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On the unique solvability of the absolute value matrix equation and its numerical solution

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Abstract In this paper, we deal with the unique solvability and numerical solution of the generalized absolute value matrix equation (GAVME) $AX - B|X| = C, (A, B, C, X \in \mathbb{R}^{n\times n})$. For its unique solvability some sufficient conditions are given. On the other hand, for its numerical solution, Picard's fixed point iterative method is proposed to compute an approximated solution of some uniquely solvable GAVME problems where its globally linear convergence is guaranteed. Finally, some numerical results are given to confirm the efficiency of our suggested approach for solving the GAVME.

1 Introduction

In this paper, we consider the generalized absolute value matrix equation (abbreviated as GAVME) of type:

$$AX - B|X| = C, \tag{1.1}$$

where A, B, C are given matrices in $\mathbb{R}^{n\times n}$, $|X|$ denotes the absolute value of the unknown matrix solution X. The GAVME is a generalization form of the following generalized absolute value equations (GAVE)

$$Ax - B|x| = b, \tag{1.2}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ is the unknown variable. When $B = I$ the identity matrix then GAVE becomes

$$Ax - |x| = b. \tag{1.3}$$

The importance of absolute value equations GAVEs is due to their broad applications in many mathematics and applied sciences domains. For instance, the linear complementarity problem, bimatrix games, mixed integer programming, system of linear interval matrix, and convex quadratic optimization can be formulated as GAVEs. Because of that reason, GAVEs attract the attention of researchers in this field. For the unique solvability of the GAVEs (1.2), there are many different types of conditions, we cite the most well-known established results until today. In [10], Mangasarian and Meyer presented a sufficient condition, namely, $1 < \sigma_{\min}(A)$ for GAVE. In [12], Rohn generalized this result to the unique solvability of GAVE where he imposed the following sufficient condition:

$$\sigma_{\max}(|B|) < \sigma_{\min}(A),$$

where $\sigma_{\max}(|B|)$ denotes the maximal singular value of matrix $|B| = (|b_{ij}|)$ and the $\sigma_{\min}(A)$ denotes the smallest singular values of matrix A. Furthermore, Lotfi and Veiseh [9], imposed other sufficient conditions that if the following matrix

$$A^T A - \||B|\|^2 I,$$

is positive definite, then GAVE (1.2) is uniquely solvable for any $b \in \mathbb{R}^n$.

In [2] Achache and Anane, have weakened the conditions of Rohn, Lotfi and, Veiseh, in assuming that the GAVE (1.2) satisfies the following sufficient conditions:

(i) $\sigma_{\min}(A) > \sigma_{\max}(B)$,

(ii) $\|A^{-1}B\| < 1$, provided A is non singular,

(iii) The matrix $A^T A - \|B\|^2 I$ is positive definite, then the GAVE (1.2) is uniquely solvable for any b.

In [14] Shubham. K and Deepmala present a sufficient condition for the unique solvability of the GAVME. They provided if $\rho(|A^{-1}| |B|) < 1$, then GAVME has an unique solution for every matrix C. It is worth mentioning that no numerical results are given by them.

In this paper, on the one hand, to guarantee the unique solvability of the GAVME (1.1). we extend those conditions given by [2] for GAVE (1.2). On the other hand, for its numerical solution, we propose a simple Picard's iterative method [8, 18], to compute numerically an approximated solution for some uniquely solvable GAVME problems. The globally linear convergence of the latter from any starting initial point is guaranteed via the sufficient condition $\|A^{-1}B\| < 1$, provided A is nonsingular. Finally, some numerical results are given to confirm the efficiency of our proposed approach for solving the GAVME.

At the end of this section, some notations are presented. Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices. The scalar product and the Euclidean norm are denoted, respectively, by $x^T y, x, y \in \mathbb{R}^n$ and $\|x\| = \sqrt{x^T x}$. Recall that a subordinate matrix norm for $A \in \mathbb{R}^{n\times n}$ is defined as follows: $\|A\| := \max\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$, this definition implies:

$$\|Ax\| \le \|A\| \|x\|, \; \|AB\| \le \|A\| \|B\|, \; \forall A, B \in \mathbb{R}^{n\times n} \text{ and } x \in \mathbb{R}^n.$$

The $sign(x)$ denotes a vector with the components equal to $-1, 0$ or 1 depending on whether the corresponding component is negative, zero, or positive. In addition, $D(x) := Diag(sign(x))$ will denote a diagonal matrix corresponding to $sign(x)$. The absolute value of a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and the vector of all ones are denoted by $|A| = (|a_{ij}|) \in \mathbb{R}^{n\times n}$ and $e \in \mathbb{R}^n$, respectively. $\sigma_{\min}(A)$, $\sigma_{\max}(A)$ represent, respectively, the smallest and the largest singular value of matrix A. As is well known, $\sigma^2_{\min}(A) = \min_{\|x\|=1} x^T A^T Ax$, and $\sigma^2_{\max}(A) = \max_{\|x\|=1} x^T A^T Ax$.

The remaining part of the paper is organized as follows. The main results are stated in Section 2. In Section 3, Picard's iterative method is suggested to provide an approximated solution for GAVME (1.1). In Section 4, some numerical results are reported. A conclusion is drawn in Section 5.

2 Main results

In this section, for our main result, the following Lemma is required.

Lemma 2.1. *If matrices A and B satisfy the following conditions*

(i) $\sigma_{\min}(A) > \sigma_{\max}(B)$,

(ii) $\|A^{-1}B\| < 1$, *provided A is non singular,*

(iii) The matrix $A^TA - \|B\|^2 I$ is positive definite.

Then the matrix $A - BD$ is non singular for all diagonal matrix D whose elements are ± 1 and 0.

Proof. The proof is similar to the one given in [2] □

Theorem 2.2. *If matrices A and B satisfy the following conditions*

(i) $\sigma_{\min}(A) > \sigma_{\max}(B)$,

(ii) $\|A^{-1}B\| < 1$, *provided A is non singular,*

(iii) The matrix $A^TA - \|B\|^2 I$ is positive definite, then the GAVME (1.1) is uniquely solvable for any matrix C.

Proof. To prove our main results, we may partition the matrices X, $|X|$ and C as follows: $X = (x^1, \cdots, x^n)$, $|X| = (|x^1|, |x^2|, \cdots, |x^n|)$ and $C = (c^1, \cdots, c^n)$ where x^l, $|x^l|$ and c^l are the l-th column of the matrices X, $|X|$ and C, respectively. Then the GAVME (1.1) can be formulated as l vectorial absolute value equations (GAVE (1.2)):

$$Ax^l - B\left|x^l\right| = c^l, l = 1, \cdots, n. \tag{2.1}$$

According to $D(x^l)x^l = |x^l|$ where $D =: Diag(sign(x^l))$, each equation in (2.1) can be rewritten as the following linear system of equations:

$$(A - BD)x^l = c^l, l = 1, \cdots, n, \tag{2.2}$$

for all diagonal matrix D with its components are ± 1 and 0. So it is clear that (2.1) is uniquely solvable if the system (2.2) has a unique solution, i.e., if the matrix $(A - BD)$ is non singular. Applying Lemma 2.1, for each equation l, then GAVME (1.1) is uniquely solvable for any C. This completes the proof. □

3 Picard's iterative method

In this section, to provide an approximated solution of some uniquely solvable GAVME problems, a simple Picard's fixed point iterative method is proposed. First, we state the Banach fixed point theorem which will be used for proving the convergence of the proposed method, one can see [6] for its details proof.

Theorem 3.1. *(Banach's fixed point theorem). Let (X, d) be a non-empty complete metric space, $0 \le \alpha < 1$ and $T : X \to X$ a mapping satisfying*

$$d(T(x), T(y)) \le \alpha d(x, y), \text{ for all } x, y \in X.$$

Then there exists a unique $x \in X$ such that $T(x) = x$. Furthermore, x can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $\{x_k\}$ by

$$x_{k+1} = T(x_k),$$

then

$$\lim_{k\mapsto\infty} x_k = x,$$

and the following inequalities hold:

$$d(x, x_{k+1}) \le \frac{\alpha}{1-\alpha} d(x_{k+1}, x_k),\ d(x, x_{k+1}) \le \alpha d(x, x_k).$$

Next, for solving the equation GAVME (1.1), we solve n equations of the following form:

$$Ax^l - B\left|x^l\right| = c^l,\ l = 1, \cdots, n. \tag{3.1}$$

based on the fixed point principle, the sequence of iterations for solving (3.1) is given by

$$x^l_{k+1} = A^{-1}B\left|x^l_k\right| + A^{-1}c^l,\ k = 0, 1, 2, \ldots \tag{3.2}$$

Next under the condition 2 (Theorem 1)[2], we provide a sufficient condition for the linearly global convergence of the fixed point iterations (3.2).

Now, we can formally describe the corresponding point fixed algorithm for solving the GAVME (1.1) as follows:

Algorithm

Input:
An accuracy $\epsilon > 0$;
for $l = 1, 2, \cdots, n$;
an initial starting point $x_0^l \in \mathbb{R}^n$;
given matrices A, B and C in $\mathbb{R}^{n\times n}$;
set $k := 0$;
while $\|x_{k+1}^l - x_k^l\| \geq \epsilon$ **do**
begin
compute x_k^l from the linear system $x_{k+1}^l = A^{-1}B\left|x_k^l\right| + A^{-1}c^l, l = 1, \cdots, n$;
update $k := k + 1$;
end.

A Picard's fixed point algorithm for the GAVME.

Theorem 3.2. *Let A be a nonsingular matrix and if*

$$\left\|A^{-1}B\right\| < 1,$$

then the sequence $\{x_k^l\}$ converges to the unique solution $x_\star^l$ of the GAVE for any arbitrary $x_0^l \in \mathbb{R}^n$. In this case the error bound is given by

$$\left\|x_{k+1}^l - x_\star^l\right\| \leq \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|}\left\|x_{k+1}^l - x_k^l\right\|. \tag{3.3}$$

Moreover, the sequence $\{x_k^l\}$ converges linearly to $x_\star^l$ as follows

$$\left\|x_{k+1}^l - x_\star^l\right\| \leq \left\|A^{-1}B\right\|\left\|x_k^l - x_\star^l\right\|, \; k = 0, 1, 2, \ldots \tag{3.4}$$

Proof. First, if the condition $\|A^{-1}B\| < 1$ holds then Theorem 2.1, implies that the GAVME (1.1) is uniquely solvable for any square matrix C. Next, to prove the convergence for the sequence $\{x_k^l\}$ to $x_\star^l$, we define the function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\varphi(x^l) = A^{-1}B\left|x^l\right| + A^{-1}b, l = 1, \cdots, n.$$

Then, it is easy to see with the help of the following inequality

$$\||x| - |y|\| \leq \|x - y\|, \text{ for all } x, y \in \mathbb{R}^n,$$

that

$$\|\varphi(x) - \varphi(y)\| \leq \left\|A^{-1}B\right\|\|x - y\|, \text{ for all } x, y \in \mathbb{R}^n.$$

Using Theorem 3.1 with $X = \mathbb{R}^n, T = \varphi, d(x, y) = \|x - y\|$ for all $x, y \in \mathbb{R}^n$ and $\alpha = \|A^{-1}B\| < 1$, we deduce the convergence of the sequence $\{x_k^l\}_{l=1,2,\ldots,n}$ given by

$$x_{k+1}^l = \varphi(x_k^l), \; k = 0, 1, 2, \ldots$$

to the unique fixed point $x_\star^l$ to $\varphi(x^l)$ which is in turn the unique solution of the GAVME (1.1). Moreover, the inequalities (3.3) and (3.4) hold which lead to the the linearly global convergence of the method. This completes the proof. □

4 Numerical experiments

In this section, we present some examples of GAVME problems where their unique solvability is checked. Also by applying Picard's iterative method, we compute an approximated solution of these examples. Our implementation is done by using the software **MATLAB 7.9** and carried out on a personal PC where we set $\epsilon = 10^{-8}$. The starting matrix and the unique solution of the GAVME are denoted, respectively, by X_0 and $X_\star$, where x_0^l and $x_\star^l$ are the l-th column of the matrices X_0 and $X_\star$, respectively. In the tables of numerical results, we display the following notations: "Iter" and "CPU" state for the number of iterations and the elapsed times. The termination of the algorithm is when the following stopping criterion:

$$\left\|x_{k+1}^l - x_k^l\right\| \leq \epsilon$$

holds where $\epsilon > 0$ is a given accuracy. Further, we can take the residue $RSD = \|Ax_k^l - B|x_k^l| - c_k^l\|$.

Example 1. Consider the GAVME (1.1) where $A, B \in \mathbb{R}^{3\times 3}$ are given by:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -4 & 4 & 0 \\ 0 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}.$$

By Theorem 2.1, and with the help of Matlab, we get $\|A^{-1}B\| = 0.7017 < 1$, so this problem is uniquely solvable for any matrix C. For this example, tha matrix C is given by

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and th initial starting matrix is defined as follows:

$$X_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The obtained numerical results are summarized in Table 1

Iter	CPU(s)	RSD
16	0.007161	$9.5430e-008$

Table 1.

The approximated unique solution of this problem is given by

$$X_\star = \begin{bmatrix} -0.2000 & 0 & 0 \\ -0.1200 & 0.3333 & 0 \\ 0.1500 & -0.1250 & 1 \end{bmatrix}.$$

Example 2. Consider the following GAVME problem, with $A, B \in \mathbb{R}^{5\times 5}$ are given by

$$A = \begin{bmatrix} 8 & 0 & -1 & 1 & -20 \\ 1 & 1 & 1 & 4 & 25 \\ 1 & -5 & 0 & 8 & -10 \\ 0 & 8 & 1 & -6 & 1 \\ 3 & 5 & -3 & 0 & 10 \end{bmatrix}, B = \begin{bmatrix} -1.5 & 0 & 1.5 & 0.5 & 0.1 \\ 0 & 0.25 & 1 & 0 & 0.5 \\ 1 & 0.6 & 1 & 0.4 & 0.5 \\ 0 & 0.3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In this example, the matrices A and B are not symmetric. Further, $\left\|A^{-1}B\right\| = 0.6859 < 1$, which implies that the problem has a unique solution for any given $C \in \mathbb{R}^{5\times 5}$. It is worth to notice that the condition given by [14] is not satisfied since $\rho(\left|A^{-1}\right| |B|) = 1.2308 > 1$. This implies that our condition is more reliable in detecting the unique solvability for this example. Here, the matrix C is given by

$$C = \begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ -1 & 1 & 2 & 1 & 0 \\ 8 & 7 & 1 & 3 & 2 \\ 0.5 & 3 & 9 & 1 & 3 \\ 2 & -1 & 0 & -2 & 1 \end{bmatrix}.$$

The initial starting matrix is defined as follows:

$$X_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix},$$

.

The iterations number and the CPU time are summarized in Table 2.

Iter	CPU(s)	RSD
35	0.006729	$5.9890e-009$

Table 2.

The unique approximated solution of this problem is given by:

$$X_* = \begin{bmatrix} -1.5915 & -0.6566 & 0.1199 & -0.1978 & 0.2450 \\ 2.7282 & 3.4302 & 3.7566 & 1.5069 & 1.5671 \\ 0.0556 & 2.5104 & 3.5740 & 1.6113 & 1.1984 \\ 2.8574 & 3.2663 & 2.7790 & 1.4819 & 1.2590 \\ -0.5053 & -0.5483 & -0.4727 & -0.2298 & -0.2532 \end{bmatrix}.$$

Example 3. In this problem, $A, B \in \mathbb{R}^{10\times 10}$ are given by

$$A = \begin{bmatrix} 101 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 102 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 103 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 104 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 105 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 106 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 107 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 108 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 109 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 110 \end{bmatrix}, B = I.$$

Here, $\|A^{-1}B\| = 0.0099 < 1$, which implies that this problem is also uniquely solvable for any matrix $C \in \mathbb{R}^{10\times 10}$. For

$$C = \begin{bmatrix} 109 & 109 & 109 & 109 & 109 & 109 & 109 & 109 & 109 & 109 \\ 108 & 108 & 108 & 108 & 108 & 108 & 108 & 108 & 108 & 108 \\ 107 & 107 & 107 & 107 & 107 & 107 & 107 & 107 & 107 & 107 \\ 106 & 106 & 106 & 106 & 106 & 106 & 106 & 106 & 106 & 106 \\ 105 & 105 & 105 & 105 & 105 & 105 & 105 & 105 & 105 & 105 \\ 104 & 104 & 104 & 104 & 104 & 104 & 104 & 104 & 104 & 104 \\ 103 & 103 & 103 & 103 & 103 & 103 & 103 & 103 & 103 & 103 \\ 102 & 102 & 102 & 102 & 102 & 102 & 102 & 102 & 102 & 102 \\ 101 & 101 & 101 & 101 & 101 & 101 & 101 & 101 & 101 & 101 \\ 100 & 100 & 100 & 100 & 100 & 100 & 100 & 100 & 100 & 100 \end{bmatrix},$$

the initial starting matrix is defined as follows:

$$X_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}.$$

The obtained iterations number and the elapsed times are summarized in Table 3.

Iter	CPU(s)	RSD
5	0.0065307	$6.7567e-008$

Table 3.

The unique approximated solution of this problem is given by:

$$X_\star = \begin{bmatrix} 1.0199 & 1.0199 & 1.0199 & 1.0199 & \ldots & 1.0199 & 1.0199 \\ 1.0201 & 1.0201 & 1.0201 & 1.0201 & \ldots & 1.0201 & 1.0201 \\ 1.0203 & 1.0203 & 1.0203 & 1.0203 & \ldots & 1.0203 & 1.0203 \\ 1.0205 & 1.0205 & 1.0205 & 1.0205 & \ldots & 1.0205 & 1.0205 \\ 1.0207 & 1.0207 & 1.0207 & 1.0207 & \ldots & 1.0207 & 1.0207 \\ -1.0016 & -1.0016 & -1.0016 & -1.0016 & \ldots & -1.0016 & -1.0016 \\ 0.9826 & 0.9826 & 0.9826 & 0.9826 & \ldots & 0.9826 & 0.9826 \\ 0.9824 & 0.9824 & 0.9824 & 0.9824 & \ldots & 0.9824 & 0.9824 \\ 0.9822 & 0.9822 & 0.9822 & 0.9822 & \ldots & 0.9822 & 0.9822 \\ 0.9821 & 0.9821 & 0.9821 & 0.9821 & \ldots & 0.9821 & 0.9821 \end{bmatrix}.$$

Example 4. Consider the GAVME (1.1) where $A, B, C \in \mathbb{R}^{n\times n}$ are given by:

$$A = \begin{bmatrix} 5 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 5 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 5 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 5 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 2 & \cdots & 0 & 0 \\ 0 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 2 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 4 & -2 & \cdots & \cdots & -2 & -2 \\ 3 & 4 & -2 & \cdots & -2 & -2 \\ 3 & 3 & \ddots & -2 & \cdots & -2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 3 & 3 & \cdots & \cdots & 4 & -2 \\ 3 & 3 & \cdots & \cdots & 3 & 4 \end{bmatrix}.$$

The initial starting matrix is defined as follows:

$$X_0 = \begin{bmatrix} 5 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 5 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 5 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 5 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 5 \end{bmatrix}.$$

The obtained numerical results for different size of n, are summarized in Table 4:

Size (n)	Iter	CPU(s)	RSD
7	39	0.007291	8.4976e-009
100	165	1.309203	8.9322e-009
500	226	10:896324	7.4673e-009
1000	233	86:611503	9.1917e-009
1500	235	624:745192	9.9543e-009
2000	237	1165.224035	9.9110e-009

Table 4.

For $n = 7$, the unique approximated solution of this problem is given by:

$$X_* = \begin{bmatrix} 1.7946 & 0.3975 & -0.1499 & -0.3254 & -0.2698 & -0.2938 & -0.2880 \\ 1.5892 & 1.7951 & 0.5504 & 0.0237 & -0.1905 & -0.1186 & -0.1360 \\ 0.7810 & 1.3914 & 1.8761 & 0.5592 & 0.0238 & -0.2035 & -0.1599 \\ 0.1647 & 0.5840 & 1.4020 & 1.9439 & 0.6268 & 0.0648 & -0.1722 \\ -0.1308 & 0.0687 & 0.5663 & 1.4572 & 2.0116 & 0.6181 & -0.0314 \\ -0.1075 & -0.1374 & 0.0698 & 0.5713 & 1.4916 & 1.8959 & 0.3555 \\ -0.1460 & -0.1374 & -0.1958 & -0.0145 & 0.5724 & 1.2072 & 1.2858 \end{bmatrix}.$$

Example 5. Consider the GAVME (1.1) where $A, B \in \mathbb{R}^{n\times n}$ are given by:

$$A = \begin{bmatrix} 50 & 5 & 0 & \cdots & 0 & 0 \\ 5 & 50 & 5 & \cdots & 0 & 0 \\ 0 & 5 & 50 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 50 & 5 \\ 0 & 0 & 0 & \cdots & 5 & 50 \end{bmatrix}, \; B = \begin{bmatrix} -25.5 & -2.5 & 0 & \cdots & 0 & 0 \\ -2.5 & -25.5 & -2.5 & \cdots & 0 & 0 \\ 0 & -2.5 & -25.5 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -25.5 & -2.5 \\ 0 & 0 & 0 & \cdots & -2.5 & -25.5 \end{bmatrix}.$$

Applying Theorem 2.1, we have, $\|A^{-1}B\| = 0.5122 < 1$, then this problem is uniquely solvable for any $C \in \mathbb{R}^{n\times n}$. For

$$C = \begin{bmatrix} 83 & 83 & \cdots & \cdots & 83 & 83 \\ 90.5 & 90.5 & \cdots & \cdots & 90.5 & 90.5 \\ 90.5 & 90.5 & \ddots & 90.5 & \cdots & 90.5 \\ \vdots & \vdots & \ddots & \ddots & 90.5 & 90.5 \\ 90.5 & 90.5 & \cdots & \cdots & 90.5 & 90.5 \\ 83 & 83 & \cdots & \cdots & 83 & 83 \end{bmatrix}.$$

The initial starting matrix is defined as follows:

$$X_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n \end{bmatrix}.$$

The obtained numerical results for different size of n, are summarized in Table 5.

Size (n)	Iter	CPU(s)	RSD
10	29	0.007484	$9.2122e-008$
100	33	0.198235	$5.6576e-008$
500	35	12.866245	$7.4837e-008$
800	36	45.073777	$6.6896e-008$
1000	36	84.541511	$8.1442e-008$
2000	37	584.544999	$7.9778e-008$
2700	38	1345.011038	$5.7970e-008$

Table 5.

The obtained approximated solution of this problem is given by:

$$X_\star = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Example 6. Consider the GAVME (1.1) where $A, B \in \mathbb{R}^{n\times n}$ are given by:

$$A = \begin{bmatrix} 6 & 0.5 & 0.5 & \dots & 0.5 & 0 \\ 0.5 & 6 & 0.5 & \dots & 0.5 & 0 \\ 0.5 & 0.5 & 6 & \dots & 0.5 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & \dots & 6 & 0 \\ 0 & 0 & \dots & 0 & 0 & 6 \end{bmatrix}, \ B = \begin{bmatrix} -1 & 0.5 & 0.5 & \dots & 0.5 & 0 \\ 0.5 & -1 & 0.5 & \dots & 0.5 & 0 \\ 0.5 & 0.5 & -1 & \dots & 0.5 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}.$$

This problem is uniquely solvable for any matrix $C \in \mathbb{R}^{n\times n}$, since $\left\|A^{-1}B\right\| = 0.2727 < 1$. For

$$C = \begin{bmatrix} 7 & 7 & \cdots & \cdots & 7 & 7 \\ 7 & 7 & \cdots & \cdots & 7 & 7 \\ 7 & 7 & \ddots & 7 & \cdots & 7 \\ \vdots & \vdots & \ddots & \ddots & 7 & 7 \\ 7 & 7 & \cdots & \cdots & 7 & 7 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}.$$

The initial starting matrix is given by:

$$X_0 = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

The obtained numerical results for different size of n, are summarized in Table 6.

Size (n)	Iter	CPU(s)	RSD
10	16	0.007875	$8.5818e-008$
40	58	0.051512	$7.9557e-008$
500	675	160.873560	$9.7311e-008$
1000	1344	4160.793441	$9.9917e-008$

Table 6.

The unique approximated solution to this problem is given by:

$$X_\star = \begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 1 & \ddots & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 & 1 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

5 Conclusion

In this paper, we have presented some weaker sufficient conditions that guarantee the unique solvability of the generalized absolute value matrix equation. Numerically, the proposed Picard's iterative method is efficient for providing an approximated solution of some uniquely solvable GAVME.

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