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Projection and commutativity of prime Banach algebras

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Abstract This article focuses on a Banach algebra denoted by $\mathcal{A}$, where we delve into two key areas. First, we present a number of results concerning the continuous projection from $\mathcal{A}$ to its center $Z(\mathcal{A})$. Second, we study in detail the conditions that promote the commutativity of $\mathcal{A}$. Moreover, we provide examples and applications to demonstrate the necessity of imposing certain restrictions on the hypotheses of our theorems.

1 Introduction

Much attention has been devoted to rings theory in order to show that certain rings must be commutative, even if the conditions imposed on them initially seem too weak to imply commutativity. In the following discussion, unless explicitly stated otherwise, we use the symbol $\mathcal{A}$ to denote a prime Banach algebra with the center $Z(\mathcal{A})$. $[s,t]$ denotes the commutator $st-ts$, and $s\circ t$ denotes the anticommutator $st+ts$.

Recall that a ring $\mathcal{R}$ is called prime if the condition $x\mathcal{R}y=\{0\}$ implies that either $x=0$ or $y=0$. On the other hand, a ring $\mathcal{R}$ is called semiprime if $x\mathcal{R}x=\{0\}$ implies that $x=0$. In addition, an additive mapping $d:\mathcal{R}\longrightarrow\mathcal{R}$ is called a derivation if the equation $d(xy)=d(x)y+xd(y)$ is satisfied for all $x,y\in\mathcal{A}$, in particular, the mapping d defined as $d(a)=[\lambda,a]$ for all $a\in\mathcal{A}$ constitutes a derivation known as an inner derivation induced by an element $\lambda\in\mathcal{A}$. Furthermore, a mapping F is said to be a generalized derivation of $\mathcal{A}$ associated with a derivation d if $F(xy)=F(x)y+xd(y)$ for all $x,y\in\mathcal{A}$.

Let M be a subspace of $\mathcal{A}$, the linear operator $P:\mathcal{A}\longrightarrow\mathcal{A}$ is said to be a projection of $\mathcal{A}$ onto M if $P(x)\in M$ for all $x\in\mathcal{A}$ and $P(x)=x$ for all $x\in M$. Consider two subsets M and N of the Banach algebra $\mathcal{A}$, and note that $M\bigoplus_{al}N$ is the direct algebraic sum of $\mathcal{A}$. We equip the subspaces M and N with the induced topology of $\mathcal{A}$, and $M\times N$ with the product topology. If the mapping $\Phi:M\times N\longrightarrow\mathcal{A}$, defined as $\phi(a,b)=a+b$, is a homeomorphism, we denote $M\bigoplus_t N$ as the topological direct sum of $\mathcal{A}$. Furthermore, if M is complemented in $\mathcal{A}$ and N is its topological complement, we observe the existence of unique continuous projection P from $\mathcal{A}$ to M.

Many studies in the literature have focused on establishing conditions for the commutativity of prime and semiprime Banach algebras. For example, Yood in [2] provided that if $\mathcal{A}$ is a Banach algebra and G_1,G_2 are non-empty open subsets of $\mathcal{A}$ such that, for every $a\in G_1$ and $b\in G_2$, there exists an integer $n=n(a,b)>1$ satisfying either $(ab)^n=a^nb^n$ or $(ab)^n=b^na^n$, then $\mathcal{A}$ must be commutative. In addition, Ali and Khan [6] established the commutativity of Banach algebras by derivation; for example, they proved that if $\mathcal{A}$ is a unital prime Banach algebra and has a nonzero continuous linear derivation $d:\mathcal{A}\longrightarrow\mathcal{A}$ such that either $d((ab)^n)-a^nb^n$ or $d((ab)^n)-b^na^n$ is in the center of $\mathcal{A}$ for an integer $n=n(a,b)>1$, then $\mathcal{A}$ is commutative. In the same context, in [5] Moumen et al. proved that a prime Banach algebra $\mathcal{A}$ is commutative if $\mathcal{A}$ has non-empty open subsets G_1,G_2 of $\mathcal{A}$ and a non-injective continuous derivation d such that for all $(x,y)\in G_1\times G_2$ there are strictly positive integers n,m such that $d(x^ny^m)+[x^n,y^m]\in Z(\mathcal{A})$, and they also proved that a prime real or complex prime Banach algebra must be commutative and $P = I$ if there exist two nonvoid open subsets G_1 and G_2 of $\mathcal{A}$ admitting a continuous projection P whose image lies in $Z(\mathcal{A})$ and satisfying the following identitie $P(x^n \star y^m) - x^n * y^m = 0$ for all $x \in G_1$, $y \in G_2$, where the symbols " $\star$ " and " $*$ " represent either the Lie product " $[.,.]$ " or the Jordan product " $\circ$ " or simple multiplication " . " of algebra.

Inspired by the above results, we will pursue the study of these problems on prime Banach algebras in two directions. First, we use projections instead of derivations, and second, we work in a more general way, using identities at the center of a Banach algebra instead of polynomial identities.

2 Preliminary Lemmas

In this section, we begin our discussion with the following well-known lemmas, which will be used extensively to prove our results.

Lemma 2.1. [4, Lemma 4] *Let b and ab be in the center of a prime ring $\mathcal{R}$. If b is not zero, then a is in $Z(\mathcal{R})$, the center of $\mathcal{R}$.*

Lemma 2.2. [3, Lemma 2.3] *Let M be a closed subspace of a Banach space $\mathcal{A}$. M is complemented if there is a continuous projection P of $\mathcal{A}$ on M, and its complement is $(I - P)(\mathcal{A})$, where I is the identity mapping on $\mathcal{A}$.*

Lemma 2.3. [2, Lemma 2] *Let $\mathcal{A}$ is a Banach algebra and M be a closed linear subspace of $\mathcal{A}$. If $p(t) = a_1 t + a_2 t^2 + ... + a_n t^n$ be a polynomial in real variable t over $\mathcal{A}$ such that $p(t) \in M$, then each $a_i \in M$.*

Lemma 2.4. [1, Lemma 1] *If $\mathcal{A}$ is a semiprime ring in which, for each x in $\mathcal{A}$ there exists a positive integer $n = n(x) > 1$ such that $(xy)^n = x^n y^n$ for all $y \in \mathcal{A}$, then $\mathcal{A}$ has no nonzero nilpotent elements.*

Lemma 2.5. [7, Theorem 2] *Suppose that there are non-empty open subsets G_1, G_2 of $\mathcal{A}$ such that for each $x \in G_1$ and $y \in G_2$ there are positive integers $n = n(x, y), m = m(x, y)$ depending on x and y, $n > 1, m > 1$, such that either $[x^n, y^m] \in Z(\mathcal{A})$ or $x^n \circ y^m \in Z(\mathcal{A})$. Then $\mathcal{A}$ is commutative if $\mathcal{A}$ is semiprime.*

3 Main Result

Throughout this paper, we assume that $\mathcal{A}$ is a real or complex prime Banach algebra. We will use the previous lemmas as proofs of our theorems. Our main goal is to establish a connection between the commutativity of a prime Banach algebra $\mathcal{A}$ and the properties of its projection.

Theorem 3.1. *Let $n > 1, m > 1$ be a fixed integer, G_1 and G_2 two opens sets of a prime Banach algebra $\mathcal{A}$. If $x^n \circ y^m \in Z(\mathcal{A})$ for all $x \in G_1, y \in G_2$, then $\mathcal{A}$ is commutative.*

Proof. Assume that

$$x^n \circ y^m \in Z(\mathcal{A}) \text{ for all } x \in G_1, y \in G_2. \tag{3.1}$$

Let $u \in \mathcal{A}$. Since G_1 is open we can replace x by $x + tu$ in (3.1) for all sufficiently small real t, we find that

$$\begin{aligned}(x+tu)^n \circ y^m &= x^n \circ y^m + t(B_1 \circ y^m) + t^2(B_2 \circ y^m) + ... + \\ &\quad t^{n-1}(B_{n-1} \circ y^m) + t^n(u^n \circ y^m) \in Z(\mathcal{A})\end{aligned} \tag{3.2}$$

where B_i, $i = 1...n-1$, are terms depending on t, x, u. Using Lemma 2.3, (3.2) gives

$$u^n \circ y^m \in Z(\mathcal{A}) \text{ for all } u \in \mathcal{A}, y \in G_2. \tag{3.3}$$

Substituting $y + tv$ for y in (3.3), where $v \in \mathcal{A}$ and t sufficiently small real, and reasoning as above we get

$$u^n \circ v^m \in Z(\mathcal{A}) \text{ for all } u, v \in \mathcal{A}. \tag{3.4}$$

Accordingly, $\mathcal{A}$ is commutative by Lemma 2.5. □

Theorem 3.2. *Let $n > 1$ be a fixed integer, $\mathcal{A}$ a prime Banach algebra, and G a nonvoid open subset of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection P whose image lies in $Z(\mathcal{A})$ and $P(x^n) - x^n \in Z(\mathcal{A})$ for all $x \in G$, then $\mathcal{A}$ is commutative and P is the identity map on $\mathcal{A}$.*

Proof. Let us consider two sets S_p and T_p, where $p > 1$ is an integer, are defined as follows :

$$S_p = \{x \in \mathcal{A} \mid P(x^p) - x^p \in Z(\mathcal{A})\} \text{ and } T_p = \{x \in \mathcal{A} \mid P(x^p) - x^p \notin Z(\mathcal{A})\}.$$

It's clear that $\left(\bigcap_{n>1} T_p\right) \cap G = \emptyset$. Now, we want to show that T_p is an open set of $\mathcal{A}$. In fact, let x_k a sequence of S_p such that ($x_k \longrightarrow x$ as $k \longrightarrow \infty$), it follows that $P(x_k^p) - x_k^p \in Z(\mathcal{A})$. Using the continuity of P, we obtain

$$\lim_{k\longrightarrow\infty} P(x_k^p) - x_k^p = P(\lim_{k\longrightarrow\infty} x_k^p) - \lim_{k\longrightarrow\infty} x_k^p = P(x^p) - x^p \in Z(\mathcal{A}).$$

This forces that $x \in S_p$, i.e. S_p, is closed, and hence T_p is open. By the Baire category theorem, there exists $p > 1$ such that T_p is not dense in $\mathcal{A}$, otherwise $\cap_{p>1} T_p$ is also dense in $\mathcal{A}$, which contradicts the fact that $(\cap_{p>1} T_p) \cap G = \varnothing$. Consequently, there exists a non-void open subset O contained in S_p, so that

$$P(x^p) - x^p \in Z(\mathcal{A}) \text{ for all } x \in O. \tag{3.5}$$

Let $x_0 \in O$ and $x \in \mathcal{A}$, then $x_0 + tx \in O$ for a sufficiently small real t. From (3.5), we get

$$P((x_0 + tx)^p) - (x_0 + tx)^p \in Z(\mathcal{A}) \text{ for all } x \in \mathcal{A}. \tag{3.6}$$

On the other hand, we have

$$(x_0 + tx)^p = X_{p,0}(x_0, x) + X_{p-1,1}(x_0, x)t + X_{p-2,2}(x_0, x)t^2 + ... + X_{0,p}(x_0, x)t^p,$$

where $X_{i,j}(x_0, x)$ denotes the sum of all terms in which x_0 appears exactly i times and x appears exactly j times such that $i + j = p$. Then, (3.6) can be rewritten as

$$\begin{aligned} P((x_0 + tx)^p) - (x_0 + tx)^p &= (P(X_{r,0}(x_0, x)) - X_{r,0}(x_0, x)) \\ &\quad +(P(X_{p-1,1}(x_0, x)) - X_{p-1,1}(x_0, x))t + \\ &\quad (P(X_{p-2,2}(x_0, x)) - X_{p-2,2}(x_0, x))t^2 + ... + \\ &\quad (P(X_{0,p}(x_0, x)) - X_{0,p}(x_0, x))t^p \in Z(\mathcal{A}). \end{aligned}$$

The coefficient of t^p in the above expression is $P(x^p) - x^p$. From Lemma (2.3), we infer that $P(x^p) - x^p \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$. Since the projection P, whose image is in $Z(\mathcal{A})$, the previous relation gives

$$x^p \in Z(\mathcal{A}) \text{ for all } x \in \mathcal{A}.$$

From the Theorem 3.1, we can easily conclude that $\mathcal{A}$ is commutative. Moreover, $Z(\mathcal{A})$ is a closed subspace of $\mathcal{A}$ and by Lemma 2.2, $(I - P)(\mathcal{A})$ its complement in $\mathcal{A}$, where I is the identity mapping on $\mathcal{A}$. The commutativity of $\mathcal{A}$ assures that $(I - P)(\mathcal{A}) = \{0\}$, which forces that $P = I$. □

Corollary 3.3. [3, Theorem 3.1] *Let $\mathcal{X}$ be a real or complex prime Banach algebra admitting a continuous projection P from $\mathcal{X}$ to $Z(\mathcal{X})$ if there is $n \in \mathbb{N}^*$ such that $P(x^n) = x^n$ for all $x \in \mathcal{H}$, where $\mathcal{H}$ is a nonvoid open subset of $\mathcal{X}$. Then, P is the identity mapping on $\mathcal{X}$ and $\mathcal{X}$ is commutative.*

Corollary 3.4. *Let $n > 1$ be a fixed integer, $\mathcal{A}$ a prime Banach algebra and G_1, G_2 are nonvoid opens subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection P whose image lies in $Z(\mathcal{A})$ such that $P((xy)^n) - x^n y^n \in Z(\mathcal{A})$ for all $x \in G_1, y \in G_2$, then $\mathcal{A}$ is commutative and P is the identity map on $\mathcal{A}$.*

Proof. Assume that $P((xy)^n) - x^n y^n \in Z(\mathcal{A})$ for all $x \in G_1, y \in G_2$, then, using the same techniques as used in the proof of Theorem 3.2 with some changes, we get $x^n y^n \in Z(\mathcal{A})$ for all $x, y \in \mathcal{A}$, so $x^n \circ y^n \in Z(\mathcal{A})$ for all $x, y \in \mathcal{A}$. Thereby, $\mathcal{A}$ is commutative and P is the identity map on $\mathcal{A}$ by Lemma 2.2. □

Theorem 3.5. *Let $n, m > 1$ are fixed integers, $\mathcal{A}$ a prime Banach algebra and G_1 and G_2 are two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection P whose image lies in $Z(\mathcal{A})$ such that $P(x^n) - y^m \in Z(\mathcal{A})$ for all $(x, y) \in G_1 \times G_2$, then $\mathcal{A}$ is commutative and P is the identity map on $\mathcal{A}$.*

Proof. Let us consider the following sets

$$H_{r,s} = \{(x, y) \in \mathcal{A}^2 \mid P(x^r) - y^s \in Z(\mathcal{A})\} \text{ and } K_{r,s} = \{(x, y) \in \mathcal{A}^2 \mid P(x^r) - y^s \notin Z(\mathcal{A})\}.$$

Obviously, $\left(\bigcap_{r,s>1} K_{r,s}\right) \cap G_1 \times G_2 = \emptyset$, otherwise there exist $(x_0, y_0) \in G_1 \times G_2$ satisfy $P(x_0^r) - y_0^s \in Z(\mathcal{A})$ for all $r, s > 1$, which contradict our hypotheses.
Lets (x_k, y_k) a sequence of $H_{r,s}$ such that ($(x_k, y_k) \longrightarrow (x, y)$ as $k \longrightarrow \infty$), then $P(x_k^n) - y_k^m \in Z(\mathcal{A})$. In view of the continuity of P, we have

$$\lim_{k\longrightarrow\infty} P(x_k^r) - y_k^s = P(\lim_{k\longrightarrow\infty} x_k^r) - \lim_{k\longrightarrow\infty} y_k^s = P(x^r) - y^s \in Z(\mathcal{A}).$$

So, $(x, y) \in H_{r,s}$ which assures that $H_{r,s}$ is a closed, and hence $K_{r,s}$ is an open. By the application of the Baire category theorem, there exist $n, m > 1$ such that $K_{n,m}$ is not dense in $\mathcal{A}$. Reasoning as above, there exists a nonvoid open subset $O \times O'$ such that

$$P(x^n) - y^m \in Z(\mathcal{A}) \text{ for all } (x, y) \in O \times O'.$$

As $P(x) \in \mathcal{A}$ for all $x \in \mathcal{A}$, we conclude that

$$y^m \in Z(\mathcal{A}) \text{ for all } y \in O'.$$

Using the same techniques used in Theorem 3.2, we find the desired result. □

Corollary 3.6. *Let $n > 1, m > 1$ are fixed integers, $\mathcal{A}$ be a prime Banach algebra, and G_1, G_2 are two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection P whose image lies in $Z(\mathcal{A})$ such that $P(x^n) - y^m = 0$ for all $(x, y) \in G_1 \times G_2$, then $\mathcal{A}$ is commutative and P is the identity mapping on $\mathcal{A}$.*

In our next result we will use the two symbols " $\star$ " and " $\diamond$ ", which represent either the Lie product " $[.,.]$ ", or the Jordan product " $\circ$ ", or the simple multiplication " $.$ " in algebra $\mathcal{A}$.

Theorem 3.7. *Let $n > 1, m > 1$ be fixed integers, $\mathcal{A}$ be a prime Banach algebra, and G_1, G_2 be two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection P whose image lies in $Z(\mathcal{A})$ such that $P(x^n \star y^m) - x^n \diamond y^m \in Z(\mathcal{A})$ for all $(x, y) \in G_1 \times G_2$, then $\mathcal{A}$ is commutative and P is the identity map on $\mathcal{A}$.*

Proof. Consider the following sets

$$U_{r,s} = \{(x, y) \in \mathcal{A}^2 \mid P(x^r \star y^s) - x^r \diamond y^s \in Z(\mathcal{A})\} \text{ and}$$
$$V_{r,s} = \{(x, y) \in \mathcal{A}^2 \mid P(x^r \star y^s) - x^r \diamond y^s \notin Z(\mathcal{A})\}.$$

Our main is to prove that $V_{r,s}$ is an open. Firstly, notice that $\left(\bigcap_{r,s>1} V_{r,s}\right) \cap G_1 \times G_2 = \emptyset$. Now, let (x_k, y_k) be a sequence of $U_{r,s}$ and suppose that ($(x_k, y_k) \longrightarrow (x, y)$ as $k \longrightarrow \infty$). Taking into account that $Z(\mathcal{A})$ is a closed subspace of $\mathcal{A}$ and that P is a continuous map, we conclude that

$$\begin{aligned} \lim_{k\longrightarrow\infty} P(x_k^n \star y_k^m) - x_k^n \diamond y_k^m &= P(\lim_{k\longrightarrow\infty} x_k^n \star y_k^m) - \lim_{k\longrightarrow\infty} x_k^n \diamond y_k^m \\ &= P(x^n \star y^m) - x^n \diamond y^m \in Z(\mathcal{A}). \end{aligned}$$

It follows that $(x, y) \in U_{r,s}$, hence $U_{r,s}$ is a closed, which allows us to deduce that $V_{r,s}$ is an open. In the light of the Baire category theorem, there exist $p > 1, q > 1$ such that $V_{p,q}$ is not dense in $\mathcal{A}$. So, necessarily there exist two nonvoid open subsets of $\mathcal{A}$, denoted O and O', such that

$$P(x^p \star y^q) - x^p \diamond y^q \in Z(\mathcal{A}) \text{ for all } (x, y) \in O \times O'.$$

Using the fact that $P(x) \in Z(\mathcal{A})$ for all $x \in \mathcal{A}$, the preceding result shows that

$$x^p \diamond y^q \in Z(\mathcal{A}) \text{ for all } (x, y) \in O \times O'.$$

Thus, $\mathcal{A}$ is commutative by Lemma 2.5. Since $Z(\mathcal{A})$ is a closed subspace of $\mathcal{A}$, then to show that $P = I$ it suffices to apply Lemma 2.2. □

Corollary 3.8. *Let $n > 1, m > 1$ be a fixed integer, $\mathcal{A}$ be a real or complex prime Banach algebra, and G_1, G_2 are two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection P whose image lies in $Z(A)$ such that $P(x^n \star y^m) - x^n \diamond y^m = 0$ for all $(x, y) \in G_1 \times G_2$, then $\mathcal{A}$ is commutative and P is the identity map on $\mathcal{A}$.*

Corollary 3.9. *Let $n, m > 1$ be a fixed integer, $\mathcal{A}$ be a real or complex prime Banach algebra, and G_1, G_2 are two nonvoid open subsets of $\mathcal{A}$. If $\mathcal{A}$ admits a continuous projection P whose image lies in $Z(\mathcal{A})$, then the following assertions are equivalent:*

(i) $P(x^n) - x^n \in Z(\mathcal{A})$ *for all* $x \in G_1$,

(ii) $P(x^n) - y^m \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(iii) $P(x^n y^m) - x^n y^m \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(iv) $P(x^n y^m) - [x^n, y^m] \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(v) $P(x^n y^m) - x^n \circ y^m \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(vi) $P([x^n, y^m]) - [x^n, y^m] \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(vii) $P([x^n, y^m]) - x^n y^m \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(viii) $P([x^n, y^m]) - x^n \circ y^m \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(ix) $P(x^n \circ y^m) - x^n \circ y^m \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(x) $P(x^n \circ y^m) - x^n y^m \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(xi) $P(x^n \circ y^m) - [x^n, y^m] \in Z(\mathcal{A})$ *for all* $x \in G_1, y \in G_2$,

(xii) $\mathcal{A}$ *is commutative and* $P = I$.

The following example shows that the condition "G_1 and G_2 are two opens of $\mathcal{A}$" is necessary in all theorems.

Example 3.10. Let $\mathbb{R}$ be the real field, and

$$\mathcal{A} = M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

be a noncommutative prime algebra with norm $\| A \|_1 = \max_{j=1,2}(\sum_{i=1}^{2} \mid a_{ij} \mid)$ for all $A \in \mathcal{A}$. Let G_1 and G_2 be two sets of $\mathcal{A}$ defined by

$$G_1 = \left\{ \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \mid s \geq 0 \right\}, G_2 = \left\{ \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \mid s < 0 \right\}$$

such that G_2 is an open and G_1 is not an open of $\mathcal{A}$.
Take

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, V_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, V_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \textit{ and } V_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, $\mathcal{B} = \{V_1, V_2, V_3, V_4\}$ is a basis of $\mathcal{A}$, and for all real λ_i, we define a linear continuous projection P on $\mathcal{A}$ to $Z(\mathcal{A}) = vect(V_4)$ by: $P(V) = \lambda_4 V_4$ for all $V = \sum_1^4 \lambda_i V_i \in \mathcal{A}$, and for all $n > 1, m > 1$ we have

(i) $P([X^n, Y^m]) - [X^n, Y^m] \in Z(\mathcal{A})$ for all $X \in G_1, Y \in G_2$,

(ii) $P(X^n \circ Y^m) - X^n \circ Y^m \in Z(\mathcal{A})$ for all $X \in G_1, Y \in G_2$,

(iii) $P([X^n, Y^m]) - X^n \circ Y^m \in Z(\mathcal{A})$ for all $X \in G_1, Y \in G_2$,

(iv) $P(X^n \circ Y^m) - [X^n, Y^m] \in Z(\mathcal{A})$ for all $X \in G_1, Y \in G_2$.

But $\mathcal{A}$ is not commutative and $P \neq I$.

Finally, we present the following example, which proves that the "primality conditions" of $\mathcal{F}$ are necessary in our theorems.

Example 3.11. Consider the product $\mathcal{F} = \mathbb{R}[X] \times M_2(\mathbb{R})$ where the addition and the multiplication are applied coordinatewise. It is clear that $\mathcal{F}$ is not a prime normed Banach algebra with norm defined by:
for all $F = (Q, M) \in \mathcal{F}$,

$$\| F \| = \| Q \| + \| M \|_1 \text{ where } \| Q \| = \| \sum_{i=0}^{n} a_i X^i \| = \sum_{i=0}^{n} | a_i | .$$

We define open sets G_1, G_2 of $\mathcal{F}$ and the projection $P : \mathcal{F} \longmapsto Z(\mathcal{F})$ by:
For all $F = (Q, M) \in \mathcal{F}$,

$$P\,(Q, M) = \left\{ \left(0, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) \mid a \in \mathbb{R} \right\},$$

and

$$G_1 = G_2 = \{(Q, 0) \mid Q(0) \neq 0\}.$$

We can verify that

(i) $P(X^n) - X^n \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$,

(ii) $P(X^n) - Y^m \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$,

(iii) $P(X^nY^m) - X^nY^m \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$,

(iv) $P([X^n, Y^m]) - [X^n, Y^m] \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$,

(v) $P(X^n \circ Y^m) - X^n \circ Y^m \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$,

(vi) $P([X^n, Y^m]) - [X^n, Y^m] \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$,

(vii) $P(X^n \circ Y^m) - X^n \circ Y^m \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$,

(viii) $P([X^n, Y^m]) - X^n \circ Y^m \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$,

(ix) $P(X^n \circ Y^m) - [X^n, Y^m] \in Z(\mathcal{F})$ for all $X \in G_1, Y \in G_2$.

But $\mathcal{F}$ is not commutative and $P \neq I$.

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