S_1 u(t_1, s)| ds \right| dt_1 \\
&\leq B_1 2^n \prod_{j=1}^n |x_j| \int_0^t \left| \int_0^x g(t_1, s) ds \right| dt_1 \\
&\leq B_1 2^n (1+t) \prod_{j=1}^n (1+|x_j|) \int_0^t \left| \int_0^x g(t_1, s) ds \right| dt_1 \\
&\leq AB_1, \quad (t,x) \in J \times \mathbb{R}^n,
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\partial}{\partial x_l} S_2 u(t,x)\right| &= \left|\int_0^t \int_0^x (t-t_1) \prod_{j=1, j\neq l}^n (x_j - s_j) g(t_1,s) S_1 u(t_1,s) ds dt_1\right| \\
&\leq \int_0^t \left|\int_0^x (t-t_1) \prod_{j=1, j\neq l}^n |x_j - s_j| g(t_1,s) |S_1 u(t_1,s)| ds\right| dt_1 \\
&\leq B_1 t 2^n \prod_{j=1, j\neq l}^n |x_j| \int_0^t \left|\int_0^x g(t_1,s) ds\right| dt_1 \\
&\leq B_1 2^n (1+t) \prod_{j=1}^n (1+|x_j|) \int_0^t \left|\int_0^x g(t_1,s) ds\right| dt_1 \\
&\leq AB_1, \quad (t,x) \in J \times \mathbb{R}^n, \quad l \in \{1,\ldots,n\}.
\end{aligned}$$

Thus, $\|S_2 u\| \leq AB_1$. This completes the proof. □

Lemma 3.5. *Suppose* $(A1)$-$(A4)$. *If* $u \in X$ *satisfies the equation*

$$S_2 u(t,x) = C, \quad (t,x) \in J \times \mathbb{R}^n, \tag{3.5}$$

for some constant C*, then* u *is a solution to the IVP* (1.1).

Proof. We differentiate two times with respect to t and two times with respect to x_l, $l \in \{1,\ldots,n\}$, the equation (3.5) and we find

$$g(t,x) S_1 u(t,x) = 0, \quad (t,x) \in J \times \mathbb{R}^n,$$

whereupon

$$S_1 u(t,x) = 0, \quad (t,x) \in (0,T] \times \left(\mathbb{R} \backslash \left(\cup_{j=1}^n \{x_j = 0\}\right)\right).$$

Since $S_1 u(\cdot,\cdot) \in \mathcal{C}(J \times \mathbb{R}^n)$, we get

$$\begin{aligned}
0 &= \lim_{t\to 0} S_1 u(t,x) \\
&= S_1 u(0,x) \\
&= \lim_{x_1 \to 0} S_1 u(t,x) \\
&= S_1 u(t,0,x_2,\ldots,x_n) \\
&\ldots \\
&= \lim_{x_n \to 0} S_1 u(t,x) \\
&= S_1 u(t,x_1,\ldots,x_{n-1},0), \quad (t,x) \in J \times \mathbb{R}^n.
\end{aligned}$$

Thus,

$$S_1 u(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Hence and Lemma 3.2, we conclude that u is a solution to the IVP (1.1). This completes the proof. □

Our main result in this section is as follows.

Theorem 3.6. *Suppose* $(A1)$-$(A4)$. *Then the IVP* (1.1) *has at least one solution in* X.

Proof. Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $Y = \overline{\widetilde{Y}}$ and

$$U = \{u \in Y : \|u\| < B \quad \text{and if} \quad \|u\| \geq \frac{B}{2}, \quad \text{then} \quad u(0,x) > \frac{B}{2}, \quad x \in \mathbb{R}^n\}.$$

For $u \in \overline{U}$ and $\epsilon > 1$, define the operators

$$\begin{aligned} Tu(t,x) &= \epsilon u(t,x), \\ Su(t,x) &= u(t,x) - \epsilon u(t,x) - \epsilon S_2 u(t,x), \quad (t,x) \in J \times \mathbb{R}^n. \end{aligned}$$

For $u \in \overline{U}$, we have

$$\begin{aligned} \|(I-S)u\| &= \|\epsilon u + \epsilon S_2 u\| \\ &\leq \epsilon\|u\| + \epsilon\|S_2 u\| \\ &\leq \epsilon B_1 + \epsilon A B_1. \end{aligned}$$

Thus, $S : \overline{U} \to X$ is continuous and $(I-S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in \overline{U}$ so that $\|u\| = B$ and

$$u = \lambda(I-S)u$$

or

$$u = \lambda\epsilon\,(u + S_2 u), \tag{3.6}$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Observe that $S_2 u(0,x) = 0$, $x \in \mathbb{R}^n$. Since $\|u\| = B > \frac{B}{2}$, we conclude that $u(0,x) > \frac{B}{2}$, $x \in \mathbb{R}^n$, and

$$u(0,x) = \lambda\epsilon u(0,x), \quad x \in \mathbb{R}^n,$$

whereupon $\lambda\epsilon = 1$, which is a contradiction. Consequently

$$\{u \in \overline{U} : u = \lambda_1 (I-S)u, \ \|u\| = B\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 2.3, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t,x) &= Tu^*(t,x) + Su^*(t,x) \\ &= \epsilon u^*(t,x) + u^*(t,x) \\ &\quad -\epsilon u^*(t,x) - \epsilon S_2 u^*(t,x), \quad (t,x) \in J \times \mathbb{R}^n, \end{aligned}$$

whereupon

$$S_2 u^*(t,x) = 0, \quad (t,x) \in J \times \mathbb{R}^n.$$

From here, u^* is a solution to the problem (1.1). From here and from Lemma 3.5, it follows that u is a solution to the IVP (1.1). This completes the proof. □

4 Existence of at Least Two Solutions

Let $X = PC^1(J, \mathcal{C}^1(\mathbb{R}^n))$ and assume that the constants B and A which appear in the conditions $(A1)$ and $(A4)$, respectively, satisfy the following inequalities:

(A5) $AB_1 < \frac{L}{5}$, where $B_1 = 2B + T\left(B + B^{1+l} + \sum_{j=1}^{n} B^{1+l_j}\right) + \sum_{j=1}^{k} B^{1+p_j}$. and L is a positive constant that satisfies the following conditions:

$$r < L < R_1 \leq B,$$

with r and R_1 are positive constants.

Our main result in this section is as follows.

Theorem 4.1. *Suppose that* $(A1)$-$(A4)$ *and* $(A5)$ *hold. Then the problem* (1.1) *has at least two nonnegative solutions in* X.

Proof. Let

$$\widetilde{P} = \{u \in X : u \geq 0 \quad \text{on} \quad J \times \mathbb{R}^n\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$T_1 v(t,x) = (1 + m\epsilon)v(t,x) - \epsilon\frac{L}{10},$$

$$S_3 v(t,x) = -\epsilon S_2 v(t,x) - m\epsilon v(t,x) - \epsilon\frac{L}{10}, (t,x) \in J \times \mathbb{R}^n,$$

where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (3.4). Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Define

$$\Omega = \mathcal{P},$$

$$U_1 = \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\},$$

$$U_2 = \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\},$$

$$U_3 = \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}.$$

(i) For $v_1, v_2 \in \Omega$, we have

$$\|T_1 v_1 - T_1 v_2\| = (1 + m\epsilon)\|v_1 - v_2\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

(ii) For $v \in \overline{\mathcal{P}_{R_1}}$, we get

$$\|S_3 v\| \leq \epsilon\|S_2 v\| + m\epsilon\|v\| + \epsilon\frac{L}{10}$$

$$\leq \epsilon\left(AB_1 + mR_1 + \frac{L}{10}\right).$$

Therefore $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is completely continuous.

(iii) Let $v_1 \in \overline{\mathcal{P}_{R_1}}$. Set

$$v_2 = v_1 + \frac{1}{m} S_2 v_1 + \frac{L}{5m}.$$

Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $J \times \mathbb{R}^n$. We have $v_2 \geq 0$ on $J \times \mathbb{R}^n$. Therefore $v_2 \in \Omega$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon S_2 v_1 - \epsilon \frac{L}{10} - \epsilon \frac{L}{10}$$

or

$$\begin{aligned}(I - T_1) v_2 &= -\epsilon m v_2 + \epsilon \frac{L}{10} \\ &= S_3 v_1.\end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

(iv) Assume that for any $v_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $v \in \partial \mathcal{P}_r \cap (\Omega + \lambda v_0)$ or $v \in \partial \mathcal{P}_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 v = (I - T_1)(v - \lambda v_0).$$

Then

$$-\epsilon S_2 v - m\epsilon v - \epsilon \frac{L}{10} = -m\epsilon (v - \lambda v_0) + \epsilon \frac{L}{10}$$

or

$$-S_2 v = \lambda m v_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 v\| = \left\| \lambda m v_0 + \frac{L}{5} \right\| \geq \frac{L}{5}.$$

This is a contradiction.

(v) Let $\epsilon_1 = \frac{2}{5m}$. Suppose that there exist a $v_1 \in \partial \mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that

$$S_3 v_1 = (I - T_1)(\lambda_1 v_1). \tag{4.1}$$

Moreover,

$$-\epsilon S_2 v_1 - m\epsilon v_1 - \epsilon \frac{L}{10} = -\lambda_1 m \epsilon v_1 + \epsilon \frac{L}{10},$$

or

$$S_2 v_1 + \frac{L}{5} = (\lambda_1 - 1) m v_1.$$

From here,

$$2\frac{L}{5} > \left\| S_2 v_1 + \frac{L}{5} \right\| = (\lambda_1 - 1) m \|v_1\| = (\lambda_1 - 1) m L$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.6 hold. Hence, the problem (1.1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

□

5 An Example

Below, we will illustrate our main results. Let $k = 2$,

$$n = T = B = 1, \quad t_1 = \frac{1}{4}, \quad t_2 = \frac{1}{2}, \quad p_1 = 2, \quad p_2 = 3, \quad l = 2.$$

and

$$R_1 = \frac{9}{10}, \quad L = \frac{3}{5}, \quad r = \frac{2}{5}, \quad m = 10^{50}, \quad A = \frac{1}{10B_1}.$$

Then

$$B_1 \quad = \quad 2 + 3 + 1 = 6.$$

Next,

$$r < L < R_1 < B, \quad AB_1 < \frac{L}{5}.$$

i.e., $(A5)$ holds. Take

$$h(s) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$h'(s) \quad = \quad \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})},$$

$$l'(s) \quad = \quad \frac{11\sqrt{2}s^{10}(1 + s^{22})}{1 + s^{44}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Therefore

$$-\infty \quad < \quad \lim_{s \to \pm\infty} (1 + s + s^2)^3 h(s) < \infty,$$

$$-\infty \quad < \quad \lim_{s \to \pm\infty} (1 + s + s^2)^3 l(s) < \infty.$$

Hence, there exists a positive constant C_1 so that

$$(1 + s + s^2)^3 \left(\tfrac{1}{44\sqrt{2}} \log \tfrac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \tfrac{1}{22\sqrt{2}} \arctan \tfrac{s^{11}\sqrt{2}}{1 - s^{22}} \right) \quad \leq \quad C_1,$$

$s \in \mathbb{R}$. Note that $\lim_{s \to \pm 1} l(s) = \frac{\pi}{2}$ and by [14] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}} \log \frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1 + s^{44})(1 + s + s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t, x) = Q(t)Q(x_1) \ldots Q(x_n), \quad t \in J, \quad x \in \mathbb{R}^n.$$

Then there exists a constant $C > 0$ such that

$$2^n (1 + t) \prod_{j=1}^{n} (1 + |x_j|) \int_0^t \left| \int_0^x g_1(\tau, z) dz \right| d\tau \leq C, \quad (t, x) \in J \times \mathbb{R}^n.$$

Let

$$g(t, x) = \frac{A}{C} g_1(t, x), \quad (t, x) \in J \times \mathbb{R}^n.$$

Then

$$2^n(1+t)\prod_{j=1}^{n}(1+|x_j|)\int_0^t\left|\int_0^x g(\tau,z)dz\right|d\tau \le A, \quad (t,x)\in J\times\mathbb{R}^n,$$

i.e., $(A4)$ holds. Therefore for the problem

$$\begin{aligned} u_t-\sqrt{t}u_x-\frac{u^2}{1+x^4} &= 0, \quad t\in[0,1], \quad x\in\mathbb{R},\\ u(t_1^+,x)-u(t_1^-,x) &= \frac{(u(t_1,x))^2}{1+x^{10}}, \quad x\in\mathbb{R},\\ u(t_2^+,x)-u(t_2^-,x) &= \frac{(u(t_2,x))^3}{1+x^{18}}, \quad x\in\mathbb{R},\\ u(0,x) &= \frac{1}{1+x^4}, \quad x\in\mathbb{R}, \end{aligned}$$

are fulfilled all conditions of Theorem 3.6 and Theorem 4.1. Note that for the above problem we can not apply the results in [11] because the Hamiltonian is not Lipschitzian in the variable t.

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