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A Comprehensive Overview of Spectral Geometry: Insights from the Laplace-Beltrami Operator on Riemannian Manifolds

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Abstract This study provides an overview of recent research concerning the geometric significance of the Laplace-Beltrami operator, inherently associated with a Riemannian manifold. Essentially, it offers an expanded examination of R. Brooks' expository paper [1], as well as closely related articles. Beginning with a basic exploration of the isospectrality of flat tori, the study progresses to elucidate Sunada's pioneering utilization of various theoretical concepts. It concludes with a concise portrayal of the fundamental aspects of isospectral deformations on Riemannian manifolds.

1 Introduction

The Laplace-Beltrami operator, a fundamental concept in differential geometry, plays a central role in understanding the geometry and topology of Riemannian manifolds [2, 3, 4, 5, 6]. This operator, canonically associated with a Riemannian manifold, encodes crucial geometric information about the manifold, including its spectral properties [7, 8, 9, 10, 11, 12]. The study of these spectral properties and their geometric implications has been a topic of significant research interest in recent years. This research aims to provide an overview of the key developments in the field of geometric analysis related to the Laplace-Beltrami operator. It builds upon the insights presented in an expository paper in [1], while also delving into closely related articles. The primary focus of this study is to explore the geometric significance of the Laplace-Beltrami operator, especially in the context of isospectrality and isospectral deformations. The journey of this research begins with an elementary treatment of the isospectrality phenomenon observed in flat tori. Understanding the isospectrality of these simple geometric objects serves as a foundation for more intricate investigations. One of the highlights of this exploration is a presentation of Sunada's groundbreaking adaptation of a number-theoretical idea. Sunada's work has profound implications for our understanding of isospectral manifolds and the connections between geometry, topology, and number theory. The research culminates with an examination of the essentials of isospectral deformations on Riemannian manifolds. Isospectral deformations are transformations that preserve the spectrum of the Laplace-Beltrami operator while altering the underlying geometry. These deformations reveal the intricate interplay between the spectral properties of the Laplace-Beltrami operator and the geometric structure of the manifold. In summary, this study provides a comprehensive overview of recent research on the geometric implications of the Laplace-Beltrami operator. It explores the isospectrality of flat tori, Sunada's contributions, and the broader context of isospectral deformations. By shedding light on these fundamental aspects of Riemannian geometry, this research contributes to our understanding of the rich interplay between analysis and geometry on manifolds.

2 Preliminaries

Definition 2.1. A Riemannian manifold is an ordered pair (M, G) where M is a smooth manifold, G is a smooth tensor field on M of bidegree $(0, 2)$ such that for each $m \in M$, G_m is positive definite.

Definition 2.2. The *Levi-Civita connection* ∇ on (M, G) is the unique torsion-free connection on M with respect to which G is parallel that is $\nabla \mathsf{G} = 0$. Given a smooth Riemannian manifold (M, G), let $\mathcal{D}(M)$ be the set of all complex valued, smooth functions on M, and $\mathcal{X}(M)$ be the set of smooth complexified vector fields on M . For each $\varphi \in \mathcal{D}(M)$ The *gradient* of φ, denoted by grad φ is the vector field on M uniquely determined by the relation.

$$\mathsf{G}(\text{ grad } \varphi, Y\ ) = Y\varphi,$$

for all $Y \in \mathcal{X}(M)$. For each $Y \in \mathcal{X}(M)$ on M, the *divergence* div Y of Y is an element of $\mathcal{D}(M)$, the value of which at each $p \in M$ is the trace of the linear map:

$$(u \longmapsto \nabla_u Y) : T_pM \longmapsto T_pM,$$

where ∇ is the Levi - Civita connection on (M, G).

Definition 2.3. The *Laplace - Beltrami* operator

$$\Delta : \mathcal{D}(M) \longmapsto \mathcal{D}(M),$$

on (M, G) is defined by,

$$\Delta\varphi = -\text{ div (grad } \varphi) \quad .$$

If x is any chart on M with

$$\mathsf{G}|_{dom(x)} = \mathsf{G}_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

then,

$$grad\varphi|_{dom(x)} = \mathsf{G}^{ij} \frac{\partial \varphi}{\partial x^j} \frac{\partial}{\partial x^j},$$

where,

$$\mathsf{G}^{ij} \mathsf{G}_{kj} = \delta^i_j,$$

and

$$\mathrm{d}iv(Y)|_{dom(x)} = \frac{\partial Y^i}{\partial x^i} + \Gamma^i_{ik} Y^k,$$

for any $\varphi \in C(M)$ and any $Y \in \mathcal{X}(M)$ where

$$Y|_{dom(x)} = Y^i \frac{\partial}{\partial x^i},$$

and Γ^i_{jk} are the Christoffel symbols of the second kind associated with the Levi-Civita connection ∇ by

$$\nabla(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

Thus,

$$\begin{aligned} \Delta\varphi|_{dom(x)} &= -\text{div}(\text{grad } \varphi)|_{dom(x)} \\ &= -\frac{\partial}{\partial x^i}(\mathsf{G}^{im} \frac{\partial \varphi}{\partial x^m}) - \Gamma^i_{ik} \mathsf{G}^{kn} \frac{\partial \varphi}{\partial x^n}, \end{aligned}$$

for any $\varphi \in C(M)$. $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of Δ if there exists $\varphi \in \mathcal{D}(M)$ such that $\Delta\varphi = \lambda\varphi$. A non-vanishing function $\varphi \in \mathcal{D}(M)$ is called an *eigenfunction* of the Laplace - Beltrami operator Δ if $\Delta\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{C}$. Under these circumstances, we refer to φ as an eigenfunction of Δ with eigenvalue λ.

Definition 2.4. Given a group Γ and finite subgroups $A, B \leq \Gamma$, A and B are said to be *equivalent in the sense Gassmann*.[22] if

$$\#(c \cap A) = \#(c \cap B),$$

for every conjugacy class c in $\nabla\Gamma$.

Remark 2.5. (a) Subgroups equivalent in the sense of Gassmann have the same cardinality. (b) If A and B are conjugate subgroups, then A and B are clearly equivalent in the sense of Gassmann.

Definition 2.6. Given a Riemannian manifold (M, G) with the heat kernel $H : M \times M \times \mathbb{R}_{.>0} \longmapsto \mathbb{R}$, the *theta function* associated with (M, G) is a function

$$\Theta = \Theta_M : \mathbb{R}_{.>0} \longmapsto \mathbb{R},$$

defined by:

$$\Theta(t) = \int_M H(m, m, t) \mathrm{d}\mu(m).$$

Let us illustrate these concepts on the basis of the following two simplest possible examples :

On the flat circle $(\frac{\mathbb{R}}{L\mathbb{Z}}, \mathrm{d}x \otimes \mathrm{d}x)$ of content $L > 0$, the Laplace - Beltrami operator reduces to

$$\Delta = -\frac{\partial^2}{\partial x^2}$$

It can routinely be checked that the eigenfunctions of Δ are of the form

$$\varphi_n(x) = e^{\frac{2\pi n i}{L} x}$$

with eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{L^2}.$$

for $n \in \mathbb{Z}$, as depicted in Figures 1 and 2.

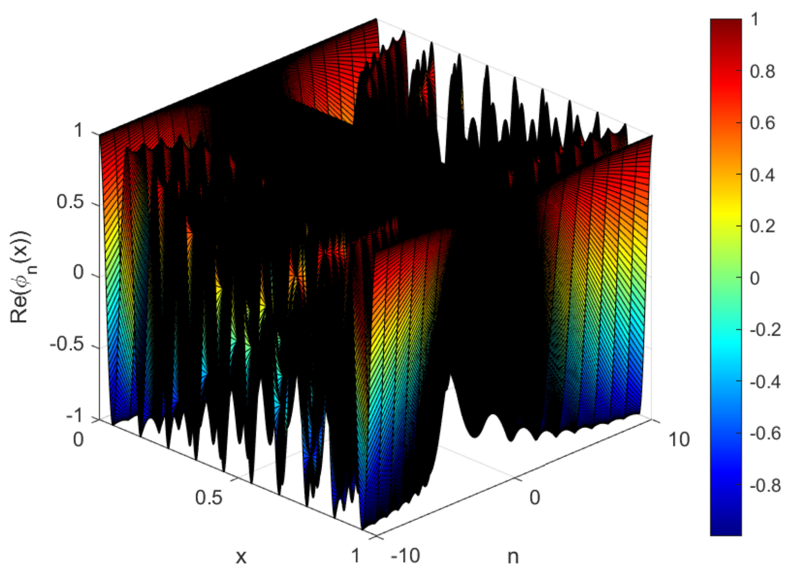

Figure 1. Eigenfunctions of Laplace-Beltrami Operator in Example 2

On the real line $(\mathbb{R}, \mathrm{d}x \otimes \mathrm{d}x)$, the Laplace-Beltrami operator is again of the form

$$\Delta = -\frac{\partial^2}{\partial x^2}.$$

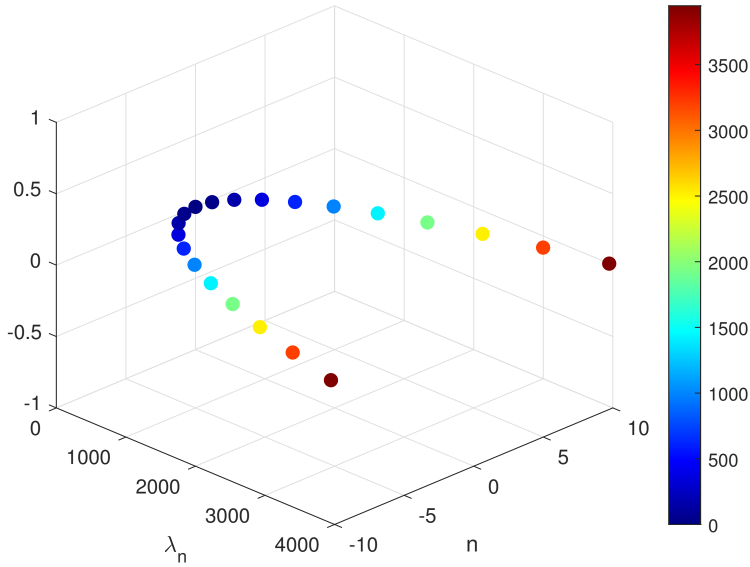

Figure 2. Eigenvalues of Laplace-Beltrami Operator in Example 2

In this case *every* real number $\lambda \in \mathbb{R}$ is an eigenvalue of Δ. Eigenfunctions with eigenvalue λ are of the form

$$\varphi(x) = \begin{cases} Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} & \text{for } A, B \in \mathbb{R} \text{ if } \lambda < 0 \\ \\ Ce^{i\sqrt{\lambda}x} + De^{-i\sqrt{\lambda}x} & \text{for } C, D \in \mathbb{R} \text{ if } \lambda > 0 \\ \\ Ex + F & \text{for } E, F \in \mathbb{R} \;\; \lambda = 0 \end{cases}$$

As it can be readily observed on the basis of the above examples, shown in Figues 3 and 4 the set of eigenvalues of the Laplace-Beltrami operator on a compact manifold has a markedly different nature from that on a non-compact manifold. We shall mostly concentrate on compact manifolds on which the set of eigenvalues of the Laplace-Beltrami operator has a well-understood and tidy structure: If (M, G) is a compact Riemannian manifold with the Laplace-Beltrami operator Δ, then the eigenvalues of Δ constitute a countable, discrete and unbounded set of non-negative real numbers. Furthermore the set of eigenfunctions corresponding to an eigenvalue λ span a *finite* dimensional subspace $\mathcal{H}_\lambda$ of $\mathcal{D}(M)$. The dimension of $\mathcal{H}_\lambda$ is the *multiplicity* of λ. $0 \in \mathbb{R}$ always occurs as an eigenvalue. However, $\mathcal{H}_0$ consists of constant functions and thus dim $\mathcal{H}_0 = 1$. In other words, the multiplicity of the eigenvalue 0 is always 1. Thus, in the case of the flat circle of content L each eigenvalue is of the form,

$$\lambda_n = \frac{4\pi^2 n^2}{L^2},$$

and

$$\mathcal{H}_{\lambda_n} = \langle e^{\frac{2\pi i n}{L}x}, e^{\frac{-2\pi i n}{L}x} \rangle,$$

where $n \in \mathbb{Z}_{\geq 0}$.

We may now offer an elucidation of the expression “spectral geometry” [17]: Spectral geometry done within the framework of Riemannian geometry is essentially the study of Riemannian manifolds on the basis of the information consisting of magnitudes and multiplicities of the eigenvalues of the Laplace-Beltrami operator. It is very important to notice that the “information” in this

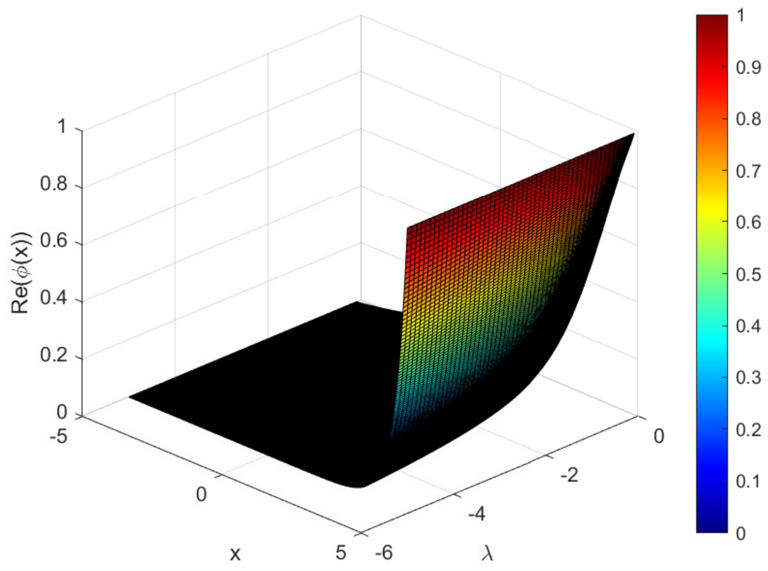

Figure 3. $\varphi(x)$ for $\lambda < 0$ in Example 2

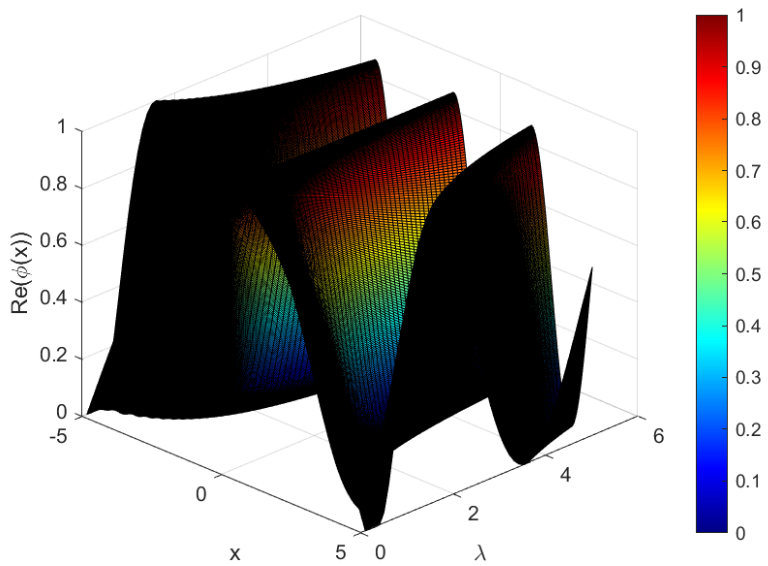

Figure 4. $\varphi(x)$ for $\lambda > 0$ in Example 2

question encompasses *not only* the magnitudes of the eigenvalues but the multiplicities thereof. Thus, given a compact Riemannian manifold (M, G) we understand the *spectrum* of (M, G) to be the set of eigenvalues of the Laplace-Beltrami operator on (M, G) with each eigenvalue tagged by a number indicating its multiplicity. We shall denote the spectrum of (M, G) by $Sp(M, \mathsf{G})$ or $Sp(M)$ unless confusion is likely. Observe that, in the case of a compact Riemannian manifold (M, G), the spectrum $Sp(M, \mathsf{G})$ may be identified with an increasing sequence

$$0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots + \infty.$$

Each eigenvalue occurs in the form of λ_n and is repeated as many times as its multiplicity. Riemannian manifolds (M, G) and (M', G') are said to be isometric if there exists a diffeomorphism

$\varphi : M \longmapsto M'$ satisfying:

$$\mathsf{G}_p(u,v) = \mathsf{G}'_{\varphi(p)}(T\varphi_p(u), T\varphi_p(v)),$$

for $p \in M,\ \ u,v \in T_pM.$
(M, G) and $(M^{'}, \mathsf{G}^{'})$ are said to be *isospectral* if

$$Sp(M,G) = Sp(M^{'},G^{'}) \quad .$$

It is clear that isometric Riemannian manifolds are isospectral. In spectral geometry of Riemannian manifolds, the principal problem is to determine the extent to which the isometry class of a Riemannian manifold is determined by its spectrum.
The broaching of the subject of "inverse spectral geometry" is popularly ascribed to M. Kac [26] who raised now the famous question "Can one hear the shape of a drum?". However, the subject seems to have come up earlier in Riemannian geometry in [16]. A negative answer to the question of whether "The isometry class of a Riemannian manifold is 'audible' " was provided by Milnor in a curt announcement [27]. Milnor's pathbreaking work was followed by a period of "sporadic counterexamples" during which diverse pairs of isospectral but non-isometric Riemannian manifolds were produced. Good examples of matured products may be found in[36, 25, 21]. Brilliantly constructed and certainly mathematically enriching as these counterexamples were, they constituted only a collection of individual instances. A general method for producing isospectrality was invented by Sunada who was inspired by the work of number theorists[35]. Sunada's method proved to be fruitful not only in understanding isospectrality but was also of fundamental importance in producing manifolds which "sounded the same while changing shape" that is, manifolds which were isometrically deformable.[23].

3 The Case of Flat Tori

The extent to which the spectrum can determine the geometry on a Riemannian manifold can be illustrated in the case of flat tori in a direct and elementary fashion.
For $N \geqslant 1,\ N \in \mathbb{Z}$, consider $\mathbb{R}^N$ with its obvious additive group and vector space structure over $\mathbb{R}$. A *lattice* in $\mathbb{R}^N$ is a discrete subgroup of $\mathbb{R}^N$ which contains a basis of $\mathbb{R}^N$. Equivalently, a lattice Λ in $\mathbb{R}^N$ is a subset of $\mathbb{R}^N$ of the form

$$\Lambda = \mathrm{B}\mathbb{Z}^N,$$

where $B \in \mathbb{R}^{N\times N}$ is a non - singular matrix. We note that lattices come in pairs: Given a lattice, $\Lambda = \mathrm{B}\mathbb{Z}^N \subseteq \mathbb{R}^N$, the *dual* of Λ is a lattice $\Lambda^* \subseteq \mathbb{R}^N$ which is defined to be

$$\Lambda^* = (\mathrm{B}^{-1})^T\mathbb{Z}^N.$$

The $dual$ Λ^* of Λ is also characterised by an important property of its elements : $\ell^* \in \mathbb{R}^N$ lies in Λ^* if and only if ${\ell^*}^T\ell$ is an integer for all $\ell \in \Lambda$. A *flat torus* is known to be isometric to

$$\mathbb{T}_\Lambda = \{\frac{\mathbb{R}^N}{\Lambda},\ \delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j\},$$

for some lattice Λ in $\mathbb{R}^N$. We shall refer to $\mathbb{T}_\Lambda$ as the flat torus determined by the lattice Λ. Notice that the content of $\mathbb{T}_\Lambda$ is exactly $\mid \det B \mid$ where $\Lambda = B\mathbb{Z}^N$. It can be routinely checked that the Laplace-Beltrami operator on $\mathbb{T}_\Lambda$ is of the form

$$\Delta = -\delta_{ij}\frac{\partial^2}{\partial x^i \partial x^j},$$

and the eigenfunctions of Δ are exactly the functions

$$\varphi_{\ell^*} = e^{2\pi i {\ell^*}^T x},$$

where $\ell^* \in \Lambda^*$. Clearly the eigenvalue corresponding to φ_{ℓ^*} is $4\pi^2|\ell^*|^2 = {\ell^*}^T\ell^*$. Consequently the spectrum $Sp(\mathbb{T}_\Lambda)$ is a sequence

$$0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_n \ldots,$$

in which, a quantity λ occurs as many times as the number of ℓ^*'s with $\lambda = |\ell^*|^2$.
The information contained in $Sp(\mathbb{T}_\Lambda)$ can be converted into geometric information as follows :
$(\mathbf{A})$ First we deposit $Sp(\mathbb{T}_\Lambda)$ into an analytic function F by writing

$$F(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t},$$

where the right hand side can be checked to be uniformly convergent on each compact subset of $\mathbb{R}_{.>0}$. (The choice of F will be justified in the next section.) Notice that by thus squeezing the spectrum of $\mathbb{T}_\wedge$ into F "no information is lost": Indeed, *given* $F(t)$, we may read off $Sp(\Lambda)$ inductively : Clearly, $\lambda_0 = 0$ and having inductively determined $\lambda_0, \dots, \lambda_n$ we have,

$$\lambda_{n+1} = sup\ \{\ m > o \mid \lim_{t\to\infty} [F(t) - \sum_{k=0}^{n} e^{-\lambda_k t}] e^{mt} = 0\}.$$

$(\mathbf{B})$ Secondly, we remember that for each symmetric, positive definite $S \in \mathbb{R}^{N\times N}$ the Jacobi theta function Θ_S is the analytic function defined by:

$$\Theta_S(z) = \sum_{n\in\mathbb{Z}^N} e^{\pi i(n^T S n)z},$$

where the series on the right-hand side is uniformly convergent on each compact subset of $\{z \in \mathbb{C} \mid Im(z) > 0\}$. Putting $\Lambda = \mathrm{B}\mathbb{Z}^N$, we easily obtain:

$$\begin{aligned} F(t) &= \sum_{\ell^*\in\Lambda^*} e^{-4\pi^2 \ell^{*T} \ell^*\ t} \\ &= \sum_{\ell^*\in B^{-1T}\mathbb{Z}^N} e^{-4\pi^2 \ell^{*T}\ell^* t} \\ &= \sum_{n\in\mathbb{Z}^N} e^{-4\pi^2 n^T B^{-1}(B^{-1})^T n\ t} \\ &= \Theta_{B^{-1}(B^{-1})^T}(4\pi i t). \end{aligned}$$

$(\mathbf{C})$ Thirdly, we employ the Jacobi Inversion Formula

$$\Theta_S(z) = (\frac{z}{i})^{\frac{-N}{2}} \frac{1}{\sqrt{detS}} \Theta_{S^{-1}}(-\frac{1}{z}),$$

with $S = (B^T B)^{-1} = B^{-1}(B^T)^{-1}$ and $z = 4\pi i t$ to obtain:

$$\begin{aligned} F(t) &= \Theta_{B^{-1}B^{-1T}}(4\pi i t) \\ &= \left(\frac{4\pi i t}{i}\right)^{\frac{-N}{2}} \frac{1}{\sqrt{(det\ B^{-1})^2}} \Theta_{B^T B}(-\frac{1}{4\pi i t}) \\ &= \frac{|\ detB|}{(4\pi t)^{\frac{N}{2}}} \sum_{n\in\mathbb{Z}^N} e^{-\pi i(n^T B^T B\ n)\frac{1}{4\pi i t}} \\ &= \frac{|detB|}{(4\pi t)^{\frac{N}{2}}} \sum_{\ell\in\Lambda} e^{\frac{-\ell^T\ell}{4t}}. \end{aligned}$$

$(\mathbf{D})$ Finally, we construct again for purposes of simple and clear exposition the sequence,

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq \dots$$

in which a quantity μ occurs as many times as the number of $\ell \in \Lambda$ with $|\ell|^2 = \mu$. We have found in $(\mathbf{C})$ that

$$F(t) = \frac{|\ \mathrm{det}B\ |}{(4\pi t)^{\frac{N}{2}}} \sum_{n=0}^{\infty} e^{-\frac{\mu_n}{4t}},$$

from which we can now read off $\mid detB \mid$ and the sequence $(\mu_k)_{k=0}^{\infty}$ as follows:
Clearly,

$$\mid \mathrm{d}etB \mid = \lim_{t \to 0^+} (4\pi t)^{\frac{N}{2}} F(t).$$

On the other hand $\mu_0 = 0$ and having inductively determined $\mu_0, \ldots, \mu_n$ along with $\mid \det B \mid$ we can compute μ_{n+1} by (2)

$$\mu_{n+1} = sup \; \{ \; m > 0 \mid \lim_{t \to 0^+} [\frac{F(t)}{\mid \mathrm{d}etB \mid}(4\pi t)^{\frac{N}{2}} - \sum_{k=0}^{n} e^{-\frac{\mu_k}{4t}} \;] \; e^{\frac{m}{4t}} = 0\}.$$

This shows us that the function $F(t)$, which is presently on our hands still without any motivation, has the peculiarity of translating spectral information consisting of the sequence $\lambda_0, \lambda_1, \ldots$, into geometric information consisting of $|\det B|$ and $\mu_0, \mu_1, \ldots, \mu_n, \ldots$

Theorem 3.1. Two 2-dimensional flat tori are isometric iff they are isospectral.

Proof. Clearly flat tori $\mathbb{T}_{\Lambda_1}, \mathbb{T}_{\Lambda_1}$ of the same dimension, say N, are isometric iff there exists an isometry of $\mathbb{R}^N$ which sends Λ_1 to Λ_2, that is iff Λ_1 and Λ_2 are isometric subsets of $\mathbb{R}^N$ with its ordinary Euclidean structure. Suppose $N = 2$ and $Sp(\mathbb{T}_{\Lambda_1}) = Sp(\mathbb{T}_{\Lambda_2})$. This means that if e_1, f_1 and e_2, f_2 are the vectors of smallest length generating Λ_1 and Λ_2 respectively then,

$$|e_1| = |e_2|,$$

$$|f_1| = |f_2|.$$

Moreover the areas of the parallelograms spanned by e_1, f_1 and e_2, f_2 are equal. Therefore these parallelograms are congruent in $\mathbb{R}^2$ with its ordinary Euclidean structure. Therefore Λ_1 is isometric to Λ_2, hence $\mathbb{T}_{\Lambda_1}$ is isometric to $\mathbb{T}_{\Lambda_2}$. Milnor's historical example [27] consisted in pointing out that on $\mathbb{R}^{16}$ there were lattices Λ_1, Λ_2, which were well known among number theorists to be non-isometric but representing integers the same number of times, which, translated into Riemannian geometry means that $\mathbb{T}_{\Lambda_1}$ is isospectral to $\mathbb{T}_{\Lambda_2}$.[37]. Although there is considerable amount of confusion, it seems to be well-established by now that the above theorem is not valid for $N > 2$. It is however known that each flat toral isospectral class contains finitely many isometry classes [31, 18].

4 The Sunada Concept

Given a Riemannian manifold (M, G) with the Laplace-Beltrami operator Δ, the *heat equation* associated with (M, G) is the partial differential equation

$$\Delta U + \frac{\partial U}{\partial t} = 0$$

where,

$$U : M \times \mathbb{R}_{.>0} \longmapsto \mathbb{R}.$$

The importance of the heat equation lies in its close relationship with the structure of the spectrum of the Laplace-Beltrami operator. For any continuous function $f : M \longmapsto \mathbb{R}$, the heat equation has a unique solution U subject to the initial condition,

$$\lim_{t \to 0^+} U(x, t) = f(x),$$

for all $x \in M$.
This existence and uniqueness result can be brought into a form which is independent of the choice of "initial conditions" by means of the concept of the *heat kernel*. Given a Riemannian manifold (M, G), the *heat kernel* is a function,

$$H : M \times M \times \mathbb{R}_{.>0} \longmapsto \mathbb{R}$$

such that,

$$\Delta|_x H + \frac{\partial}{\partial t} H = 0$$

and

$$\lim_{t\to 0^+}\int_M H(x,y,t)f(y)\mathrm{d}\mu(y) = f(x),$$

for any continuous $f : M \longmapsto \mathbb{R}$ where μ stands for the Lebesque measure on M induced by the Riemannian metric G. Given a continuous $f : M \longmapsto \mathbb{R}$, it is clear that

$$U(x) = \int_M H(x,y,t)f(y)\mathrm{d}\mu(y)$$

is the unique solution of the heat equation subject to $\lim_{t\to 0^+} U(x,t) = f(x)$.
Quite generally, given a compact Riemannian manifold (M,G) where the Laplace-Beltrami operator has the spectrum,

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots \lambda_n \le \dots$$

which correspond to the respective normalized eigenfunctions,

$$\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n, \dots$$

When it exists the heat kernel is unique and can be verified by direct computation that when (M,G) has a heat kernel H_M

$$H_M(x,y,t) = \sum_{n=0}^{\infty} \varphi_n(x)\overline{\varphi}_n(y)e^{-\lambda_n t}.$$

□

Consider the Euclidean space:

$$\left(R^N, \delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j\right).$$

It can be verified by direct computation that,

$$H_{\mathbb{R}^N}(x,y,t) = \frac{1}{4\pi t}^{\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}.$$

is the heat kernel for

$$\left(R^N, \delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j\right) \quad .$$

Consider the case of the flat torus
$\mathbb{T}_\Lambda = (\mathbb{R}^n/\Lambda, \delta_{ij}\ \mathrm{d}x^i \otimes \mathrm{d}x^j)$. In view of the remark preceding the above example.

$$H(x,y,t) = \sum_{\ell^* \in \Lambda^*} e^{2\pi \ell^{*T} x} e^{-2\pi \ell^{*T} y} e^{-4\pi |\ell|^{*2} t}.$$

Of basic importance for the theory which is to be introduced in this section there is another technique for constructing heat kernels in quotient manifolds. Consider a smooth manifold $\overline{M}$ acted upon by a group Γ properly discontinuously. Let $M = \overline{M}/\Gamma$. Clearly, M is a smooth manifold and the quotient map $p : \overline{M} \longmapsto M$ is a covering projection. We put Riemannian metrics G,$\overline{\mathsf{G}}$ on M,$\overline{M}$ respectively, so that $p : \overline{M} \longmapsto M$ becomes a local isometry. It is possible to do this by choosing $\overline{\mathsf{G}}$ to be invariant under the action of Γ and by defining Γ to be the quotient Riemannian tensor. Equivalently we may take a Riemannian metric G on M and lift it to $\overline{M}$ via p to attain $\overline{\mathsf{G}}$ which is then automatically invariant under Γ. We shall also make the provision that G be sufficiently generic to exclude non-trivial isometries between open subsets of M. Such a Riemannian metric is folklorically referred to as “bumpy”: The fact that on smooth paracompact manifolds bumpy metrics exist (in fact abundantly) has been formulated and proven by Sunada.[35].
We state the theorem and omit the proof which is rather technical.

Theorem 4.1. Given a smooth paracompact manifold M, there exists a Riemannian metric G on M such that for any disjoint open subsets U,$V \subseteq M$, no map $\varphi : U \longmapsto V$ can be an isometry with respect to G.

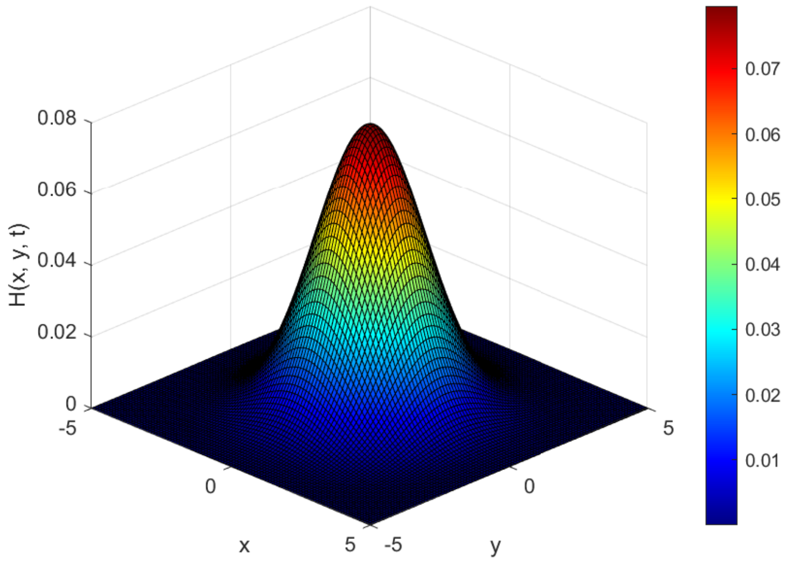

Figure 5. Heat Kernel $H_{\mathbb{R}^2}(x, y, t)$ for $N = 2$ and $t = 1$ in Example 3

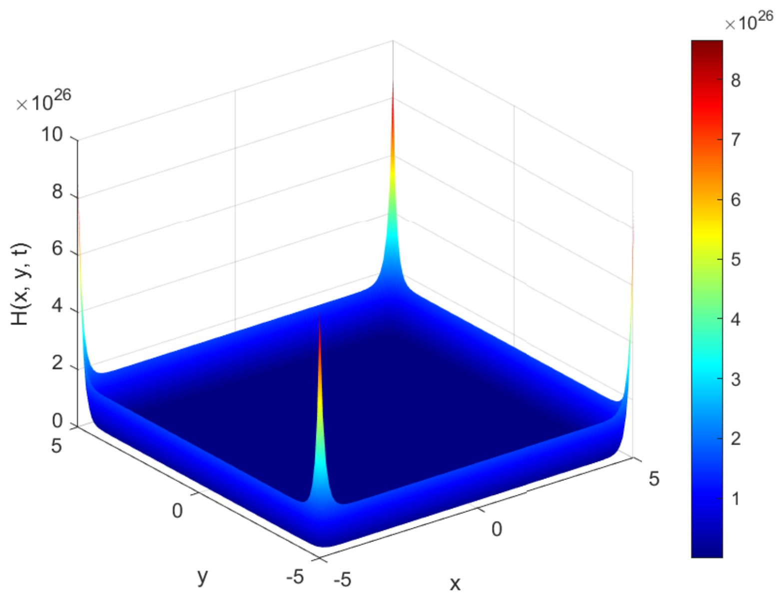

Figure 6. Heat Kernel $H(x, y, t)$ in Example 3

Proposition 4.2. $\overline{H}$ is invariant under the action of Γ in the sense that

$$\overline{H}(g\overline{m}_1, g\overline{m}_2, t) = \overline{H}(\overline{m}_1, \overline{m}_2, t)$$

for any $\overline{m}_1$,$\overline{m}_2 \in \overline{M}$ and $g \in \Gamma$.

Proof. This is a direct consequence of the uniqueness clause for the heat kernel. Indeed, given $g \in \Gamma$ if we define

$$\overline{H}^{'} : M \times M \times \mathbb{R}_{.>0} \longmapsto \mathbb{R},$$

to be,

$$\overline{H}^{'}(\overline{m}_1, \overline{m}_2, t) = \overline{H}(g\overline{m}_1, g\overline{m}_2, t),$$

for $\overline{m}_1, \overline{m}_2 \in \overline{M}$, $\overline{H}^{'}$ satisfies all the conditions for the heat kernel:

$$\Delta|_{m_1}\overline{H}^{'} + \frac{\partial \overline{H}^{'}}{\partial t} = 0.$$

It is obvious by the $\Gamma-$invariance of $\overline{\mathsf{G}}$ and hence that of Δ. Trivially

$$\overline{H}^{'}(\overline{m}_1, \overline{m}_2, t) = \overline{H}^{'}(\overline{m}_2, \overline{m}_1, t).$$

Finally, given any continuous $f : M \longmapsto \mathbb{R}$ we have:

$$\begin{aligned}
\lim_{t\to 0^+} \int_{\overline{M}} \overline{H}^{'}(\overline{m}_1, \overline{m}_2, t) f(\overline{m}_2) \mathrm{d}\mu(\overline{m}_2) &= \lim_{t\to 0^+} \int_{\overline{M}} \overline{H}(g\overline{m}_1, g\overline{m}_2, t) f(\overline{m}_2) \mathrm{d}\mu(\overline{m}_2) \\
&= \lim_{t\to 0^+} \int_{\overline{M}} \overline{H}(g\overline{m}_1, \overline{m}_2, t) f(g^{-1}\overline{m}_2) \mathrm{d}\mu(g^{-1}\overline{m}_2) \\
&= \lim_{t\to 0^+} \int_{\overline{M}} \overline{H}(g\overline{m}_1, \overline{m}_2, t) f(g^{-1}\overline{m}_2) \mathrm{d}\mu(\overline{m}_2) \\
&= f(g^{-1}g\overline{m}_1) = f(\overline{m}_1).
\end{aligned}$$

By the uniqueness of heat kernels, we conclude that, $\overline{H}^{'} = \overline{H}$. □

Proposition 4.3. The heat kernel H on M satisfies

$$H(m_1, m_2, t) = \sum_{g\in\Gamma} \overline{H}(g\overline{m}_1, \overline{m}_2, t),$$

where $\overline{m}_1 \in p^{-1}(\overline{m}_1)$, $\overline{m}_2 \in p^{-1}(\overline{m}_2)$ *provided* that the sum at the right hand side is meaningful.

Proof. It should be noted that the right hand side quantity is well-defined in the sense that it is independent of the choice of $\overline{m}_1 \in p^{-1}(\overline{m}_1)$. Indeed for any $\overline{m}_1{}^{'} \in p^{-1}(\overline{m}_1), \overline{m}_2{}^{'} \in p^{-1}(\overline{m}_2)$ there exist g_1,$g_2 \in \Gamma$ such that $\overline{m}_2{}^{'} = g_2\overline{m}_2$, $\overline{m}_1{}^{'} = g_1\overline{m}_1$ and

$$\begin{aligned}
H(m_1, m_2, t) &= \sum_{g\in\Gamma} \overline{H}(g\overline{m}_1{}^{'}, \overline{m}_2{}^{'}, t) \\
&= \sum_{g\in\Gamma} \overline{H}(gg_1\overline{m}_1, g_2\overline{m}_2, t) \\
&= \sum_{g\in\Gamma} \overline{H}(g_2{}^{-1}gg_1\overline{m}_1, \overline{m}_2, t) \\
&= \sum_{g\in\Gamma} \overline{H}(g\overline{m}_1, \overline{m}_2, t).
\end{aligned}$$

□

Consider $\mathbb{T}_\Lambda = (\mathbb{R}^N/\Lambda, \delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j)$ where $\Lambda = B\mathbb{Z}^N$. Since the heat kernel of $\mathbb{R}^N$ is

$$H_{\mathbb{R}^2}(x, y, t) = \frac{1}{4\pi t} e^{\frac{-|x-y|^2}{4t}},$$

we obtain:

$$H_{\mathbb{T}_\Lambda}(x, y, t) = \frac{1}{4\pi t} \sum_{l\in\Lambda} e^{\frac{-|x-y+l|^2}{4t}}.$$

, as shown in Figure 7
For the flat torus $\mathbb{T}_\Lambda$ the theta function is easily computed to be:

$$\Theta(t) = \sum_{\ell^*\in\Lambda^*} e^{-4\pi|\ell^*|^2 t},$$

which, at long last, justifies our choice of the function F in section 5. The fundamental observation of Sunada [35] is that, when Γ is a finite group,the relationship between H and $\overline{H}$ allows us to obtain the theta function $\Theta = \Theta_M$ of M in terms of $\overline{H}$ and Γ. For any set K, let $\#(K)$ denote the cardinality of K:

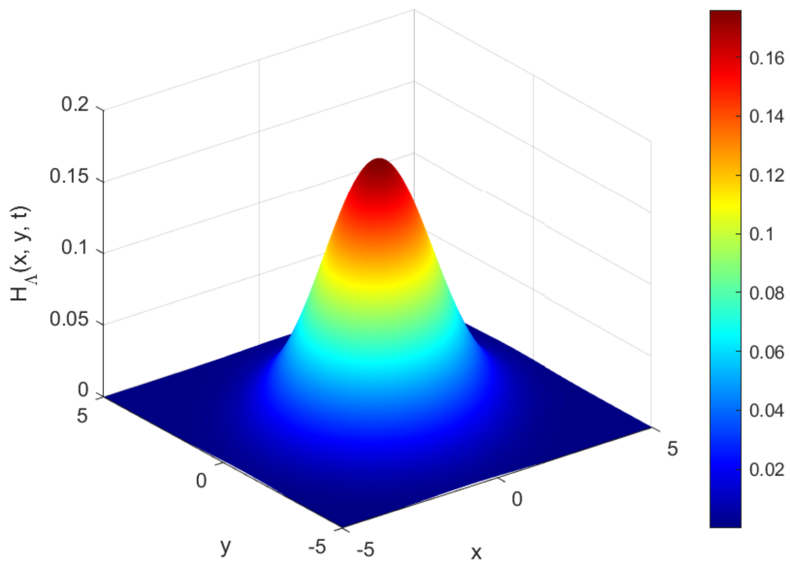

Figure 7. Heat kernel in Example 3

Lemma 4.4.

$$\Theta(t) = \int_M H(m, m, t)\mathrm{d}\mu(m) = \sum_{g\in\Gamma} \frac{1}{\#(\Gamma)} \mathcal{I}_g(t)$$

where,

$$\mathcal{I}_g(t) = \int_{\overline{M}} \overline{H}(g\overline{m}, \overline{m}, t)\mathrm{d}\mu(\overline{m}).$$

Proof. Obvious. It is important to notice that $\mathcal{I}_g(t)$ is constant across conjugacy classes in Γ: That is, given $g, g^{'} \in \Gamma$ with $g^{'} = h^{-1}gh$ for some $h \in \Gamma$, we have,

$$\begin{aligned}
\mathcal{I}_{g'}(t) &= \int_{\overline{M}} \overline{H}(g^{'}\overline{m}, \overline{m}, t)\mathrm{d}\mu(\overline{m}) \\
&= \int_{\overline{M}} \overline{H}(h^{-1}gh\overline{m}, \overline{m}, t)\mathrm{d}\mu(\overline{m}) \\
&= \int_{\overline{M}} \overline{H}(h^{-1}gh\overline{m}, \overline{m}, t)\mathrm{d}\mu(\overline{m}) \\
&= \int_{\overline{M}} \overline{H}(gh\overline{m}, h\overline{m}, t)\mathrm{d}\mu(\overline{m}) \\
&= \int_{\overline{M}} \overline{H}(gh\overline{m}, h\overline{m}, t)\mathrm{d}\mu(h\overline{m}) \\
&= \int_{\overline{M}} \overline{H}(g\overline{m}, \overline{m}, t)\mathrm{d}\mu(\overline{m}) \\
&= \mathcal{I}_g(t)
\end{aligned}$$

Let's denote the set of conjugacy classes in Γ by $Con(\Gamma)$ and put

$$\mathcal{I}[c](t) = \mathcal{I}_g(t)$$

for $c \in Con(\Gamma)$ where $g \in c$. The left hand side is well-defined owing to the constancy of $\mathcal{I}_g$ across conjugacy classes. □

Lemma 4.5.

$$\Theta(t) = \sum_{c \in Con(\Gamma)} \frac{\#(c)}{\#(\Gamma)} \mathcal{I}[c](t).$$

Proof. This is quite simple in view of the previous lemma and the above observations:

$$\begin{aligned}\Theta(t) &= \frac{1}{\#(\Gamma)} \sum_{g \in \mathsf{G}} \mathcal{I}_g(t) \\ &= \frac{1}{\#(\Gamma)} \sum_{c \in Con(\Gamma)} \#(c)\mathcal{I}[c](t) \\ &= \sum_{c \in Con(\Gamma)} \frac{\#(c)}{\#(\Gamma)} \mathcal{I}[c](t).\end{aligned}$$

Given a subgroup $A \leq \Gamma$, let $M_A = \overline{M}/A$, with its canonical Riemannian tensor G_A obtained either by lifting G on $M = \overline{M}/\Gamma$ or by lowering $\overline{\mathsf{G}}$ on $\overline{M}$.
We can now obtain the theta function $\Theta_A(t)$ of (M_A, G_A) by employing similar arguments : □

Lemma 4.6.

$$\Theta_A(t) = \sum_{c \in Con(A)} \frac{\#(c \cap A)}{\#(A)} \mathcal{I}[c](t).$$

Proof. Clearly the heat kernel H_A of $M_A = \overline{M}/A$ satisfies,

$$H_A(m_1, m_2, t) = \sum_{a \in A} \overline{H}(a\overline{m}_1, \overline{m}_2, t),$$

with $\overline{m}_1 \in {p_A}^{-1}(m_1), \overline{m}_1 \in {p_A}^{-1}(m_2)$ where $p_A : \overline{M} \longmapsto M_A = \overline{M}/A$ is the obvious covering projection. Let μ_A denote the Lebesque measure induced on M_A by G_A. We have,

$$\begin{aligned}\Theta_A(t) &= \int_{M_A} H(m, m, t) \mathrm{d}\mu_A(m) \\ &= \frac{1}{\#(A)} \int_{\overline{M}} \sum_{a \in A} \overline{H}(a\overline{m}, a\overline{m}, t) \mathrm{d}\mu(\overline{m}). \\ &= \sum_{a \in A} \frac{1}{\#(A)} \mathcal{I}_a(t).\end{aligned}$$

Once again we notice that $I_a(t)$ is constant across conjugacy classes and consequently,

$$\Theta_A(t) = \sum_{c \in Con(\Gamma)} \frac{\#(c)}{\#(A)} \mathcal{I}[c](t),$$

hence,

$$\Theta_A(t) = \sum_{c \in Con(\Gamma)} \frac{\#(c \cap A)}{\#(A)} \mathcal{I}[c](t).$$

□

Theorem 4.7. Let G be a "bumpy" Riemannian metric on $M = \overline{M}/\Gamma, A, B \leq \Gamma$ be subgroups of Γ, let $M_A = \overline{M}/A, M_B = \overline{M}/B$ with respective natural Riemannian metrics.

(i) M_A is isometric to M_B iff A and B are conjugate subgroups of Γ .

(ii) $\Theta_A(t) = \Theta_B(t)$ if A and B are equivalent in the sense of Gassmann.

Proof. ii is now clear by the observations preceding this theorem. As for i : Each element of $M = \overline{M}/\Gamma$ is an equivalence class under the equivalence relation $\sim_\Gamma$ where $\overline{m}_1 \sim_\Gamma \overline{m}_2$ if $\exists\, g \in \Gamma$ such that $\overline{m}_2 = g\overline{m}_1$. Let's denote the $\sim_\Gamma$ the equivalence class containing $\overline{m}$ by $[\overline{m}]_\Gamma$. Similarly, we introduce the notations $[\overline{m}]_A$, $[\overline{m}]_B$. Suppose first that A and B are conjugate groups.There exists $\gamma \in \Gamma$ such that

$$B = \gamma A \gamma^{-1}.$$

Define $F_\gamma : M_A \longmapsto M_B$ by $F_\gamma([\overline{m}]_A) = [\gamma\, \overline{m}]_B$ First check that this is well defined. Indeed if $\overline{m}_1 \sim_A \overline{m}_2$ then there exists $a \in M$ such that $\overline{m}_2 = a\overline{m}_1$. Hence,

$$\gamma\overline{m}_2 = \gamma a\, \overline{m}_2 = \gamma a \gamma^{-1} \gamma \overline{m}_1.$$

As $\gamma\ a\gamma^{-1} \in \gamma\ A\gamma^{-1} = B$, we conclude $\gamma\overline{m}_2 \sim_B \gamma\overline{m}_1$. Therefore F_γ is well-defined. As $\overline{m} \longmapsto \gamma\ \overline{m}$ is an isometry, so is $F_\gamma : M_A \longmapsto M_B$ an isometry. Suppose, conversely, that M_A, M_B are isometric, that is there exists an isometry $F : M_A \longmapsto M_B$. Let $\hat{p}_A : M_A = \overline{M}/A \longmapsto M = M_\Gamma = \overline{M}/\Gamma$ be the obvious covering projection sending $[\overline{m}]_A$ into $[\overline{m}]_\Gamma$. Similarly define $\hat{p}_B : M_B \longmapsto M$. Take $[\overline{m}]_A \in M_A$ and assume that $F([\overline{m}]_A) = [\overline{m}']_B$. If $m = \hat{p}_A([\overline{m}]_A) = [\overline{m}]_\Gamma \neq [\overline{m}']_\Gamma = \hat{p}_B([\overline{m}]_B) = m'$ then we may choose open disjoint neighbourhoods U,V of m,m' and open neighbourhoods $\hat{U}$,$\hat{V}$ of $[\overline{m}]_A$, $[\overline{m}]_\Gamma$ such that:

$$\hat{p}_A|_{\hat{U}} : \hat{U} \longmapsto U,$$

$$F|_{\hat{U}} : \hat{U} \longmapsto \hat{V},$$

$$\hat{p}_B|_{\hat{V}} : \hat{V} \longmapsto V,$$

are isometries. Consequently,

$$\hat{p}_B|_{\hat{V}} \circ F|_{\hat{U}} \circ (\hat{p}_A|_{\hat{U}})^{-1} : U \longmapsto V$$

is an isometry. This is impossible since G is a "bumpy" metric. We conclude $m = m'$ that is

$$[\overline{m}]_\Gamma = \hat{p}_A([\overline{m}]_A) = \hat{p}_B([\overline{m}']_B) = [\overline{m}']_\Gamma.$$

This being true for any $[\overline{m}]_A \in M_A$,we conclude that F has to be a lifting of the identity map on M, that is

$$\hat{p}_B \circ F = \hat{p}_A$$

Thus for a given $[\overline{m}]_A \in M_A$, if $F([\overline{m}]_A) = [\overline{m}']_B$ then $[\overline{m}]_\Gamma = [\overline{m}']_\Gamma$ and there exists $\gamma \in \Gamma$ with

$$\overline{m}' = \gamma\overline{m}.$$

By the uniqueness of liftings with a given action on a single point we conclude that,

$$F = F_\gamma.$$

and

$$B = \gamma A \gamma^{-1}.$$

□

For any commutative ring R with 1, let $U(R)$ denote the multiplicative group of units in R. Given groups G, H where G acts upon H on the left by $\varphi : G \longmapsto Aut(H)$, let us write for simplicity gh instead of $\varphi(g)(h)$. The semi-direct product $G \ltimes_\varphi H$ (or simply $G \ltimes H$) of G and H consists of pairs $(g, h) \in G \times H$ with the binary operation.

$$(g, h)(g', h') = (gg', h(gh')).$$

The group structure can be routinely checked.
Consider now the group $\mathbb{Z}/8\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$ of integers modulo 8. $U(\mathbb{Z}/8\mathbb{Z}) = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$ acts on $\mathbb{Z}/8\mathbb{Z}$ by multiplication. Consider the group $\Gamma = U(\mathbb{Z}/8\mathbb{Z}) \ltimes \mathbb{Z}/8\mathbb{Z}$. In plain language, Γ is the group consisting of pairs (32 in all!),

$$(x, y) \in \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\} \times \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\},$$

with a binary operation,

$$(x,y)(x',y') = (xx', y + xy').$$

It can be readily checked that $(\bar{1},\bar{0}) \in \Gamma$ is the identity element, that is,

$$(\bar{1},\bar{0})(x,y) = (\bar{1}x, 0 + \bar{1}y) = (x,y).$$

$$(x,y)(\bar{1},\bar{0}) = (x\bar{1}, y + x\bar{0}) = (x,y)$$

and the inverse element,

$$(x,y)^{-1} = (x^{-1}, -x^{-1}y).$$

Now we consider $A, B \leq \Gamma$ where,

$$A = \{\bar{1},\bar{3},\bar{5},\bar{7}\} \times \{\bar{0}\}$$

$$B = \{(\bar{1},0),(\bar{3},\bar{4}),(\bar{5},\bar{4}),(\bar{7},\bar{0})\}.$$

A routine tabulation of the conjugacy classes in Γ shows that A and B are equivalent in the sense of Gassmann.

Let S_n be the symmetric group of degree n which consists of the permutations of n objects. Every finite group G of order n can be embedded as a subgroup of the group of permutations $S_G \simeq S_n$ of the carrier set of G by means of the so-called Cayley representation which is a group monomorphism.

$$G \overset{i}{\longmapsto} S_G,$$

where $i(g)$ is defined to be:

$$i(g)(x) = g(x),$$

for all $x \in G$.
Let p be a prime number and consider the groups:

$$A = \mathbb{Z}/\mathrm{p}\mathbb{Z} \oplus \mathbb{Z}/\mathrm{p}\mathbb{Z} \oplus \mathbb{Z}/\mathrm{p}\mathbb{Z}.$$

$$B = \{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \mid x,y,z \in \mathbb{Z}/\mathrm{p}\mathbb{Z}\} \quad .$$

which are both finite groups of order p^3 which we consider to be subgroups of S_{p^3} by means of the above described procedure.
Clearly the order of each non-zero element of A is p. The same holds for the elements of B since,

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & na & nb + \dfrac{n(n-1)}{2}ac \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{bmatrix},$$

which can be obtained by a simple induction as,

$$\begin{bmatrix} 1 & na & nb + \dfrac{n(n-1)}{2}ac \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^n$$

$$= \begin{bmatrix} 1 & (n+1)a & (n+1)b + \dfrac{n(n+1)}{2}ac \\ 0 & 1 & (n+1)c \\ 0 & 0 & 1 \end{bmatrix}.$$

This means that all non-identity elements of $A, B \leq S_{p^3}$ are products p^2 disjoint cycles of length p. Consequently, given $g \in S_{p^3}$, either $g = e$ and

$$\#([g] \cap A) = \#([g] \cap B) = 1$$

or $g \neq e$ and

$$\#([g] \cap A) = \#([g] \cap B) = p^3 - 1 \quad .$$

Therefore A, B are subgroups of S_{p^3} which are equivalent in the sense of Gassmann. On the other hand, let alone being conjugate, A, B are not even isomorphic since A is Abelian but B is not.

Let, $n \geq 3 \quad G = SL(n, \mathbb{Z}/\mathrm{p}\mathbb{Z})$ and consider:

$$A = \{[a_{ij}]_{1 \leq i,j \leq n} \in SL(n, \mathbb{Z}/\mathrm{p}\mathbb{Z}) \mid a_{i1} = 0 \text{ for } i \geq 2\}$$

$$B = \{[b_{ij}]_{1 \leq i,j \leq n} \in SL(n, \mathbb{Z}/\mathrm{p}\mathbb{Z}) \mid b_{1j} = 0 \text{ for } j \geq 2\} \quad .$$

Let

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in (\mathbb{Z}/\mathrm{p}\mathbb{Z})^n \quad .$$

Clearly $x \in A$ iff $xa = \lambda a$ for some $\lambda \in \mathbb{Z}/\mathrm{p}\mathbb{Z} - \{0\}$. In other words, $x \in A$ is a (right) eigenvector of x. Similarly $y \in B$ iff $a^T y = \mu a^T$ for some $\mu \in \mathbb{Z}/\mathrm{p}\mathbb{Z} - \{0\}$. Again, this means that $y \in B$ iff a^T occurs as a (left) eigenvector of y. Since, for any $g \in SL(n, \mathbb{Z}/\mathrm{p}\mathbb{Z})$ a occurs as a right eigenvector of xgx^{-1} if and only if a^T occurs as a left eigenvector of $(x^{-1})^T g$. These show that A and B are not conjugate but equivalent in the sense of Gassmann. Since any finite group is known to arise as the fundamental group of a compact, smooth manifold of dimension 4 (in fact as the fundamental group of a compact complex projective algebraic surface) [34], in view of theorem 4.7, examples 3, 3 and 3 above provides us with isospectral but non-isometric Riemannian manifolds.

5 Isospectral Deformations on Nilmanifolds

Having produced isospectral Riemannian manifolds which are not isometric, it is natural to ask whether it is possible to find continuous families of isospectral manifolds which are naturally isometrically distinct. Rephrased in the manner of Kac, we ask, whether it is possible for a Riemannian manifold to change its shape continuously while sounding the same. To be precise, the problem is to find a continuously parametrised family G_t of Riemannian metrics on a manifold M for $t \in [0, 1]$ such that (M, G_t) and $(M, \mathsf{G}_{t'})$ are isospectral for any $t, t' \in [0, 1]$, isometric only when $t = t'$. The manifolds (M, G_t) are said to constitute *isospectral deformations* from G_0 to G_1. To emphasize the requirement that (M, G_t) and $(M, \mathsf{G}_{t'})$ are not isometric for $t \neq t'$, one may talk about *non-trivial* isospectral deformations. Non-trivial isospectral deformations are known not to exist on flat tori[31] and the so called Heisenberg manifolds[32] and compact manifolds of negative sectional curvature [24, 28] In this section we present a method allied to that of Sunada by means of which it is possible to produce non-trivial isospectral deformations on a special but large class of Riemannian manifolds. Given a Lie group G with Lie algebra $\mathcal{G}$, the group $Aut(G)$ of Lie group automorphisms of G is a Lie group. If G is connected then $Aut(G)$ can be naturally immersed in $Aut[\mathcal{G}]$. If G is connected and simply connected then $Aut(G)$ may be naturally identified with $Aut[\mathcal{G}]$. The Lie algebra of $Aut[\mathcal{G}]$ and hence that of $Aut(G)$ can be identified with the algebra $Der[\mathcal{G}]$ of *derivations* of $\mathcal{G}$, that is, of vector space endomorphisms.

$$\varphi : \mathcal{G} \longmapsto \mathcal{G}$$

such that:

$$\varphi([X,Y]) = [\theta X, Y] + [X, \theta Y] \quad .$$

$\varphi \in Aut(G)$ is said to be an *inner automorphism* if there exists $g \in \mathcal{G}$ such that

$$\varphi(x) = gxg^{-1},$$

for all $x \in G$. We shall denote such $\varphi \in Aut(G)$ by i_g. It can be readily checked that inner automorphisms of G constitute a normal Lie subgroup $Inn(G)$ of G. The Lie algebra of $Inn(G)$ can be identified with the ideal $ad[\mathcal{G}]$ of $Der[\mathcal{G}]$ which consists of derivations of the form $ad[X]$, for $X \in \mathcal{G}$ where $ad[X] : \mathcal{G} \longmapsto \mathcal{G}$ is defined by:

$$ad[X](Y) = [X, Y] \quad .$$

The fact that $ad[X]$ is a derivation is tantamount to the Jacobi identity for Lie brackets. If $Z, \mathfrak{z}$ denote the centers of $G, \mathcal{G}$ respectively, that is,

$$Z = \{x \in G \mid xg = gx \quad \forall\, g \in G\},$$

$$\mathfrak{z} = \{X \in \mathcal{G} \mid [X,Y] = 0 \quad \forall\, Y \in \mathcal{G}\},$$

then it can be easily checked that,

$$Inn(G) \simeq G/Z$$

$$ad[\mathcal{G}] \simeq \mathcal{G}/\mathfrak{z} \quad .$$

Let's abbreviate the matrix

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix},$$

by $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and consider the Heisenberg group of degree 1 defined by:

$$G = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R}\}.$$

It can be easily checked that the Lie algebra $\mathcal{G}$ of G is generated by the left-invariant vector fields,

$$X = \frac{\partial}{\partial x},$$

$$Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},$$

$$Z = \frac{\partial}{\partial z},$$

with

$$[X,Y] = Z,$$

$$[Y,Z] = [Z,X] = 0.$$

Clearly,

$$Z = \{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \mid z \in \mathbb{R}\} \;\leq\; G.$$

$$\mathfrak{z} =< Z > \leq \mathcal{G}$$

It can be readily checked that each $\theta \in Der[\mathcal{G}]$ is of the form

$$\theta(X) = aX + bY + cZ$$

$$\theta(Y) = a^{'}X + b^{'}Y + c^{'}Z$$

$$\theta(Z) = (a + b^{'})Z.$$

Therefore

$$dim(Aut(G)) = dimAut[\mathcal{G}] = dim(Der[\mathcal{G}]) = 6$$

whereas

$$dim(Inn(G)) = dim(Ad[\mathcal{G}])$$
$$= dim(G) - dim(Z) = dim(\mathcal{G}) - dim(\mathfrak{z})$$
$$= 3 - 1 = 2 \quad .$$

Proceeding more directly we can easily check that each $\varphi \in Aut[\mathcal{G}]$ is of the form:

$$(*) \begin{cases} \varphi(X) = aX + bY + cZ \\ \varphi(Y) = a^{'}X + b^{'}Y + c^{'}Z \\ \varphi(Z) = (ab^{'} - a^{'}b)Z \end{cases} ,$$

where $ab^{'} - a^{'}b \neq 0$, from which we observe once again that,

$$dim(Aut[\mathcal{G}]) = dim(Aut(G)) = dim(Der[\mathcal{G}]) = 6 \quad .$$

In this particular example where G is connected, simply connected and nilpotent, the relationship between G and $\mathcal{G}$ is very simple. The maps:

$$exp : \mathcal{G} \longmapsto \mathcal{G}$$

$$log : G \longmapsto \mathcal{G}$$

are diffeomorphisms and can be explicitly given by:

$$exp(aX + bY + cZ) = \begin{bmatrix} a \\ b \\ c + \frac{1}{2}ab \end{bmatrix},$$

$$log(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = xX + yY + (z - \frac{xy}{2})Z.$$

This allows us to write down the general form of the automorphisms of G: Indeed each $F \in Aut(G)$ is of the form $F = F_{\varphi} \in Aut(G)$ where,

$$F_{\varphi} = exp \circ \varphi \circ log,$$

for some $\varphi \in Aut(G)$. Explicitly, we employ the general form of φ as given in $(*)$ and put

$$F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = exp \circ \varphi \circ log(\begin{bmatrix} x \\ y \\ z \end{bmatrix}),$$

$$= exp \circ \varphi[xX + yY + (z - \frac{xy}{2})Z]$$
$$= exp[x(aX + bY + cZ) + y(a^{'}X + b^{'}Y + c^{'}Z) + (z - \frac{xy}{2})(ab^{'} - a^{'}b)Z]$$
$$= exp\{(ax + a^{'}y)X + (bx + b^{'}y)Y + [cx + c^{'}y + \Delta(z - \frac{xy}{2})]Z\},$$

where $\Delta = ab^{'} - a^{'}b$. Thus,

$$F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} ax + a^{'}y \\ bx + b^{'}y \\ cx + c^{'}y + \Delta z - (ab^{'} - a^{'}b)\frac{xy}{2} + \frac{1}{2}abx^2 + \frac{1}{2}ab^{'}xy + \frac{1}{2}a^{'}bxy + \frac{1}{2}a^{'}b^{'}y^2 \end{bmatrix}$$
$$= \begin{bmatrix} ax + a^{'}y \\ bx + b^{'}y \\ cx + c^{'}y + \Delta z + \frac{1}{2}(abx^2 + 2a^{'}bxy + a^{'}b^{'}y^2) \end{bmatrix}.$$

[23, 20]

Let's abbreviate the matrix,

$$\begin{bmatrix} 1 & x_1 & x_2 & z_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 & z_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

by

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix},$$

and consider the Lie group,

$$G = \{ \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix} \mid x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}\}.$$

It can be readily checked that,

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \\ z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} x_1 + x_1' \\ x_2 + x_2' \\ y_1 + y_1' \\ y_2 + y_2' \\ z_1 + z_1' + x_1 y_1' + x_2 y_2' \\ z_2 + z_2' + x_1 y_2' \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix}^{-1} = \begin{bmatrix} -x_1 \\ -x_2 \\ -y_1 \\ -y_2 \\ -z_1 + x_1 y_1 + x_2 y_2 \\ -z_2 + x_1 y_2 \end{bmatrix} .$$

The Lie algebra $\mathcal{G}$ of G is generated by the left invariant vector fields:

$$X_1 = \frac{\partial}{\partial x_1},$$

$$X_2 = \frac{\partial}{\partial x_2},$$

$$Y_1 = \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial z_1},$$

$$Y_2 = \frac{\partial}{\partial y_2} + x_2 \frac{\partial}{\partial z_1} + x_1 \frac{\partial}{\partial z_2},$$

$$Z_1 = \frac{\partial}{\partial z_1},$$

$$Z_2 = \frac{\partial}{\partial z_2},$$

which obey:

$$[X_1, Y_1] = Z_1,$$

$$[X_2, Y_2] = Z_1,$$

$$[X_1, Y_2] = Z_2,$$

all the remaining brackets being zero. It can be readily checked that,

$$Z = \{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ z_1 \\ z_2 \end{bmatrix} \mid z_1, z_2 \in \mathbb{R} \},$$

and

$$\mathfrak{z} =< Z_1, Z_2 > \quad .$$

Consequently

$$dim\ Inn(G) = dim\ Ad[\mathcal{G}] = dim G/Z = dim\ \mathcal{G}/\mathfrak{z} = 6 - 2 = 4.$$

It is not easy to express the elements of $Aut(G)$ explicitly. On the other hand, it is relatively easier to investigate the elements of $Aut(\mathcal{G})$ and to conclude that,

$$dim(Aut(G)) = dim(Aut[\mathcal{G}]) = dim(Aut[\mathcal{G}]) = 22 \quad .$$

An inner automorphism i_g of G is of the form:

$$i_g(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix}) = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 + a_1 y_1 + a_2 y_2 - b_1 x_1 - b_2 x_2 \\ z_2 + a_1 y_2 - b_2 x_1 \end{bmatrix},$$

where,

$$g = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix} \quad .$$

The logarithm and the exponential can be computed routinely :

$$exp[a_1 X_1 + a_2 X_2 + b_1 Y_1 + b_2 Y_2 + c_1 Z_1 + c_2 Z_2]$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 + \frac{1}{2}(a_1 b_1 + a_2 b_2) \\ c_2 + \frac{1}{2} a_1 b_2 \end{bmatrix} \quad .$$

$$log(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix}) = x_1 X_1 + x_2 X_2 + y_1 Y_1 + y_2 Y_2 + (z_1 - \frac{x_1 y_1 + x_2 y_2}{2}) Z_1 + (z_2 - \frac{x_1 y_2}{2}) Z_2.$$

Intuitively each derivation $\theta \in Der[\mathcal{G}]$ of $\mathcal{G}$ is an"infinitesimal" automorphism of $\mathcal{G}$, hence of G. Similarly elements of $Ad[\mathcal{G}]$ may be regarded as "infinitesimal" inner automorphisms. Now, let us revisit the case of finite G. Let $\varphi : G \longmapsto G$ be an automorphism with the property that for each $g \in G$, there exists $x = x(g) \in G$ with $\varphi(g) = xgx^{-1}$. It can be easily checked that for any subgroup $A \leq G$, A is equivalent in the sense of Gassmann to $\varphi(A) \leq G$. An obvious analogue of the above situation can be formulated as follows: [23] Given a Lie group G with Lie algebra $\mathcal{G}$, $\varphi \in Aut(G)$ is said to be an *almost inner automorphism* if for each $g \in G$, there exists $x = x(g) \in G$ such that $\varphi(g) = xgx^{-1}$. A derivative $\xi \in Der[\mathcal{G}]$ is called an *almost inner derivative* if for each $Y \in \mathcal{G}$, there exists $X = X(Y) \in \mathcal{G}$ such that

$$\xi(Y) = [X, Y] \quad .$$

We denote the set of almost inner automorphisms of G by $AIA(G)$, the set of almost inner derivatives of $\mathcal{G}$ by $AID(\mathcal{G})$ Clearly

$$Inn(G) \subseteq AIA(G) \subseteq Aut(G),$$

$$Ad[\mathcal{G}] \subseteq AID[\mathcal{G}] \subseteq Der[\mathcal{G}].$$

The Heisenberg group admits no non-trivial almost inner automorphism. To be precise, each almost inner automorphism in the Heisenberg group is an inner automorphism. To see this, note that for any

$$g = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix},$$

$$\begin{aligned} i_g(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) &= \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} -\alpha \\ -\beta \\ -\gamma + \alpha\beta \end{bmatrix} \\ &= \begin{bmatrix} \alpha + x \\ \beta + y \\ \gamma + z + \alpha y \end{bmatrix} \begin{bmatrix} -\alpha \\ -\beta \\ -\gamma + \alpha\beta \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \\ z + \alpha y + \alpha\beta + (\alpha + x)(-\beta) \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \\ z + \alpha y - \beta x \end{bmatrix}. \end{aligned}$$

Given any automorphism φ of G defined by:

$$\varphi(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} ax + a^{'}y \\ bx + b^{'}y \\ cx + c^{'}y + \Delta z + \frac{1}{2}[abx^2 + 2a^{'}bxy + a^{'}b^{'}y^2] \end{bmatrix}.$$

In order for φ to be an almost inner automorphism it is clear that $a = b^{'} = 1, a^{'} = b = 0$ in which case

$$\varphi = i_g,$$

where g may be chosen to be of the form:

$$g = \begin{bmatrix} c^{'} \\ -c \\ \gamma \end{bmatrix}.$$

for arbitrary $\gamma \in \mathbb{R}$.

In the six-dimensional nilpotent Lie group of Example 2 it is fairly easy to work out the general form of an almost inner derivative : Since all Lie brackets take their values in $\mathfrak{z} =< Z_1, Z_2 >$, we conclude that for an almost inner derivative $\delta \in AID[\mathfrak{z}],\ \delta(X) \in \mathfrak{z}$ for all $X \in \mathcal{G}$. On the other hand $\delta\mid_{\mathfrak{z}} \equiv 0$. Therefore an almost inner derivative δ must have the form:

$$\delta : \begin{cases} X_1 \longmapsto a_{11}Z_1 + a_{12}Z_2 \\ X_2 \longmapsto a_{21}Z_1 + a_{22}Z_2 \\ Y_1 \longmapsto b_{11}Z_1 + b_{12}Z_2 \\ Y_2 \longmapsto b_{21}Z_1 + b_{22}Z_2 \\ Z_1 \longmapsto 0 \\ Z_2 \longmapsto 0 \end{cases}$$

However, as Lie brackets containing X_2 or Y_1 can take values only in $< Z_1, Z_2 >$ we conclude that $a_{22} = b_{22} = 0$. It is easy to check now that an endomorphism δ of $\mathcal{G}$ is an almost inner derivative iff δ is of the form:

$$\delta : \begin{cases} X_1 \longmapsto a_{11}Z_1 + a_{12}Z_2 \\ X_2 \longmapsto a_{21}Z_1 \\ Y_1 \longmapsto b_{11}Z_1 \\ Y_2 \longmapsto b_{21}Z_1 + b_{22}Z_2 \\ Z_1 \longmapsto 0 \\ Z_2 \longmapsto 0 \end{cases}$$

This shows that $AID[\mathcal{G}]$ is a six dimensional subalgebra of $Der[\mathcal{G}]$. In view of the four linearly independent inner derivatives,

$$Ad[X_1] : \begin{cases} X_1 \longmapsto 0 \\ X_2 \longmapsto 0 \\ Y_1 \longmapsto Z_1 \\ Y_2 \longmapsto Z_2 \\ Z_1 \longmapsto 0 \\ Z_2 \longmapsto 0 \end{cases}$$

$$Ad[X_2] : \begin{cases} X_1 \longmapsto 0 \\ X_2 \longmapsto 0 \\ Y_1 \longmapsto 0 \\ Y_2 \longmapsto Z_1 \\ Z_1 \longmapsto 0 \\ Z_2 \longmapsto 0 \end{cases}$$

$$Ad[Y_1] : \begin{cases} X_1 \longmapsto -Z_1 \\ X_2 \longmapsto 0 \\ Y_1 \longmapsto 0 \\ Y_2 \longmapsto 0 \\ Z_1 \longmapsto 0 \\ Z_2 \longmapsto 0 \end{cases}$$

$$Ad[Y_2] : \begin{cases} X_1 \longmapsto -Z_2 \\ X_2 \longmapsto -Z_1 \\ Y_1 \longmapsto 0 \\ Y_2 \longmapsto 0 \\ Z_1 \longmapsto 0 \\ Z_2 \longmapsto 0 \end{cases}$$

it is clear that,

$$AID[\mathcal{G}] =< Ad[X_1], Ad[X_2], Ad[Y_1], Ad[Y_2], \delta, \varepsilon >$$

where,

$$\delta : \begin{cases} X_1 \longmapsto Z_2 \\ X_2 \longmapsto 0 \\ Y_1 \longmapsto 0 \\ Y_2 \longmapsto 0 \\ Z_1 \longmapsto 0 \\ Z_2 \longmapsto 0 \end{cases}$$

$$\varepsilon : \begin{cases} X_1 \longmapsto 0 \\ X_2 \longmapsto 0 \\ Y_1 \longmapsto 0 \\ Y_2 \longmapsto Z_2 \\ Z_1 \longmapsto 0 \\ Z_2 \longmapsto 0 \end{cases}$$

It can be directly checked that automorphisms of G obtained by exponentiation from elements of $AID[\mathcal{G}]$ are almost inner automorphisms. At this stage, we content ourselves by noticing that the maps

$$\varphi = exp \circ exp_{\mathcal{G}}(\delta) \circ log = exp \circ (I + \delta) \circ log$$

$$\psi = exp \circ exp_{\mathcal{G}}(\varepsilon) \circ log = exp \circ (I + \varepsilon) \circ log$$

are both almost inner automorphisms. Indeed,

$$\varphi(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix}) = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 + x_1 \end{bmatrix} = i_g(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix}),$$

where,

$$g = \begin{bmatrix} 0 \\ 0 \\ x_2/x_1 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

if $x_1 \neq 0$, otherwise $g = e_G$.
Again

$$\psi(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix}) = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 + y_2 \end{bmatrix} = i_h(\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix}),$$

where

$$h = \begin{bmatrix} 1 \\ -y_1/y_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

if $y_2 \neq 0$, otherwise $h = e_G$.

Theorem 5.1. Almost inner automorphisms of G constitute a normal subgroup of $Aut(G)$.

Proof. Clearly $I_{d_G} \in AIA(G)$. Suppose $\varphi, \psi \in Aut(G)$. Given $x \in G$, $\exists \ g = g(x)$ and $h = h(x)$ such that

$$\varphi(x) = gxg^{-1}$$
$$\psi(x) = hxh^{-1} \quad .$$

hence

$$\psi \circ \varphi(x) = \psi(\varphi(x))$$
$$= \psi(gxg^{-1})$$
$$= \psi(g)\psi(x)\psi(g)^{-1}$$
$$= \psi(g)hx(\psi(g)h)^{-1}.$$

This being true for arbitrary $x \in G$, it is seen that $\psi \circ \varphi \in AIA(G)$. Therefore $AIA(G) \leq Aut(G)$.

As for normality:

Given $\varphi \in AIA(G)$ and $\alpha \in Aut(G)$, consider $x \in G$ and choose $g = g(\alpha^{-1}(x))$ Thus

$$\alpha \circ \varphi \circ \alpha^{-1}(x) = \alpha \circ \varphi(\alpha^{-1}(x))$$
$$= \alpha(g\alpha^{-1}(x)g^{-1})$$
$$= \alpha(g)\ x\ \alpha(g)^{-1}.$$

This being true for each $x \in G$, we conclude that $\alpha \circ \varphi \circ \alpha^{-1} \in AIA(G)$. Consequently $AIA(G) \trianglelefteq Aut(G)$ □

Theorem 5.2. Let $\mathcal{G}^*$ stands for the dual of $\mathcal{G}$. Given a simply connected Lie group $\mathcal{G}$ and $\varphi \in Aut(G)$, the following are equivalent:

(i) φ is an almost inner automorphism.

(ii) For each $X \in \mathcal{G}$ there exists $g = g(x) \in G$ such that

$$\varphi_*(X) = (i_g)_*(X).$$

(iii) For each $\xi \in \mathcal{G}^*$ there exists $g = g(\xi) \in G$ such that

$$\xi \circ \varphi_* = \zeta \circ (i_g)_*$$

Proof. $(i) \Longrightarrow (ii)$: Given $X \in \mathcal{G}$

$$\varphi_*(X)(e) = T_e\varphi(X(e)) = \frac{\mathrm{d}}{\mathrm{d}t}\,|_{t=0}\ \varphi(\gamma(t)),$$

where $\gamma : (-\varepsilon, \varepsilon) \longrightarrow G$ is any smooth path with $\gamma(0) = e$ and $\dot{\gamma}(0) = X(e)$. We choose $\gamma(t), t \in \mathbb{R}$ to be the one-parameter subgroup of G generated by $X \in \mathcal{G}$. There exists $g \in G$ such that

$$\varphi(\gamma(1)) = i_g(\gamma(1)).$$

It can be checked that

$$\varphi(\gamma(1)) = i_g(\gamma(t)),$$

for all $t \in \mathbb{R}$. Consequently

$$\varphi(X) = (i_g)_*(X).$$

$(ii) \Longrightarrow (iii)$: Choose a left-invariant Riemannian metric G on G. Equivalently G may be understood to be an innerproduct on $\mathcal{G}$. Given $\xi \in \mathcal{G}^*$, there exists unique $X \in \mathcal{G}$ such that

$$\xi = \mathsf{G}(X, .).$$

Therefore, for any $Y \in \mathcal{G}$,

$$\xi \circ \varphi_*(Y) = \mathsf{G}(X, \varphi_*(Y))$$
$$= \mathsf{G}(\varphi_*(X), Y)$$
$$= \mathsf{G}((i_g)_*(X), Y)$$

for some $g = g(X) \in G$. This being true for arbitrary $Y \in \mathcal{G}$, we find

$$\xi \circ \varphi_* = \xi \circ (i_g)_*.$$

It is obvious that the dual of this argument allows us to conclude that $(iii) \Longrightarrow (ii)$.

$(ii) \Longrightarrow (i)$: Consider $x \in G$. Choose $X \in \mathcal{G}$ such that $\gamma(1) = x$ where γ(t) is the one-parameter subgroup generated by $X \in \mathcal{G}$. Choose $g = g(X) \in G$ with

$$\varphi_*(X) = (i_g)_*(X).$$

It can now be checked that $\varphi(\gamma(t))$ is the one parameter subgroup generated by $(i_g)_*(X)$ and

$$\varphi(x) = \varphi(\gamma(1)) = g\gamma(1)g^{-1} = gxg^{-1}.$$

□

Theorem 5.3. In a connected, simply connected Lie group G, $AIA(G)$ is a Lie subgroup of $Aut(G)$.

Proof. By a standard theorem of E. Cartan a closed subgroup of a real Lie group is a Lie subgroup [29]. We have already shown that $AIA(G)$ is a subgroup (in fact a normal subgroup) of $Aut(G)$. Therefore, it will be sufficient to show that $AIA(G)$ is closed in $Aut(G)$. To this end notice that G is nilpotent and there exists $m \in \mathbb{N}$ such that for all $X \in \mathcal{G}$, $Ad[X]^m \equiv 0$. Consequently $exp(Ad[X])$ is a polynomial in $Ad[X]$ which has an order independent of $X \in \mathcal{G}$. As a result, the orbits of $Inn(G)$ on $\mathcal{G}$ are closed in $\mathcal{G}$. Since the orbits of $AIA(G)$ on $\mathcal{G}$ coincide with those of $Inn(G)$ we conclude that the orbits of $AIA(G)$ on $\mathcal{G}$ are closed in $\mathcal{G}$. Therefore $AIA(G)$ is closed in $Aut(G) \simeq Aut[\mathcal{G}]$.

□

Theorem 5.4. Given a connected, simply connected nilpotent Lie group G, the Lie subalgebra corresponding to the Lie subgroup $AIA(G)$ of almost inner derivations.

Proof. Assume G is of nilpotence length m. Let $\{\mathcal{G}^k\}_{k=0}^m$ be the central series of $\mathcal{G}$ consisting of the iterated derived subalgebras $\mathcal{G}^k$ defined inductively by

$$\mathcal{G}^0 = \mathcal{G} \ldots \mathcal{G}^{k+1} = [\mathcal{G}, \mathcal{G}^k].$$

Consider any $\varphi \in AIA(G)$. Clearly,

$$(\varphi_* - I)\mathcal{G}^k \subseteq \mathcal{G}^{k+1},$$

for each $k \geq 0$. In other words, $\varphi_* - I$ is nilpotent on $\mathcal{G}$. If $\varphi = exp_{\mathcal{G}} D$ for some $D \in Der[\mathcal{G}]$. Again we have,

$$D\mathcal{G}^k \subseteq \mathcal{G}^{k+1},$$

for each $k \geq 0$. Given $X \in \mathcal{G}$, there exists $Y \in \mathcal{G}$ such that,

$$\varphi_* X = exp_{\mathcal{G}}(Ad[Y])X.$$

Consequently,

$$exp_{\mathcal{G}}(-Ad[Y])exp_{\mathcal{G}}(D)X = X. \quad (*),$$

Now, remember that $Ad[\mathcal{G}]$ is an ideal in $Der[\mathcal{G}]$,in fact $[D, Ad[Y]]_{\mathcal{G}} = Ad[D(y)]$. Employing the Hausdorff-Campbell formula on the left hand side of $(*)$ we obtain

$$exp_{\mathcal{G}}(D - Ad[Z])X = X, \qquad (**),$$

where $Z = Y - \frac{1}{2}D(Y) + \cdots$ which terminates after finitely many terms involving powers of D. But, $D - Ad[Z]$ is again nilpotent on $\mathcal{G}$ and $f(D - Ad[Z])$ is invertible where $f : \mathbb{C}\mathbb{C}$ is the entire function defined by

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{for z} \neq 0 \\ 1 & \text{for z} = 0 \,. \end{cases}$$

By $(**)$ we have,

$$f(D - Ad[Z])(D - Ad[Z])X$$
$$= exp_{\mathcal{G}}(D - Ad[Z])X = X,$$

from which we conclude by the invertibility of $f(D - Ad[Z])$ that,

$$(D - Ad[Z])X = 0,$$

or equivalently

$$DX = Ad[Z]X,$$

which, being true for arbitrary X implies that,

$$D \in AID[\mathcal{G}].$$

We conclude that,

$$AIA(G) \subseteq exp(AID[\mathcal{G}]).$$

A similar argument can be employed to reverse this inclusion.

□

Remark 5.5. The proof of the above theorem, rephrased in slightly greater detail will allow us to observe that for a nilpotent Lie group G of nilpotence length m, $AIA(G)$ is a nilpotent group of nilpotence length at most $m - 1$. A similar result is valid for $AID[\mathcal{G}]$.

Remark 5.6. In view of the fact that $AIA(G)$ is normal, we conclude that $AID[\mathcal{G}]$ is a Lie ideal in $Der[\mathcal{G}]$. A *nilmanifold* M is a quotient manifold of right cosets $M = \Gamma\backslash G$ where G is a simply connected nilpotent Lie group, Γ is a discrete cocompact subgroup of G.

Notice that a right coset of Γ in G is nothing but an orbit of the action of Γ on G by multiplication on the left. Consequently a tangent vector of $M = \Gamma\backslash G$ can be regarded as an equivalence class of tangent vectors on G where $u \in T_xG$ and $v \in T_yG$ are are understood to be equivalent if there exists $\gamma \in \Gamma$ such that

$$y = \gamma x,$$

and

$$v = T_xL_g(u).$$

The equivalence class containing $u \in T_xG$ which we denote by $[u]_\Gamma$ can be identified with a tangent vector at $\Gamma_x \in M = \Gamma\backslash G$. Thus, a left-invariant Riemannian metric G on G will naturally induce a Riemannian metric G_Γ on $M = \Gamma\backslash G$ defined by

$$\mathsf{G}_\Gamma([u]_\Gamma, [v]_\Gamma) = G(u, v).$$

By an obvious and convenient use of notation we shall simply write u , G instead of $[u]_\Gamma, \mathsf{G}_\Gamma$ in the sequel.
A *Riemannian nilmanifold* is a pair $(\Gamma\backslash G, \mathsf{G})$ where G is a Riemannian metric which is induced by a left invariant one on G in the manner described above.

Lemma 5.7. On a Lie group G with left invariant metric G, the Laplacian $\Delta = \Delta_{\mathsf{G}}$ is of the form

$$\Delta f = \sum_{i=1}^{n}\{-E_iE_if + (\nabla_{E_i}E_i)f\},$$

for each $f \in \mathcal{D}(G)$, where $\{E_i\}_{1\leq i\leq n}$ is any basis for $\mathcal{G}$ that is orthonormal with respect to G and ∇ is the Levi-Civita connection attached to G.

Proof. Notice that for any smooth $f : G \longmapsto \mathbb{R}$ we have

$$grad(f) = \sum_{i=1}^{n}(E_if)E_i \quad .$$

Now, given $A = A^i E_i \in \mathcal{X}(G)$,

$$\begin{aligned} div(A) &= trace\{E_r \nabla_{E_r}(A^i E_i)\} \\ &= trace\{E_r(E_r A^i)E_i + (A^i \Gamma_{r_i}{}^m)E_m\}, \end{aligned}$$

where we put $\Gamma_{r_i}{}^m E_m$ for $\nabla_{E_r}(E_i)$. Thus we conclude,

$$div(A) = E_r A^r + A^i \Gamma_{r_i}{}^r.$$

The result follows from the observation $\Gamma_{ij}{}^k = -\Gamma_{ik}{}^j$ for any $1 \leq j, k \leq n$, which can be derived by noticing that $\nabla \mathsf{G} = 0$, G is left invariant and hence

$$G(\nabla_{E_i} E_j, E_k) + G(E_j, \nabla_{E_i} E_k) = 0.$$

□

The following theorems are best derived by means of standard but heavy techniques. We offer sketches of proofs.

Theorem 5.8. If G is a bi-invariant Riemannian metric on the nilpotent Lie group G, then the Riemannian nilmanifolds $(\Gamma\backslash G, \mathsf{G})$ and $(\Gamma\backslash G, \varphi^* G)$ are isospectral for each $\varphi \in AIA(G)$.

Proof. When G on the nilpotent Lie group G is biinvariant(such tensors exist on G since G admits cocompact subgroups!) then the Levi-Civita connection ∇ obeys $\nabla_X Y = 0$ for all $X, Y \in \mathcal{G}$ and Δ reduces to the form

$$\Delta f = -\sum_{i=1}^{n} E_i E_i f,$$

for each $f \in C(G)$ where $\{E_i\}_{1 \leq i \leq n}$ is any basis for $\mathcal{G}$ which is orthonormal with respect to G. By Kirillov's theory on the unitary representations of nilpotent Lie groups [33], each unitary irreducible representation of G on the Hilbert space of functions which are square integrable by the Lebesque measure induced by G, are parametrised by elements of $\mathcal{G}$, any almost inner automorphism induces well-behaved unitary transformations between these. Since, in the presence of a biinvariant metric, the effect of each $X \in \mathcal{G}$ on $\mathcal{D}(G)$ can be expressed in terms of irreducible representations, the same applies to the above mentioned E_i's and to Δ.

□

Theorem 5.9. Riemannian nilmanifolds $(\Gamma\backslash G, \mathsf{G}_1)$ and $(\Gamma\backslash G, \mathsf{G}_2)$ are isometric iff there exists $g \in G$ and $\varphi \in Aut(G)$ with $\varphi(\Gamma) = \Gamma$ such that

$$\mathsf{G}_2 = (i_g \circ \varphi)^* \mathsf{G}_1.$$

Proof. Suppose $G_2 = (i_g \circ \varphi)^* G_1$ where $g \in G$ and $\varphi \in Aut(G)$ with $\varphi(\Gamma) = \Gamma$. Clearly

$$i_g \circ \varphi = L_g \circ R_{g^{-1}} \circ \varphi,$$

where L_x and R_y represent multiplications in G by x and y on the left and right respectively. G_1 being left invariant we find

$$G_2 = (R_{g^{-1}} \circ \varphi)^* G_1.$$

But $R_{g^{-1}} \circ \varphi = \overline{f}$ is the lifting of a map $f : \Gamma\backslash G \longmapsto \Gamma\backslash G$ which is an isometry from $(\Gamma\backslash G, \mathsf{G}_1)$ to $(\Gamma\backslash G, \mathsf{G}_2)$.

Conversely suppose

$$f : (\Gamma\backslash G, \mathsf{G}_1) \longmapsto (\Gamma\backslash G, \mathsf{G}_2)$$

is an isometry. f lifts to an isometry

$$\overline{f} : (\Gamma\backslash G, \mathsf{G}_1) \longmapsto (\Gamma\backslash G, \mathsf{G}_2).$$

By a standard result of [13] $\mathsf{G}_1 = \Psi^*\mathsf{G}_2$, for some $\Psi \in Aut(G)$ with $\Psi(\Gamma) = \Gamma$. Thus $\overline{f} \circ \Psi^{-1}$ is an isometry of $(\Gamma\backslash G, \mathsf{G}_2)$. Let

$$\sigma = L_g \circ \overline{f} \circ \Psi^{-1},$$

where $g^{-1} = \overline{f} \circ \Psi^{-1}(e) \in \Gamma$. Therefore σ is an isometry of $(\Gamma\backslash G, \mathsf{G}_2)$. with $\sigma(e) = e$. Again from [14, 15], we conclude that $\sigma \in Aut(G)$. Consequently

$$\sigma \circ \Psi = L_g \circ R_{g^{-1}}(R_g \circ \overline{f}).$$

$$= i_g \circ (R_g \circ \overline{f}).$$

But $R_g \circ \overline{f}(\Gamma) = \Gamma$.

□

Remark 5.10. It is possible to give a direct but not quite as natural a proof for theorem 5.9 in the case where G is of nilpotence length 2[20].

Remark 5.11. The theorems 5.8 and 5.9 indicate that on nilmanifolds like that in Example 5 on which inner automorphisms have sufficiently large codimension inside the group of almost inner automorphisms, one can trivially obtain non-trivial isospectral deformations.

6 Conclusion

This study highlighted that the spectrum gives fairly detailed and complete information on large families of Riemannian manifolds such as flat tori, Heisenberg spaces and manifolds of negative sectional curvature. This is what should perhaps be called the realm of “spectral rigidity”. Moreover, away from spectral rigidity, (which we consider to be “generic”) that is where the possibility of spectral deformations take over, a meagre category of highly structured manifolds await exploration. It would be interesting to develop criteria distinguishing these situations more clearly.

7 Competing interests

The authors declare no competing interests regarding the publication of this manuscript.

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