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Numerical solution of Fredholm integro-differential equations using finite element method

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Abstract The aim of this work is to apply the standard Galerkin finite element method to solve the linear integro-differential equation using Lagrange shape functions. In this approach, the approximate solution is sought in a finite-dimensional space, reducing the solution of the given problem to the solution of a linear system of algebraic equations. Finally, we illustrate examples that prove the reliability and efficiency of our method.

1 Introduction

Linear integro-differential equations are important in many branches of functional analysis, applied mathematics and engineering sciences for example, in physics, chemistry, biology and mechanics. In general the solution of the linear integro-differential equations is difficult analytically, therefore a numerical method is required.

Many different types of methods are used to obtain the numerical solution of linear integro-differential equations, such as Biazar and Porshokouhi [7], which use the Adomian decomposition method to solve initial or boundary conditioned values of the equation. Hosseini and Shahmorad [12], replaced the operator matrix representation for the differential part of the equation using the operational Tau method. Danfu and Xufeng [9], used the CAS wavelet approximating technique to simplify the integro-differential equation into algebraic equations. Darania and Ebadian [10], applied the differential transform method based on Taylor series expansion. Kajani et al. [14], compared the homotopy perturbation method with the sine–cosine wavelet method. Atabakan et al [1], presented the composite Chebyshev finite difference method. Jafri et al. [13], obtained the operational matrix of derivative by introducing hybrid third-kind Chebyshev polynomials and Block-pulse functions. Ghomanjani [11], used the bezier curve method. Linz [17], used Nystrom's method to establish numerical procedures. Yusufoglu [19], applied an improvement of HPM to initial value problems. Mredula and Vakaskar [18], used the wavelet collocation method on differential equation. Benyoucef and Rahmoune [5] proposed a Legendre spectral collocation method for the numerical solution of a class of linear Fredholm integro-differential equations on the half-line. Atkinson et al. [2, 3, 4], tried to purify some results of the discrete Galerkin method and the discrete iterated Galerkin method for Fredholm integral equations.

In this paper, we apply the finite element method of degree one or two to solve the boundary value problem of Fredholm integro-differential equations of first order (FIDEs) with non-homogeneous conditions. In other words, looking for a function $u : [0, 1] \to \mathbb{R}$ such that,

$$(CP)\begin{cases} -u'(x) + \displaystyle\int_0^1 k(x,t)u(t)dt = f(x), \quad 0 < x < 1, \\ u(0) = \alpha, \ u(1) = \beta, \end{cases} \tag{1.1}$$

where k and f are given continuous functions on $[0,1]\times[0,1]$ and $[0,1]$ respectively, u is unknown function to be determined. writing the abbreviation of Problem (1.1) by,

$$\begin{cases} -\mathcal{A}u+\mathcal{T}u=f \\ u(0)=\alpha,\ u(1)=\beta, \end{cases} \tag{1.2}$$

where $\mathcal{A}$ is the differential operator and $\mathcal{T}$ is the integral one, given by

$$\mathcal{A}u(x)=u'(x),\quad \mathcal{T}u(x)=\int_0^1 k(x,t)u(t)dt,\quad x,t\in[0,1]. \tag{1.3}$$

2 Variational formulation

The conditions of Problem (1.1) are non-homogeneous. Firstly, we transform this problem into a problem with homogeneous conditions. Put $u(x)=\tilde{u}(x)+(\beta-\alpha)x+\alpha$, by replacing in (1.1), we obtain a homogeneous problem

$$(\widetilde{CP})\begin{cases} -\tilde{u}'(x)+\displaystyle\int_0^1 k(x,t)\tilde{u}(t)dt=g(x),\quad 0<x<1, \\ \tilde{u}(0)=\tilde{u}(1)=0, \end{cases} \tag{2.1}$$

where

$$g(x)=f(x)+(\beta-\alpha)-\int_0^1 k(x,t)\left[(\beta-\alpha)t+\alpha\right]dt.$$

Multiplying the equation in (2.1) by a test function $v\in H_0^1\left(]0,1[\right)$, vanish at the end-points of the interval $]0,1[$ and integrate, we get

$$-\int_0^1 \tilde{u}'(x)v(x)dx+\int_0^1\left(\int_0^1 k(x,t)\tilde{u}(t)dt\right)v(x)dx=\int_0^1 g(x)v(x)dx, \tag{2.2}$$

this expression is called variational formulation of Problem (2.1), and due to the Dirichlet boundary conditions, we seek the unknown function $\tilde{u}$ in the space $H_0^1\left(]0,1[\right)$. Next, let $\mathbb{V}=H_0^1(]0,1[)$ be a Hilbert space, we note the bilinear form $a(\tilde{u},v)$ on $\mathbb{V}\times\mathbb{V}$ by

$$a(\tilde{u},v)=-\int_0^1 \tilde{u}'(x)v(x)dx+\int_0^1\left(\int_0^1 k(x,t)\tilde{u}(t)dt\right)v(x)dx, \tag{2.3}$$

and the linear form $l(v)$ on $\mathbb{V}$ by

$$l(v)=\int_0^1 g(x)v(x)dx. \tag{2.4}$$

Hence, the variational formulation reads

$$(\widetilde{VP})\begin{cases} \text{Find } \tilde{u}\in\mathbb{V},\ \text{such that} \\ a(\tilde{u},v)=l(v),\ \forall v\in\mathbb{V} \\ \tilde{u}(0)=\tilde{u}(1)=0. \end{cases} \tag{2.5}$$

3 Lagrange Finite element P_1 and P_2

The approximation of variational problem $(\widetilde{VP})$ using Lagrange finite element of degree one or two, i.e, approximation by Lagrange polynomial of degree equal to one (LFE1) or equal two (LFE2) respectively.

Let N be a positive integer and $i=0,1,\ldots,N$. Define the points $x_0,x_1,\ldots,x_N$ on the interval $[0,1]$ such that $0=x_0<x_1<\cdots<x_N=1$. The points x_i are called vertices. We also

introduce N sub-intervals $I_i = [x_{i-1}, x_i]$, $i = 1, 2, \dots, N$, where I_i are called elements. The size of the elements is given by $h_i = x_i - x_{i-1}$ and we denote $h = \max\ h_i$.

$$\begin{cases} x_0 = 0,\ x_N = 1, \\ h = \frac{1}{N+1}, \\ x_{i+1} = x_i + h,\ i = 0, 1, \dots, N-1, \\ x_{i+\frac{1}{2}} = x_i + \frac{h}{2},\ i = 0, 1, \dots, N-1. \end{cases}$$

Let $\mathbb{V}_h^d$ denote the vector space of all piecewise linear ($d = 1$) or quadratic ($d = 2$) continuous functions v_h defined on $[0, 1]$,

$$\mathbb{V}_h^d = \left\{ v_h :\ v_h \in H^1 \text{ and } v_h|_{I_i} \in P_d(I_i) \right\}, \qquad \text{for } d = 1 \text{ or } 2, \tag{3.1}$$

and $\mathbb{V}_{h,0}^d$ denote the vector space of $\mathbb{V}_h^d$ with $v_h(0) = v_h(1) = 0$,

$$\mathbb{V}_{h,0}^d = \left\{ v_h \in \mathbb{V}_h^d :\ v_h(0) = v_h(1) = 0 \right\}, \qquad \text{for } d = 1 \text{ or } 2, \tag{3.2}$$

where the dimension of the space $\mathbb{V}_h^d$ is finite (i.e. equal to $dN + 1$).
Let $\varphi_{i/d}$ be basis of the space $\mathbb{V}_h^d$, satisfying,

$$\varphi_{i/d}^d(x_{j/d}) = \delta_{i/d, j/d}, \qquad i, j = 0, 1, \dots, N.d,$$

where $\delta_{..}$ is the Kronecker delta.

The basis functions $\left\{ \varphi_{i/d}^d \right\}_{i=0}^{dN}$ characterizing the nodes of the mesh on the interval $[0, 1]$, are defined by the formula

- For $d = 1$

$$\varphi_i^1(x) := \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x \in I_i, \\ \frac{x - x_{i+1}}{x_i - x_{i+1}} & \text{if } x \in I_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

- For $d = 2$

$$\varphi_i^2(x) := \begin{cases} \frac{(x - x_{i-1})(x - x_{i-1/2})}{(x_i - x_{i-1})(x_i - x_{i-1/2})} & \text{if } x \in I_i, \\ \frac{(x - x_{i+1})(x - x_{i+1/2})}{(x_i - x_{i+1})(x_i - x_{i+1/2})} & \text{if } x \in I_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\varphi_{i+1/2}^2(x) := \begin{cases} \frac{(x - x_i)(x - x_{i+1})}{(x_{i+1/2} - x_i)(x_{i+1/2} - x_{i+1})} & \text{if } x \in I_{i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that the support of φ_i^d is $I_i \cup I_{i+1}$. The exception is the two basis functions φ_0^d and φ_N^d at the leftmost and rightmost nodes a $x_0 = 0$ and $x_N = 1$ with support only on one interval (i.e, I_1 and I_N respectively).

In terms of basis functions this means that a basis for $\mathbb{V}_{h,0}$ is obtained by deleting the two basis φ_0^d and φ_N^d from the usual set $\{\varphi_{i/d}\}_{i=0}^{dN}$ of basis functions spanning $\mathbb{V}_h$.

4 Approximation variational formulation

The idea is to approximate in the finite dimensional space $\mathbb{V}_{h,0}^d$. Let v be in the space of approximation $\mathbb{V}_{h,0}^d$, we multiplied the integro-differential equations of continuous Problem (2.1) by the function v and integrated between 0 and 1, we get

$$-\int_0^1 \tilde{u}'(x) v(x) dx + \int_0^1 \left(\int_0^1 k(x,t) \tilde{u}(t) dt \right) v(x) dx = \int_0^1 g(x) v(x) dx.$$

Moreover, an integration by parts (Green's formula) and using the homogeneous conditions, then the approximate problem is given by

$$\begin{cases} \text{Find } \tilde{u} \in \mathbb{V}_{h,0}^d, \text{ such that} \\ a(\tilde{u}, v) = l(v), \ \forall v \in \mathbb{V}_h^d, \\ \tilde{u}(0) = \tilde{u}(1) = 0, \\ \text{where } a(\tilde{u}, v) = \displaystyle\int_0^1 \left[\tilde{u}(x) v'(x) + \left(\int_0^1 k(x,t) \tilde{u}(t) dt \right) v(x) \right] dx, \\ l(v) = \displaystyle\int_0^1 g(x) v(x) dx. \end{cases}$$

Now, the L^2-projection $\mathcal{P}_h^d \tilde{u} \in \mathbb{V}_{h,0}^d$ of $\tilde{u}$ is defined by

$$\int_I (\tilde{u} - \mathcal{P}_h^d \tilde{u}) v dx = 0, \quad \forall v \in \mathbb{V}_{h,0}^d. \tag{4.1}$$

In order to actually compute the L^2-projection $\mathcal{P}_h^d \tilde{u}$, we first note that the formula (4.1) is equivalent to

$$\int_I (\tilde{u} - \mathcal{P}_h^d \tilde{u}) \varphi_{i/d}^d dx = 0, \quad i = 1, \dots, dN - 1. \tag{4.2}$$

Then, since $\mathcal{P}_h^d \tilde{u}$ belongs to $\mathbb{V}_{h,0}^d$ it can be written as the linear combination

$$\tilde{u}_N^d := \mathcal{P}_h^d \tilde{u} = \sum_{j=1}^{dN-1} u_j \varphi_{j/d}^d, \tag{4.3}$$

and choosing $v(x) = \varphi_{i/d}^d(x), \ i = 1, \dots, dN - 1$, i.e $d = 1$ or $d = 2$.
Next, the approximate variational formulation is obtained by substituting approximate functions $\tilde{u}$, then the expression given by

$$\begin{cases} \text{Find } \tilde{u} \in \mathbb{V}_{h,0}^d([0,1]), \text{ such that} \\ \displaystyle\sum_{j=1}^{dN-1} \int_0^1 u_j \varphi_{j/d}^d(x) {\varphi_{i/d}^d}'(x) dx + \sum_{j=1}^{dN-1} \int_0^1 \int_0^1 u_j k(x,t) \varphi_{j/d}^d(t) \varphi_{i/d}^d(x) dt dx \\ \qquad = \displaystyle\int_0^1 g(x) \varphi_{i/d}^d(x) dx, \quad i = 1, \dots, dN - 1. \end{cases}$$

5 Matrix representations of method

Now, we approximate the functions k and g in the spaces $\mathbb{V}_h^d \times \mathbb{V}_h^d$ and $\mathbb{V}_h^d$ respectively, as

$$k(x,t) \approx \sum_{p=0}^{dN} \sum_{q=0}^{dN} k_{pq} \varphi_{p/d}^d(x) \varphi_{q/d}^d(t), \qquad k_{pq} = k(x_{p/d}, x_{q/d}),$$

and

$$g(x) \approx \sum_{j=0}^{dN} g_j \varphi_{j/d}^d(x), \qquad g_j = g(x_{j/d}).$$

This implies,

$$\sum_{j=1}^{dN-1} u_j \left[\int_0^1 \varphi_{j/d}^d(x) {\varphi_{i/d}^d}'(x) dx + \sum_{p=0}^{dN} \sum_{q=0}^{dN} k_{pq} \left(\int_0^1 \varphi_{p/d}^d(x) \varphi_{i/d}^d(x) dx \right) \left(\int_0^1 \varphi_{q/d}^d(t) \varphi_{j/d}^d(t) dt \right) \right]$$
$$= \sum_{j=0}^{dN} g_j \int_0^1 \varphi_{j/d}^d(x) \varphi_{i/d}^d(x) dx, \quad i = 1, \dots, dN - 1, \tag{5.1}$$

we can write the system as follows

$$\sum_{j=1}^{dN-1}\left[a_{ji}+\sum_{p=0}^{dN}\sum_{q=0}^{dN}b_{ip}k_{pq}b_{qj}\right]u_j=\sum_{j=0}^{dN}b_{ij}g_j, \qquad i=1,\ldots,dN-1, \tag{5.2}$$

where, $a_{ji}=\int_0^1 \varphi^d_{i/d}(x)\varphi^{d\,\prime}_{j/d}(x)dx$ and $b_{ji}=\int_0^1 \varphi^d_{i/d}(x)\varphi^d_{j/d}(x)dx$.
Then, (5.2) leads to the matrix system

$$(\mathbf{A}+\mathbf{B}\mathbf{K}\mathbf{B}^t)U=\mathbf{B}G, \tag{5.3}$$

where
$U=[u_1,\ldots,u_{dN-1}]^t$, $G=[g(x_0),g(x_{1/d}),\ldots,g(x_{dN-1/d}),g(x_{dN})]^t$, $\mathbf{K}=[k(x_{i/d},x_{j/d})]_{0\le i,j\le dN}$,
if $d=1$,

$$\mathbf{A}=[a_{ji}]_{1\le j,i\le N-1}=\begin{pmatrix}0 & \frac{-1}{2} & & & 0\\ \frac{1}{2} & 0 & \frac{-1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & \frac{-1}{2}\\ 0 & & & \frac{1}{2} & 0\end{pmatrix},\ \mathbf{B}=[b_{ji}]_{\substack{1\le j\le N-1\\ 0\le i\le N}}=\frac{h}{6}\begin{pmatrix}1 & 4 & 1 & & & 0\\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & 4 & 1\end{pmatrix},$$

and if $d=2$,

$$\mathbf{A}=[a_{ji}]_{1\le j,i\le 2N-1}=\frac{1}{6}\begin{pmatrix}0 & -4 & & & & & & & \\ 4 & 0 & -4 & 1 & & & 0 & & \\ & 4 & 0 & -4 & & & & & \\ & -1 & 4 & 0 & -4 & & & & \\ & & & 4 & 0 & -4 & 1 & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & 4 & 0 & -4 & 0\\ & 0 & & & & -1 & 4 & 0 & -4\\ & & & & & & & 4 & 0\end{pmatrix},$$

$$\mathbf{B}=[b_{ji}]_{\substack{1\le j\le 2N-1\\ 0\le i\le 2N}}=\frac{h}{30}\begin{pmatrix}2 & 16 & 2 & & & & 0 & & \\ -1 & 2 & 8 & 2 & -1 & & & & \\ & & 2 & 16 & 2 & & & & \\ & & -1 & 2 & 8 & 2 & -1 & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & 2 & 8 & 2 & -1\\ & 0 & & & & & 2 & 16 & 2\end{pmatrix}.$$

Which is a linear system of $(dN-1)$ algebraic equations for $(dN-1)$ unknown coefficients $u_1,u_2,\ldots,u_{dN-1}$. System (5.3) can be summarized as

$$\mathbf{M}U=\mathbf{G} \text{ or } \left[\ \mathbf{M}\ \middle|\ \mathbf{G}\ \right], \tag{5.4}$$

where

$$\mathbf{M}=\mathbf{A}-\mathbf{B}\mathbf{K}\mathbf{B}^t \text{ and } \mathbf{G}=\mathbf{B}G,$$

then,

$$U=\mathbf{M}^{-1}\mathbf{G}. \tag{5.5}$$

Finally, after finding the unknown coefficients, we substitute them into formula (4.3) to obtain the general solution for Problem (2.1). Then can be write the solution of Problem (1.1) as

$$u^d_N(x)=\sum_{i=1}^{dN-1}u_i\varphi^d_{i/d}(x)+(\beta-\alpha)x+\alpha. \tag{5.6}$$

6 Error analysis

In this section, we perform error analysis in the space $\mathbb{V}_h^d$, because $\mathbb{V}_{h,0}^d \subset \mathbb{V}_h^d$ and we have,

$$u(x) - u_N^d(x) = \tilde{u}(x) - \tilde{u}_N^d(x).$$

Firstly, we start with some propositions for interpolation.

Proposition 6.1. *The interpolant $\mathcal{I}_N^d u$ in the space $\mathbb{V}_h^d$ satisfies the estimates*

$$\|u - \mathcal{I}_N^d u\|_{L^2(I_1)} \le Ch^{d+1}\|u^{(d+1)}\|_{L^2(I_1)}, \tag{6.1}$$

$$\|(u - \mathcal{I}_N^d u)'\|_{L^2(I_1)} \le Ch^{d}\|u^{(d+1)}\|_{L^2(I_1)}, \tag{6.2}$$

where C is a constant positive, and $h = x_1 - x_0$.

Proof. (see [15] page 6). Indication, for $d = 2$ we have $(\mathcal{I}_N^2 u)^{(3)} = 0$. □

Proposition 6.2. *The interpolant $\mathcal{I}_N^d u$ satisfies the estimates*

$$\|u - \mathcal{I}_N^d u\|_{L^2(I)}^2 \le C\sum_{i=1}^{N} h_i^{2d+2}\|u^{(d+1)}\|_{L^2(I_i)}^2, \tag{6.3}$$

$$\|(u - \mathcal{I}_N^d u)'\|_{L^2(I)}^2 \le C\sum_{i=1}^{N} h_i^{2d}\|u^{(d+1)}\|_{L^2(I_i)}^2, \tag{6.4}$$

where C is a constant positive, and $h_i = x_i - x_{i-1}$.

Proof. Using the Triangle inequality and Proposition 6.1, we have

$$\begin{aligned}
\|u - \mathcal{I}_N^d u\|_{L^2(I)}^2 &= \int_I (u(x) - \mathcal{I}_N^d u(x))^2 dx \\
&= \sum_{i=1}^{N} \int_{I_i} (u(x) - \mathcal{I}_N^d u(x))^2 dx \\
&= \sum_{i=1}^{N} \|u - \mathcal{I}_N^d u\|_{L^2(I_i)}^2 \\
&\le C\sum_{i=1}^{N} h_i^{2d+2}\|u^{(d+1)}\|_{L^2(I_i)}^2.
\end{aligned}$$

This confirms the first estimate. We follow the same steps to prove the second estimate. □

Now, in flowing the error estimate for projection.

Lemma 6.3 ([15]). *The L_2-projection $\mathcal{P}_N^d u$, defined by (4.1), satisfies the best approximation result*

$$\|u - \mathcal{P}_N^d u\|_{L^2(I)}^2 \le \|u - v\|_{L^2(I)}^2, \qquad \forall v \in \mathbb{V}_h^d, \tag{6.5}$$

$$\|(u - \mathcal{P}_N^d u)'\|_{L^2(I)}^2 \le \|(u - v)'\|_{L^2(I)}^2, \qquad \forall v \in \mathbb{V}_h^d. \tag{6.6}$$

Proof. (see [15], page 12). □

Lemma 6.4. *The L^2-projection $\mathcal{P}_N^d u$ satisfies the estimate*

$$\|u - \mathcal{P}_N^d u\|_{L^2(I)}^2 \le C\sum_{i=1}^{N} h_i^{2d+2}\|u^{(d+1)}\|_{L^2(I_i)}^2, \tag{6.7}$$

$$\|(u - \mathcal{P}_N^d u)'\|_{L^2(I)}^2 \le C\sum_{i=1}^{N} h_i^{2d}\|u^{(d+1)}\|_{L^2(I_i)}^2. \tag{6.8}$$

Proof. Commencing with the optimal approximation result, selecting $v = \mathcal{I}_N^d u$ as the interpolant for u, and applying the interpolation error estimate inequality (6.1) from Proposition 6.1, we obtain

$$\begin{aligned} \|u - \mathcal{P}_N^d u\|^2_{L^2(I)} &\leq \|u - \mathcal{I}_N^d u\|^2_{L^2(I)} && (6.9)\\ &\leq \sum_{i=1}^{N} \|u - \mathcal{I}_N^d u\|^2_{L^2(I_i)} && (6.10)\\ &\leq C\sum_{i=1}^{N} h_i^{2d+2}\|u^{(d+1)}\|^2_{L^2(I_i)}, && (6.11)\end{aligned}$$

which proves the estimate. Using the same steps, we obtain the second inequality by applying inequality (6.2). □

Corollary 6.5. *Recalling the definition $h = \max h_i$ we conclude that,*

$$\|u - \mathcal{P}_N^d u\|_{L^2(I)} \leq Ch^{d+1}\|u^{(d+1)}\|_{L^2(I)}, \tag{6.12}$$

$$\|(u - \mathcal{P}_N^d u)'\|_{L^2(I)} \leq Ch^{d}\|u^{(d+1)}\|_{L^2(I)}. \tag{6.13}$$

Theorem 6.6. *Let u the exact solution of to Problem (1.1) and the approximated solution u_N be obtained by using the finite element method (5.6). If $u \in H_0^1(]0,1[)$, then for*

$$\|(u - u_N^d)'\|_{L_2(I)} \leq Ch^{d+1}\left(\|g^{(d+1)}\|_{L^2(I)} + \lambda_{d,N}\|\tilde{u}^{(d+1)}\|_{L^2(I)} + \theta_{d,N}\|\tilde{u}\|_{L_2(I)}\right),$$

where

$$\lambda_{d,N} = \max_{0\leq i,j\leq dN} |k(x_i, x_j)|,$$

$$\theta_{d,N} = \max_{x,t\in I} |\partial_t^{d+1} k(x,t)|.$$

Proof. Let the Fredholm integro-differential equations in (1.1),

$$-u'(x) + \int_I k(x,t)u(t)dt = f(x), \tag{6.14}$$

while using the transformation we apply (2.1),

$$-\tilde{u}'(x) = g(x) - \int_I k(x,t)\tilde{u}(t)dt. \tag{6.15}$$

Now, we have the approximation solution of this equation as,

$$-(\tilde{u}_N^d)'(x) = \mathcal{P}_N^d g(x) - \int_I \mathcal{P}_{N,N}^{d,d} k(x,t)\tilde{u}_N^d(t)dt. \tag{6.16}$$

Subtracting (6.16) from (6.15), we get the error equation

$$\tilde{u}'(x) - (\tilde{u}_N^d)'(x) = \mathcal{P}_N^d g(x) - g(x) + \int_I k(x,t)\tilde{u}(t)dt - \int_I \mathcal{P}_{N,N}^{d,d} k(x,t)\mathcal{P}_N\tilde{u}(t)dt, \tag{6.17}$$

which can be rewritten as,

$$\begin{aligned} (\tilde{u} - \tilde{u}_N^d)'(x) &= \mathcal{P}_N^d g(x) - g(x) + \int_I \mathcal{P}_{N,N}^{d,d} k(x,t)(\tilde{u}(t) - \mathcal{P}_N^d\tilde{u}(t))dt \\ &\quad + \int_I \left(k(x,t) - \mathcal{P}_{N,N}^{d,d} k(x,t)\right)\tilde{u}(t)dt \\ &= E_1(x) + E_2(x) + E_3(x), \end{aligned}$$

where

$$E_1(x) = \mathcal{P}_N^d g(x) - g(x) \tag{6.18}$$

$$E_2(x) = \int_I \mathcal{P}_{N,N}^{d,d} k(x,t)(\tilde{u}(t) - \tilde{u}_N^d(t))dt \tag{6.19}$$

$$E_3(x) = \int_I \Big(k(x,t) - \mathcal{P}_{N,N}^{d,d} k(x,t)\Big)\, \tilde{u}(t)dt. \tag{6.20}$$

We have by the triangle inequality

$$\|(\tilde{u} - \tilde{u}_N^d)'\|_{L_2(I)} \le \|E_1\|_{L_2(I)} + \|E_2\|_{L_2(I)} + \|E_3\|_{L_2(I)}. \tag{6.21}$$

Corollary 6.5 implies directly that

$$\|E_1\|_{L_2(I)} = \|g(x) - \mathcal{P}_N g(x)\|_{L_2(I)} \le Ch^{d+1}\|g^{(d+1)}\|_{L^2(I)}. \tag{6.22}$$

By using Cauchy-Schwarz together with Corollary 6.5, we have

$$\begin{aligned}
|E_2(x)| &= \left|\int_I \mathcal{P}_{N,N}^{d,d} k(x,t)(\tilde{u}(t) - \tilde{u}_N^d(t))dt\right| \\
&\le \left(\int_I |\mathcal{P}_{N,N}^{d,d} k(x,t)|^2 dt\right)^{1/2} \left(\int_I |\tilde{u}(t) - \tilde{u}_N^d(t)|^2 dt)\right)^{1/2} \\
&\le Ch^{d+1}\|\tilde{u}^{(d+1)}\|_{L^2(I)} \left(\int_I |\mathcal{P}_{N,N}^{d,d} k(x,t)|^2 dt\right)^{1/2},
\end{aligned}$$

then

$$\begin{aligned}
\|E_2(x)\|_{L^2(I)} &= \left(\int_I |E_2(x)|^2 dx\right)^{1/2} \\
&\le Ch^{d+1}\|\tilde{u}^{(d+1)}\|_{L^2(I)} \left(\int_I\int_I |\mathcal{P}_{N,N}^{d,d} k(x,t)|^2 dtdx\right)^{1/2} \\
&\le Ch^{d+1} \max_{0\le i,j\le dN} |k(x_i,x_j)|.\|\tilde{u}^{(d+1)}\|_{L^2(I)}.
\end{aligned}$$

Additionally, employing Cauchy-Schwarz once more

$$\begin{aligned}
|E_3(x)| &= \left|\int_I \Big(k(x,t) - \mathcal{P}_{N,N}^{d,d} k(x,t)\Big)\, \tilde{u}(t)dt\right| \\
&\le \left(\int_I \left|k(x,t) - \mathcal{P}_{N,N}^{d,d} k(x,t)\right|^2 dt\right)^{1/2} \left(\int_I |\tilde{u}(t)|^2 dt\right)^{1/2} \\
&\le Ch^{d+1}\|\tilde{u}\|_{L_2(I)} \left(\int_I |\partial_t^{d+1} k(x,t)|^2 dt\right)^{1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|E_3\|_{L^2(I)} &= \left(\int_I |E_3(x)|^2 dx\right)^{1/2} \\
&\le Ch^{d+1}\|\tilde{u}\|_{L_2(I)} \left(\int_I\int_I |\partial_t^{d+1} k(x,t)|^2 dtdx\right)^{1/2} \\
&\le Ch^{d+1} \max_{x,t\in I} |\partial_t^{d+1} k(x,t)|.\|\tilde{u}\|_{L_2(I)}.
\end{aligned}$$

In conclusion, the theorem's assertion is a consequence of the triangle inequality. □

7 Illustrative Examples

In this section, we show the numerical results and absolute errors for some examples are given. All computations were carried out by software MATLAB. Now we define the maximum absolute error for $u_N^d(x)$ as

$$E_N := \|u(x) - u_N^d(x)\|_\infty = \max_{i=0,100} \left|u(x_i) - u_N^d(x_i)\right|, \qquad 0 \le x_i \le 1.$$

Example 7.1. Let us first consider the integro-differential equation

$$\begin{cases} -u'(x) + \displaystyle\int_0^1 xtu(t)dt = \frac{-7}{4}x, \; 0 < x < 1, \\ u(0) = 0, \; u(1) = 1, \end{cases} \tag{7.1}$$

whose exact solution is $u(x) = x^2$. We apply the methods that was explained in previous Section for $N = 3$.

Firstly, for LFE1 method, the vrertices is $\{x_i\}_{i=0}^3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Then the augmented matrix is

$$\left[\begin{array}{c|c} \mathbf{M} & \mathbf{G} \end{array}\right] = \left[\begin{array}{cc|c} 1/81 & -77/162 & 11/108 \\ 85/162 & 4/81 & -7/54 \end{array}\right].$$

By solving this system,

$$U = [-55/243 \quad -107/486]^T,$$

then the approximate solution of Problem (7.1) is given by

$$\begin{aligned} u_3^1(x) &= \sum_{i=1}^{2} u_N \varphi_i^1(x) + x \\ &= -55/243.\varphi_1^1(x) - 107/486.\varphi_2^1(x) + x \\ &= \begin{cases} 26x/81 & \text{if } 0 \le x \le \frac{1}{3}, \\ 55x/54 - 113/486 & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ 269x/162 - 107/162 & \text{if } \frac{2}{3} \le x \le 1. \end{cases} \end{aligned}$$

Now, for LFE2 method, the vrertices is $\{x_i\}_{i=0}^6 = \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\}$. Then the augmented matrix is,

$$\left[\begin{array}{c|c} \mathbf{M} & \mathbf{G} \end{array}\right] = \left[\begin{array}{ccccc|c} \frac{1}{729} & \frac{-485}{729} & \frac{1}{243} & \frac{2}{729} & \frac{5}{729} & \frac{47}{324} \\ \frac{487}{729} & \frac{1}{729} & \frac{-161}{243} & \frac{247}{1458} & \frac{5}{729} & \frac{11}{324} \\ \frac{1}{243} & \frac{163}{243} & \frac{1}{81} & \frac{-160}{243} & \frac{5}{243} & \frac{-1}{108} \\ \frac{2}{729} & \frac{-239}{1458} & \frac{164}{243} & \frac{4}{729} & \frac{-476}{729} & \frac{-7}{162} \\ \frac{5}{729} & \frac{5}{729} & \frac{5}{243} & \frac{496}{729} & \frac{25}{729} & \frac{-53}{324} \end{array}\right].$$

By solving this system,

$$U = [-5/36 \quad -2/9 \quad -1/4 \quad -2/9 \quad -5/36]^T,$$

then the approximate solution of Problem (7.1) is given by

$$\begin{aligned} u_3^2(x) &= -5/36\varphi_{1/2}^2(x) - 2/9\varphi_1^2(x) - 1/4\varphi_{3/2}^2(x) - 2/9\varphi_2^2(x) - 5/36\varphi_{5/2}^2(x) + x \\ &= x^2, \end{aligned}$$

it is also the exact solution to the problem. The results in Table 1 shows the numerical solutions are in a very good agreement with the exact solution.

Example 7.2. [19] Consider the integro-differential

$$-u'(x) + \int_0^1 xu(t)dt = x - (x+1)e^x, \; 0 < x < 1, \tag{7.2}$$

x	Exact Solution	LFE1($N=3$)	LFE2($N=3$)
0.0	0.00000000	0.00000000	0.00000000
0.1	0.01000000	0.03209877	0.01000000
0.2	0.04000000	0.06419753	0.04000000
0.3	0.09000000	0.09629630	0.09000000
0.4	0.16000000	0.17489712	0.16000000
0.5	0.25000000	0.27674897	0.25000000
0.6	0.36000000	0.37860082	0.36000000
0.7	0.49000000	0.50185185	0.49000000
0.8	0.64000000	0.66790123	0.64000000
0.9	0.81000000	0.83395062	0.81000000
1.0	1.00000000	1.00000000	1.00000000

Table 1. Approximate and exact solutions of Example 7.1.

with $u(0)=0$, $u(1)=e^1$. The exact solution of this problem is $u(x)=xe^x$. In Table 2, absolute errors in solutions obtained by Lagrange finite element P_1 and P_2 for $N=64$ are compared with differential transfer method [10], Hybrid function method [13] and Improved homotopy perturbation method [19]. In Fig.1 **a**, we plot the exact solution against the numerical solution obtained by LFE1 and LFE2 with $N=4$, and in Fig.1 **b**, we depict the rate of convergence.

				Present Methods	
x	DTM [10]	HFM [13]	IHPM [19]	LFE1	LFE2
0.0	$0.00000e+00$	$1.11022e-16$	$0.00000e+00$	$0.00000e+00$	$0.00000e+00$
0.1	$1.00118e-02$	$9.07270e-03$	$2.31481e-05$	$4.11506e-06$	$6.09093e-05$
0.2	$2.78651e-02$	$1.13773e-02$	$9.25926e-05$	$9.21840e-05$	$4.10075e-05$
0.3	$5.08730e-02$	$9.84041e-03$	$2.08333e-04$	$8.35173e-05$	$4.20896e-05$
0.4	$7.55356e-02$	$6.87421e-03$	$3.70370e-04$	$3.25261e-05$	$6.69726e-05$
0.5	$9.71888e-02$	$4.30919e-03$	$5.78704e-04$	$3.38208e-10$	$2.37256e-06$
0.6	$1.09551e-01$	$3.31778e-03$	$8.33333e-04$	$6.67817e-05$	$7.39818e-05$
0.7	$1.04133e-01$	$4.32872e-03$	$1.13426e-03$	$3.86874e-05$	$5.18724e-05$
0.8	$6.94512e-02$	$6.93055e-03$	$1.48148e-03$	$2.22010e-05$	$5.53054e-05$
0.9	$1.00034e-02$	$9.76311e-03$	$1.87500e-03$	$1.36465e-04$	$8.96383e-05$
1.0	$1.55147e-01$	$1.03954e-02$	$2.31481e-03$	$0.00000e+00$	$0.00000e+00$

Table 2. A comparison of absolute errors between DTM, HFM, IHPM and present methods ($N=64$) of Example 7.2.

Example 7.3. Consider the integro-differential

$$\begin{cases} -u'(x)+\displaystyle\int_0^1 (x^2+t)u(t)dt = f(x),\ 0<x<1, \\ u(0)=1,\ u(1)=\ln(2), \end{cases}$$

where $f(x)$ is chosen so that the exact solution is $u(x)=\ln(x+1)$. Table 3 we show the maximum absolute errors obtained by using present methods LFE1 and LFE2 for different values of N. The absolute errors are presented in Fig.2 for $N=8$.

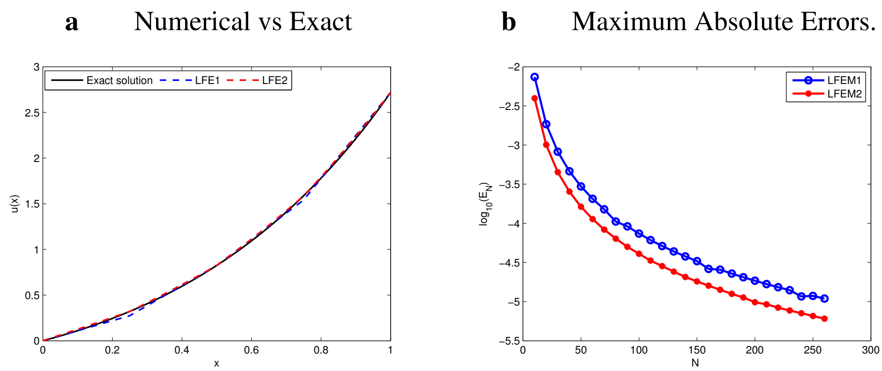

Figure 1. (**a**) The exact and numerical solution of Example 7.2 with $N = 4$. (**b**) Maximum absolute errors E_N of Example 7.2 for different values of N.

n	LFE1	LFE2
4	$4.36448e-03$	$1.78173e-03$
8	$1.24007e-03$	$4.72214e-04$
16	$3.32892e-04$	$1.21201e-04$
32	$8.63913e-05$	$3.06506e-05$
64	$2.19744e-05$	$7.71055e-06$
128	$5.50337e-06$	$1.93355e-06$
256	$1.37258e-06$	$4.84120e-07$

Table 3. The maximum errors E_N for different values of N for Example 7.3.

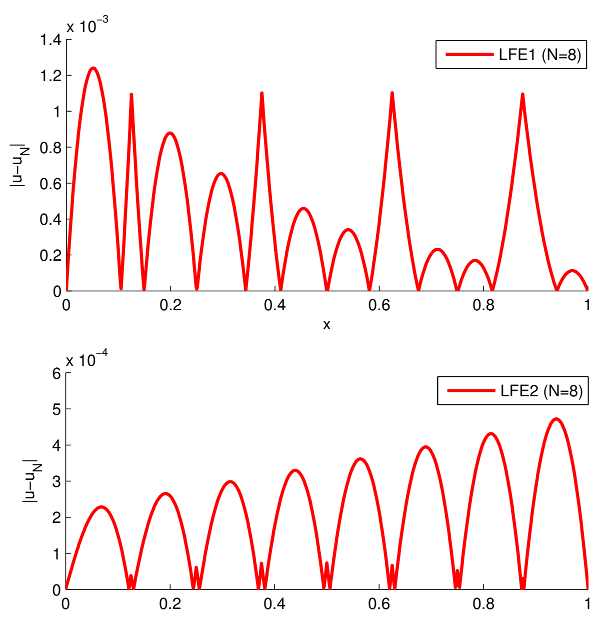

Figure 2. The absolute error of Example 7.3 for $N = 8$.

8 Conclusion

We transformed Problem (1.1) into a problem with homogeneous conditions. By employing the standard Galerkin method with Lagrange finite element, we derived a system of $(dN-1)$ equations with $(dN-1)$ unknowns. We easily verified that the coercivity of the bilinear form implies the invertibility of the system's matrix. Consequently, we are assured of solving Problem (1.1). The convergence of the finite element solutions to the exact one is guaranteed, and we tested and compared the efficiency of this method by solving several examples.

References

[1] Z. P. Atabakan, A. K. Nasab, and A. Kilikman, *On Solution of Fredholm Integro-differential Equations Using Composite Chebyshev Finite Difference Method*, Abstract and Applied Analysis, Abstr Appl Anal, 1–12, (2013).

[2] K. Atkinson and A. Bogomolny, *The Discrete Galerkin Method for Integral Equations*, in Mathematics of Computation, Academic Press, 595–616, (1987).

[3] K.E. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, 3rd edition, Springer-Verlag, New York, 2009.

[4] K. E. Atkinson and F. A. Potra, *On the Discrete Galerkin Method for Fredholm Integral Equations of the Second Kind*, in IMA Journal of Numerical Analysis, Academic Press, 385–403, (1989).

[5] S. Benyoucef and A. Rahmoune, *Efficient spectral-collocation methods for a class of linear Fredholm integro-differential equations on the half-line*, in J. Comput. Appl. Math., Academic Press, **377** 112–894, (2020).

[6] M. I. Berenguer, M. V. Fernàndez Munoz, A. I. Garralda-Guillem and M. Ruiz Galan, *A sequential approach for solving the Fredholm integro-differential equation*, in Applied Numerical Mathematics, **62(4)**, pp. 297–304, (2012).

[7] J. Biazar and M. G. Porshokouhi, *Application of variational iteration method for linear and nonlinear integro-differential-difference equations*, International Mathematical Forum, **5(67)**, pp. 3335–3341, (2010).

[8] J. Chaskalovic, *Finite element methods for engineering sciences*, Springer-Verlag, New York, 2008.

[9] H. Danfu and S. Xufeng, *Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration*, Applied Mathematics and Computation, **194(2)**, pp. 460–466, (2007).

[10] P. Darania and A. Ebadian, *A method for the numerical solution of the integro-differential equations*, Applied Mathematics and Computation, **188 (1)**, pp. 657–668, (2007).

[11] F. Ghomanjani, *Numerical solution for nonlinear Volterra integro-differential equations*. Palest. J. Math. **9**, 164–169, (2020).

[12] S. M. Hosseini and S. Shahmorad, *Numerical piecewise approximate solution of Fredholm integro-differential equations by the Tau method*, Applied Mathematical Modelling, **29**, 1005–1021, (2005).

[13] R. Jafri, R.Ezzati and K. Maleknejad, *Numerical Solution of Fredholm Integro-differential Equations By Using Hybrid Function Operational Matrix of Differentiation*, International Journal of Industrial Mathematics, **9**, 349–358, (2017).

[14] M. T. Kajani, M. Ghasemi and E. Babolian, *Comparison between the homotopy perturbation method and the sine-cosine wavelet method for solving linear integro-differential equations*, Computers and Mathematics with Applications, **54**, 1162–1168, (2007).

[15] M. G. Larson and F. Bengzon, *The finite element method: Theory, Implementation, and applications*, Springer-Verlag Berlin Heidelberg 2013.

[16] P. Linz, *A Method for the Approximate Solution of Linear Integro-differential Equations*, in SIAM Journal on Numerical Analysis, Academic Press, (1974), 137–144.

[17] P. Linz, *Convergence of a Discretization Method for Integro-differential Equations*, Numer. Math., Academic Press, (1975), 103–107.

[18] K. P. Mredula and D. C. Vakaskar. *Collocation methods general approach for ordinary differential equations*. Palest. J. Math. **7(2)**, 650-662, (2018).

[19] E. Yusufoglu (Agadjanov), *Improved homotopy perturbation method for solving Fredholm type integro-differential equations*, Chaos, Solitons and Fractals, **41(1)**, 28–37, 2009.

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