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Generalizations of prime intuitionistic fuzzy ideals of a lattice

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Abstract As a generalization of the concepts of an intuitionistic fuzzy prime ideal and a prime intuitionistic fuzzy ideal, the concepts of an intuitionistic fuzzy 2-absorbing ideal and a 2-absorbing intuitionistic fuzzy ideal of a lattice are introduced. Some results on such intuitionistic fuzzy ideals are proved. It is shown that the radical of an intuitionistic fuzzy ideal of L is a 2-absorbing intuitionistic fuzzy ideal if and only if it is a 2-absorbing primary intuitionistic fuzzy ideal of L. We also introduce and study these concepts in the product of lattices.

1 Introduction

The concept of intuitionistic fuzzy sets was introduced by Atanassov [5, 6, 7] as a generalization of fuzzy sets previously introduced by Zadeh [25]. Atanassov and Stoeva [8] generalised this concept by taking the evaluation set as a lattice. After a few years, Thomas and Nair [22] studied intuitionistic fuzzy sublattice, intuitionistic fuzzy ideals, and intuitionistic fuzzy filters on a lattice. For more details, we refer to [1, 2, 3, 13, 14, 16, 19]. Milles, Zedam and Rak in [18] introduced the notion of prime intuitionistic fuzzy ideal and filter and studied many characterizations of these notions.

The notion of a 2-absorbing ideal of a commutative ring was introduced by Badawi [9]. A proper ideal I of a commutative ring R is said to be a 2-absorbing, if whenever $a, b, c \in R$ such that $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. This concept was generalised by Anderson and Badawi [4], Badawi and Darani [10], Wasadikar and Gaikwad [23, 24] in other mathematical structures such as semirings, semigroups, submodules and lattices.

In this paper, we introduce the concepts of an intuitionistic fuzzy 2-absorbing ideal and a 2-absorbing intuitionistic fuzzy ideal of a lattice L. This is a generalization of the concepts of an intuitionistic fuzzy prime ideal and a prime intuitionistic fuzzy ideal of L introduced by Hur et al. [16] and Milles et al. [18] respectively. Also, we define a primary intuitionistic fuzzy ideal and the radical of an intuitionistic fuzzy ideal of L. Some properties of these intuitionistic fuzzy ideals are proven. We also introduce and study these concepts in the context of product of lattices.

2 Preliminaries

Throughout in this paper, $L = (L, \wedge, \vee)$ denotes a bounded lattice with least element 0_L and greatest element 1_L. We recall some concepts and results.

Definition 2.1. ([5, 6, 7]) An intuitionistic fuzzy set (IFS) A in L can be represented as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in L\}$, where the functions $\mu_A : L \to [0, 1]$ and $\nu_A : L \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in L$ to A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in L$.

Remark 2.2. ([7, 13, 20])

(i) When $\mu_A(x) + \nu_A(x) = 1, \forall x \in L$. Then A is called a fuzzy set in L.

(ii) An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is briefly written as $A(x) = (\mu_A(x), \nu_A(x))$, $\forall x \in L$. We denote by $IFS(L)$ the set of all IFSs of L.

(iii) If $p, q \in [0,1]$ such that $p+q \leq 1$. Then $A \in IFS(L)$ defined by $\mu_A(x) = p$ and $\nu_A(x) = q$, for all $x \in L$, is called a constant intuitionistic fuzzy set of L. Any IFS of L defined other than this is referred to as a non-constant intuitionistic fuzzy set.

If $A, B \in IFS(L)$, then $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in L$ and $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. For any subset S of L, the intuitionistic fuzzy characteristic function χ_S is an intuitionistic fuzzy set of L, defined as $\chi_S(x) = (1, 0), \forall x \in S$ and $\chi_S(x) = (0, 1), \forall x \in L \backslash S$. Let $\alpha, \beta \in [0, 1]$ with $\alpha+\beta \leq 1$. Then the crisp set $A_{(\alpha,\beta)} = \{x \in L : \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta\}$ is called the (α, β)-level cut subset of A [20]. Further, if $A, B \in IFI(L)$. Then $A \cap B$ and $A \cup B$ represent the intersection and union of intuitionistic fuzzy sets A and B respectively. These are defined as $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$; $\nu_{A\cap B}(x) = \nu_A(x) \vee \nu_B(x)$, for all $x \in L$ and $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$; $\nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x)$, for all $x \in L$ [13].

Definition 2.3. ([16, 18]) Let $L = L_1 \times L_2$ be the direct product of lattices L_1 and L_2. Let $A_1 \in IFS(L_1)$ and $A_2 \in IFS(L_2)$. Then their direct product is denoted by $A_1 \times A_2$ and is an intuitionistic fuzzy set of L defined by

$$\mu_{A_1\times A_2}(x, y) = \mu_{A_i}(x) \wedge \mu_{A_2}(y) \text{ and } \nu_{A_1\times A_2}(x, y) = \nu_{A_i}(x) \vee \nu_{A_2}(y), \forall (x, y) \in L.$$

Definition 2.4. ([22]) Let $A \in IFS(L)$. Then A is called an intuitionistic fuzzy lattice (IFL) of L, if for all $x, y \in L$, the followings are satisfied

(i) $\mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(ii) $\mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(iii) $\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$;

(iv)$\nu_A(x \wedge y) \leq \max\{\nu_A(x), \nu_A(y)\}$.

Definition 2.5. ([22]) Let $A \in IFS(L)$. Then A is called an intuitionistic fuzzy ideal (IFI) of L, if for all $x, y \in L$, the followings are satisfied

(i) $\mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(ii) $\mu_A(x \wedge y) \geq \max\{\mu_A(x), \mu_A(y)\}$;

(iii) $\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$;

(iv)$\nu_A(x \wedge y) \leq \min\{\nu_A(x), \nu_A(y)\}$.

Note that $\mu_A(0_L) \geq \mu_A(x) \geq \mu_A(1_L), \mu_A(0_L) \leq \mu_A(x) \leq \mu_A(1_L), \forall x \in L$. The set of all intuitionistic fuzzy ideals of L is denoted by $IFI(L)$.

Theorem 2.6. *([1, 18]) Let L be a lattice and $A \in IFS(L)$. Then it holds that A is an IFI on L if and only if the following two conditions are satisfied:*

(i) $\mu_A(x \vee y) = \min\{\mu_A(x), \mu_A(y)\}$;

(ii) $\nu_A(x \vee y) = \max\{\nu_A(x), \nu_A(y)\}$, *for any* $x, y \in L$.

Theorem 2.7. *([1, 18]) Let L be a lattice and $A \in IFI(L)$. Then it holds that A is an intuitionistic fuzzy prime ideal (IFPI) on L if and only if the following two conditions are satisfied:*

(i) $\mu_A(x \wedge y) = \max\{\mu_A(x), \mu_A(y)\}$;

(ii) $\nu_A(x \wedge y) = \min\{\nu_A(x), \nu_A(y)\}$, *for any* $x, y \in L$.

Theorem 2.8. *([16]) Let $L = L_1 \times L_2 \times .... \times L_k$ be the direct product of lattices $L_1, L_2, ..., L_k$. If $A_i \in IFS(L_i)$, $(i = 1, 2, ..., k)$. Then $A_1 \times A_2 \times .... \times A_k \in IFI(L_1 \times L_2 \times .... \times L_k)$ and is defined as $\mu_{A_1\times A_2\times....\times A_k}(x_1, x_2, ..., x_k) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge ....\mu_{A_k}(x_k)$ and $\nu_{A_1\times A_2\times....\times A_k}(x_1, x_2, ..., x_k) = \nu_{A_1}(x_1) \vee \nu_{A_2}(x_2) \vee .... \vee \nu_{A_k}(x_k)$, for all $(x_1, x_2, ..., x_k) \in L_1 \times L_2 \times .... \times L_k$.*

3 Intuitionistic fuzzy prime ideals and prime intuitionistic fuzzy ideal of a lattice

Definition 3.1. ([17]) A non-empty subset I of a lattice L is called an ideal if for $a, b \in L$, the following conditions holds

(i) If $a, b \in I$, $a \vee b \in I$ and

(ii) If $a \leq b$ and $b \in I$, then $a \in I$

A proper ideal I (i.e., $I \neq L$) is called a prime ideal, if $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$.

On the line of Koguep et al. [17], we will define prime intuitionistic fuzzy ideal (PIFI) of a lattice as follow:

Definition 3.2. A proper IFI P of a lattice L is called a prime intuitionistic fuzzy ideal (PIFI) of L if for any two IFIs A and B of L

$$A \cap B \subseteq P \text{ implies that either } A \subseteq P \text{ or } B \subseteq P$$

From the definition of PIFI, following results are easy to derive

Theorem 3.3. *Let I be an ideal of L and χ_I denote the IF characteristic function of I. Then*
(i) I is a prime ideal of L if and only if χ_I is an IFPI of L;
(ii) I is a prime ideal of L if and only if χ_I is a PIFI of L.

Proof. Clearly, χ_I is an IFI of L.
(i) Suppose that I is a prime ideal of L. Let $a, b \in L$, we need to show that

$$\mu_{\chi_I}(a \wedge b) = \mu_{\chi_I}(a) \vee \mu_{\chi_I}(b) \text{ and } \nu_{\chi_I}(a \wedge b) = \nu_{\chi_I}(a) \wedge \nu_{\chi_I}(b).$$

If $a, b \in I$, then $a \wedge b \in I$ and we have

$$\mu_{\chi_I}(a \wedge b) = 1 = 1 \vee 1 = \mu_{\chi_I}(a) \vee \mu_{\chi_I}(b) \text{ and } \nu_{\chi_I}(a \wedge b) = 0 = 0 \wedge 0 = \nu_{\chi_I}(a) \wedge \nu_{\chi_I}(b).$$

If $a, b \notin I$, then as I is a prime ideal $a \wedge b \notin I$ and we have

$$\mu_{\chi_I}(a \wedge b) = 0 = 0 \vee 0 = \mu_{\chi_I}(a) \vee \mu_{\chi_I}(b) \text{ and } \nu_{\chi_I}(a \wedge b) = 1 = 1 \wedge 1 = \nu_{\chi_I}(a) \wedge \nu_{\chi_I}(b).$$

If only one of a or b is in I, say $a \in I$ and $b \notin I$, then $a \wedge b \in I$, we have

$$\mu_{\chi_I}(a) = 1, \nu_{\chi_I}(a) = 0, \mu_{\chi_I}(b) = 0, \nu_{\chi_I}(b) = 1 \text{ and } \mu_{\chi_I}(a \wedge b) = 1, \nu_{\chi_I}(a \wedge b) = 0.$$

Thus $\mu_{\chi_I}(a \wedge b) = 1 = 1 \vee 0 = \mu_{\chi_I}(a) \vee \mu_{\chi_I}(b)$ and $\nu_{\chi_I}(a \wedge b) = 0 = 0 \wedge 1 = \nu_{\chi_I}(a) \wedge \nu_{\chi_I}(b)$. Therefore, χ_I is an IFPI of L.

Conversely, suppose that χ_I is an IFPI of L. Let $a \wedge b \in I$. Then
$\mu_{\chi_I}(a \wedge b) = 1 = \mu_{\chi_I}(a) \vee \mu_{\chi_I}(b)$ and $\nu_{\chi_I}(a \wedge b) = 0 = \nu_{\chi_I}(a) \wedge \nu_{\chi_I}(b)$............$(*)$

If both $a, b \notin I$, then $\mu_{\chi_I}(a) = \mu_{\chi_I}(b) = 0$ and $\nu_{\chi_I}(a) = \nu_{\chi_I}(b) = 1$ implies that $\mu_{\chi_I}(a) \vee \mu_{\chi_I}(b) = 0$ and $\nu_{\chi_I}(a) \wedge \nu_{\chi_I}(b) = 1$, which contradict $(*)$. Hence I must be a prime ideal of L.

(ii) Suppose that I is a prime ideal of L. Let $A, B \in IFI(L)$. Suppose that $A \cap B \subseteq \chi_I$. If $A \nsubseteq \chi_I$, $B \nsubseteq \chi_I$, then there exists $a, b \in L$ such that $\mu_{\chi_I}(a) < \mu_A(a), \nu_{\chi_I}(a) > \nu_A(a)$ and $\mu_{\chi_I}(b) < \mu_A(b), \nu_{\chi_I}(b) > \nu_A(b)$. Then by definition, we conclude that $a, b \notin I$. For, if say $a \in I$, then $\mu_{\chi_I}(a) = 1, \nu_{\chi_I}(a) = 0$ leads to $\mu_A(a) > 1, \nu_A(a) < 0$, which is not possible.

Since I is a prime ideal of L, we get $a \wedge b \notin I$. Hence $\mu_{\chi_I}(a \wedge b) = 0, \nu_{\chi_I}(a \wedge b) = 1$. Since A, B are IFIs of L, we have $\mu_A(a) \leq \mu_A(a \wedge b), \nu_A(a) \geq \nu_A(a \wedge b)$ and $\mu_B(b) \leq \mu_B(a \wedge$

$b), \nu_B(b) \geq \nu_B(a \wedge b)$. As the image of any element under an IFS is a non-zero number. From the above, we get

$$\begin{aligned} \mu_{\chi_I}(a \wedge b) &= 0 \\ &\leq \mu_{\chi_I}(a) \wedge \mu_{\chi_I}(b) \\ &< \mu_A(a) \wedge \mu_B(b) \\ &\leq \mu_A(a \wedge b) \wedge \mu_B(a \wedge b) \\ &= \mu_{A \cap B}(a \wedge b) \\ &\leq \mu_{\chi_I}(a \wedge b) \\ &= 0. \end{aligned}$$

Thus we get $0 < 0$. Similarly, we can show $1 > 1$, which is not possible. Hence either $A \subseteq \chi_I$ or $B \subseteq \chi_I$.

Conversely, suppose that χ_I is a PIFI of L. Suppose that for some $a, b \in L, a \wedge b \in I$, but $a, b \notin I$. Define IFSs A and B of L as follows

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in (a] \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in (a] \\ 1, & \text{otherwise} \end{cases} .$$

and

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in (b] \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x \in (b] \\ 1, & \text{otherwise} \end{cases} .$$

Then $A \cap B \subseteq \chi_I$, a contradiction. Hence I is a prime ideal of L. □

The following example shows that the condition of "primeness" in Theorem 3.3 is necessary.

Example 3.4. Consider the lattice as shown in the figure 1:
We note that the ideal $I = (0]$ is not a prime ideal of L, as $a \wedge b = 0 \in I$, but $a \notin I$ and $b \notin I$.

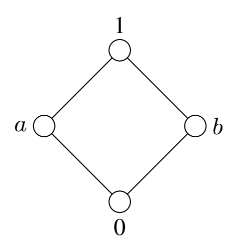

Figure 1.

(i) We know that $\mu_{\chi_I}(a \wedge b) = 1, \mu_{\chi_I}(a) = \mu_{\chi_I}(b) = 1; \nu_{\chi_I}(a \wedge b) = 0, \nu_{\chi_I}(a) = \nu_{\chi_I}(b) = 0$. Thus $\mu_{\chi_I}(a \wedge b) \nleq \mu_{\chi_I}(a) \vee \mu_{\chi_I}(b)$ and $\nu_{\chi_I}(a \wedge b) \ngeq \nu_{\chi_I}(a) \wedge \nu_{\chi_I}(b)$.
Hence χ_I is not an IFPI of L.
(ii) Define IFIs A and B of L as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x = a \\ 0, & \text{if } x = b, 1. \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.4, & \text{if } x = a \\ 1, & \text{if } x = b, 1. \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.3, & \text{if } x = b \\ 0, & \text{if } x = a, 1. \end{cases} ; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.6, & \text{if } x = b \\ 1, & \text{if } x = a, 1. \end{cases}$$

Then $A \cap B \subseteq \chi_I$ but neither $A \subseteq \chi_I$ nor $B \subseteq \chi_I$. Thus χ_I is not a PIFI of L.

Theorem 3.5. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. If P is an IFI of L, then there exist IFIs P_1, P_2 of L_1, L_2 respectively such that $P = P_1 \times P_2$. Moreover, if P is an IFPI, then so are P_1 and P_2.*

Proof. Define $P_i \in IFS(L_i)$, $i = 1, 2$. by $P_1(x) = P(x, 0)$ and $P_2(y) = P(0, y)$.
Let $x_1, x_2 \in L_1$, we have

$$\mu_P[(x_1, 0) \wedge (x_2, 0)] = \mu_P(x_1 \wedge x_2, 0) = \mu_{P_1}(x_1 \wedge x_2)\ ;$$
$$\nu_P[(x_1, 0) \wedge (x_2, 0)] = \nu_P(x_1 \wedge x_2, 0) = \nu_{P_1}(x_1 \wedge x_2)$$

and

$$\mu_P[(x_1, 0) \vee (x_2, 0)] = \mu_P(x_1 \vee x_2, 0) = \mu_{P_1}(x_1 \vee x_2)\ ;$$
$$\nu_P[(x_1, 0) \vee (x_2, 0)] = \nu_P(x_1 \vee x_2, 0) = \nu_{P_1}(x_1 \vee x_2).$$

Hence $\mu_{P_1}(x_1 \wedge x_2) \wedge \mu_{P_1}(x_1 \vee x_2) = \mu_P[(x_1, 0) \wedge (x_2, 0)] \wedge \mu_P[(x_1, 0) \vee (x_2, 0)]$ and $\nu_{P_1}(x_1 \wedge x_2) \vee \nu_{P_1}(x_1 \vee x_2) = \nu_P[(x_1, 0) \vee (x_2, 0)] \vee \nu_P[(x_1, 0) \vee (x_2, 0)]$.
As P is an IFI of L, we have

$$\begin{aligned} \mu_{P_1}(x_1 \wedge x_2) \wedge \mu_{P_1}(x_1 \vee x_2) &= \mu_P[(x_1, 0) \wedge (x_2, 0)] \wedge \mu_P[(x_1, 0) \vee (x_2, 0)] \\ &\geq \mu_P(x_1, 0) \wedge \mu_P(x_2, 0) \\ &= \mu_{P_1}(x_1) \wedge \mu_{P_1}(x_2). \end{aligned}$$

Thus, $\mu_{P_1}(x_1 \wedge x_2) \wedge \mu_{P_1}(x_1 \vee x_2) \geq \mu_{P_1}(x_1) \wedge \mu_{P_1}(x_2)$............................(**)
Similarly, we can show that $\nu_{P_1}(x_1 \wedge x_2) \vee \mu_{P_1}(x_1 \vee x_2) \leq \nu_{P_1}(x_1) \vee \nu_{P_1}(x_2)$.
Also, $\mu_{P_1}(x_1 \vee x_2) = \mu_P[(x_1, 0) \vee (x_2, 0))] = \mu_P(x_1, 0) \wedge \mu_P(x_2, 0) = \mu_{P_1}(x_1) \wedge \mu_{P_1}(x_2)$.
Similarly, we can have $\nu_{P_1}(x_1 \vee x_2) = \nu_{P_1}(x_1) \vee \nu_{P_1}(x_2)$. Therefore, from (**) we get $\mu_{P_1}(x_1 \wedge x_2) \geq \mu_{P_1}(x_1) \wedge \mu_{P_1}(x_2)$. Similarly, we can show that $\nu_{P_1}(x_1 \wedge x_2) \leq \nu_{P_1}(x_1) \vee \nu_{P_1}(x_2)$.
Thus P_1 is an IFI of L_1. Similarly, we can show that P_2 is an IFI of L_2.
Next, let $x_1 \in L_1, y_1 \in L_2$, we have

$$\begin{aligned} \mu_P(x_1, y_1) &= \mu_P[(x_1, 0) \vee (0, y_1)] \\ &= \mu_P(x_1, 0) \wedge \mu_P(0, y_1) \\ &= \mu_{P_1}(x_1) \wedge \mu_{P_2}(y_1) \\ &= \mu_{P_1 \times P_2}(x_1, y_1). \end{aligned}$$

Similarly, we can show that $\nu_P(x_1, y_1) = \nu_{P_1 \times P_2}(x_1, y_1)$. This implies that $P = P_1 \times P_2$.
Further, suppose that P is an IFPI of L. Let $x_1, x_2 \in L_1$. Then

$$\begin{aligned} \mu_{P_1}(x_1) \vee \mu_{P_1}(x_2) &= \mu_P(x_1, 0) \vee \mu_P(x_2, 0) \\ &= \mu_P[(x_1, 0) \wedge (x_2, 0)] \\ &= \mu_P(x_1 \wedge x_2, 0) \\ &= \mu_{P_1}(x_1 \wedge x_2). \end{aligned}$$

Similarly, we can show that $\nu_{P_1}(x_1) \wedge \nu_{P_1}(x_2) = \nu_{P_1}(x_1 \wedge x_2)$.
This implies that P_1 is an IFPI of L_1. In a same way, we can show that P_2 is an IFPI of L_2. □

The following examples shows that the converse of Theorem 3.5 may not be true.

Example 3.6. Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let P_1, P_2 be IFPIs of L_1, L_2 respectively. Then $P = P_1 \times P_2$ need not be an IFPI of L.

Proof. Consider the lattices L_1, L_2 as shown below:

Define IFSs $P_1 \in IFS(L_1)$ and $P_2 \in IFS(L_2)$ as follows:

$$P_1(x) = \begin{cases} (1, 0), & \text{if } x = 0, b \\ (0.5, 04), & \text{if } x = a \\ (0, 1), & \text{if } x = 1. \end{cases} \quad ; \quad P_2(x) = \begin{cases} (1, 0), & \text{if } x = 0 \\ (0, 1), & \text{if } x = 1. \end{cases}$$

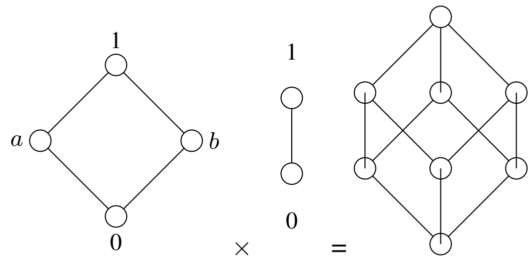

Figure 2. Product lattice

We note that P_1 is an IFPI of L_1 and P_2 is an IFPI of L_2. We consider $P \in IFS(L_1 \times L_2)$ defined by

$$\mu_P(x, y) = \mu_{P_1}(x) \wedge \mu_{P_2}(y) \text{ and } \nu_P(x, y) = \mu_{P_1}(x) \vee \nu_{P_2}(y).$$

i.e., $P = P_1 \times P_2$. We have

$$P(x, y) = \begin{cases} (1, 0), & \text{if } (x, y) = (0, 0), (b, 0) \\ (0.5, 04), & \text{if } (x, y) = (a, 0) \\ (0, 1), & \text{otherwise .} \end{cases}$$

Now, $\mu_P[(0, 1) \wedge (1, 0)] = \mu_P(0, 0) = 1$ and $\nu_P[(0, 1) \wedge (1, 0)] = \nu_P(0, 0) = 0$.
Also, $\mu_P(0, 1) = 0, \mu_P(1, 0) = 0, \nu_P(0, 1) = 1, \nu_P(1, 0) = 1$ implies that

$$\mu_P[(0, 1) \wedge (1, 0)] \nleq \mu_P(0, 1) \vee \mu_P(1, 0) \text{ and } \nu_P[(0, 1) \wedge (1, 0)] \ngeq \nu_P(0, 1) \wedge \nu_P(1, 0).$$

Hence P is not an IFPI of L. □

In Example (3.6), we have shown that a product of two IFPIs need not be an IFPI. However we have the following theorem.

Theorem 3.7. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let P_1 be an IFI of L_1. Then the product $P_1 \times \chi_{L_2}$ is an IFPI of L if and only if P_1 is an IFPI of L_1.*

Proof. Suppose that P_1 is an IFPI of L_1. We have

$$\begin{aligned} \mu_{P_1 \times \chi_{L_2}}[(x_1, y_1) \wedge (x_2, y_2)] &= \mu_{P_1 \times \chi_{L_2}}[(x_1 \wedge x_2, y_1 \wedge y_2)] \\ &= \mu_{P_1}(x_1 \wedge x_2) \wedge \mu_{\chi_{L_2}}(y_1 \wedge y_2) \\ &= \mu_{P_1}(x_1 \wedge x_2) \wedge 1 \\ &= \mu_{P_1}(x_1 \wedge x_2) \\ &= \mu_{P_1}(x_1) \vee \mu_{P_1}(x_2) \\ &= [\mu_{P_1}(x_1) \wedge 1] \vee [\mu_{P_1}(x_2) \wedge 1] \\ &= [\mu_{P_1}(x_1) \wedge \mu_{\chi_{L_2}}(y_1)] \vee [\mu_{P_1}(x_2) \wedge \mu_{\chi_{L_2}}(y_2)] \\ &= \mu_{P_1 \times \chi_{L_2}}(x_1, y_1) \vee \mu_{P_1 \times \chi_{L_2}}(x_2, y_2). \end{aligned}$$

Similarly, we can show that $\nu_{P_1 \times \chi_{L_2}}[(x_1, y_1) \wedge (x_2, y_2)] = \nu_{P_1 \times \chi_{L_2}}(x_1, y_1) \wedge \nu_{P_1 \times \chi_{L_2}}(x_2, y_2)$.
Hence $P_1 \times \chi_{L_2}$ is an IFPI of L.
The converse part can be similarly proved. □

Theorem 3.8. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let P_2 be an IFI of L_2. Then the product $\chi_{L_1} \times P_2$ is an IFPI of L if and only if P_2 is an IFPI of L_2.*

Proof. Straightforward. □

Theorem 3.9. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let P_i , Q_j be IFIs of L_1 and L_2 respectively. Let $R_{ij} = P_i \times Q_j$. Then $\cap R_{ij} = (\cap P_i) \times (\cap Q_i)$.*

Proof. Let $(x, y) \in L$, we have

$$\begin{aligned}
\mu_{\cap R_{ij}}(x,y) &= \wedge_{ij}[\mu_{P_i \times Q_j}(x,y)] \\
&= \wedge_{ij}[\mu_{P_i}(x) \wedge \mu_{Q_j}(y)] \\
&= [\wedge_{ij}\{\mu_{P_i}(x)\}] \wedge [\wedge_{ij}\{\mu_{Q_j}(y)\}] \\
&= [\wedge_{i}\{\mu_{P_i}(x)\}] \wedge [\wedge_{j}\{\mu_{Q_j}(y)\}] \\
&= [\mu_{\cap P_i}(x)] \wedge [\mu_{\cap Q_j}(y)] \\
&= \mu_{\cap P_i \times \cap Q_j}(x,y).
\end{aligned}$$

Similarly, we can show that $\nu_{\cap R_{ij}}(x,y) = \nu_{\cap P_i \times \cap Q_j}(x,y)$.
Hence $\cap R_{ij} = (\cap P_i) \times (\cap Q_i)$. □

4 Intuitionistic fuzzy primary ideals and primary intuitionistic fuzzy ideal of a lattice

Definition 4.1. [24] "Let L be a lattice with 0. An ideal I of L is called a primary ideal, if for $a, b \in L, a \wedge b \in I$ implies that either $a \in I$ or $b \in \sqrt{I}$, where $\sqrt{I}$ denotes the radical of I (i.e., the intersection of all prime ideals of L containing I).

If there does not exist a prime ideal containing an ideal I in a lattice L, then we have $\sqrt{I} = L$."

We define the radical of an IFI. Since there are two concepts of primeness (namely an IFPI and a PIFI), we can introduce two concepts, of the radical and primeness. For the radical of an IFS, we use the notation $\sqrt{A}$. The content will decide the radical (i.e., whether IF prime radical or prime IF radical).

Definition 4.2. Let Q be an IFI of a lattice L. We define the IF prime radical (respectively, prime IF radical) of Q as the intersection of all IFPIs (respectively, PIFIs) containing Q and we denote it by $\sqrt{Q}$.

We note that for an IFI Q of L always $Q \subseteq \sqrt{Q}$. It can be shown that for an I of L we have $\sqrt{\chi_I} = \chi_{\sqrt{I}}$.

Definition 4.3. A proper IFI Q of a lattice L is called an IF primary ideal of L, if for $a, b \in L$ the following holds:

$$\mu_Q(a \wedge b) \le \mu_Q(a) \vee \mu_{\sqrt{Q}}(B) \text{ and } \nu_Q(a \wedge b) \ge \nu_Q(a) \wedge \nu_{\sqrt{Q}}(b).$$

Lemma 4.4. *Let I be a proper ideal of L. Then I is a primary ideal of L if and only if χ_I is an IF primary ideal of L.*

Proof. Suppose that I is a primary ideal of L. Let $a, b \in L$

(i) If $a \wedge b \in I$, then as I is a primary ideal of L, either $a \in I$ or $b \in \sqrt{I}$. Thus, we have

$$\mu_{\chi_I}(a \wedge b) \le \mu_{\chi_I}(a) \vee \mu_{\chi_{\sqrt{I}}}(b) \text{ and } \nu_{\chi_I}(a \wedge b) \ge \nu_{\chi_I}(a) \wedge \nu_{\chi_{\sqrt{I}}}(b).$$

(ii) If $a \wedge b \notin I$, then clearly $a \notin I$ and $b \notin I$. In this case also, we have

$$\mu_{\chi_I}(a \wedge b) \le \mu_{\chi_I}(a) \vee \mu_{\chi_{\sqrt{I}}}(b) \text{ and } \nu_{\chi_I}(a \wedge b) \ge \nu_{\chi_I}(a) \wedge \nu_{\chi_{\sqrt{I}}}(b).$$

Hence χ_I is an IF primary ideal of L.

Conversely, suppose that χ_I is an IF primary ideal of L. Let $a \wedge b \in I$. Then

$$\mu_{\chi_I}(a \wedge b) \le \mu_{\chi_I}(a) \vee \mu_{\chi_{\sqrt{I}}}(b) \text{ and } \nu_{\chi_I}(a \wedge b) \ge \nu_{\chi_I}(a) \wedge \nu_{\chi_{\sqrt{I}}}(b)$$

implies that either $\mu_{\chi_I}(a) = 1, \nu_{\chi_I}(a) = 0$ or $\mu_{\chi_{\sqrt{I}}}(b) = 1, \nu_{\chi_{\sqrt{I}}}(b) = 0$.
This further implies that either $a \in I$ or $b \in \sqrt{I}$. Hence I is a primary ideal of L. □

Now we give a relationship between an IFPI and an IF primary ideal.

Lemma 4.5. *If Q is an IFPI of L, then Q is an IF primary ideal.*

Proof. Let Q be an IFPI of L. For all $a, b \in L$, we have

$$\mu_Q(a \wedge b) = \mu_Q(a) \vee \mu_Q(b) \text{ and } \nu_Q(a \wedge b) = \nu_Q(a) \wedge \nu_Q(b).$$

Since $Q \subseteq \sqrt{Q}$, we get $\mu_Q(b) \leq \mu_{\sqrt{Q}}(b)$ and $\nu_Q(b) \geq \nu_{\sqrt{Q}}(b)$. Thus we have

$$\mu_Q(a \wedge b) \leq \mu_Q(a) \vee \mu_{\sqrt{Q}}(b) \text{ and } \nu_Q(a \wedge b) \geq \nu_Q(a) \wedge \nu_{\sqrt{Q}}(b)$$

Hence Q is an IF Primary ideal. □

The following example shows that the converse of the Lemma (4.5) does not hold.

Example 4.6. Consider the ideal $I = (a]$ of the following lattice as shown in figure 3. We note

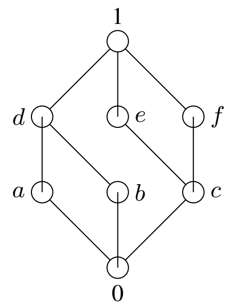

Figure 3.

that $J = (d]$ is the only prime ideal of L containing I. Hence $\sqrt{I} = J$. We know that for any ideal K of L, $\sqrt{\chi_K} = \chi_{\sqrt{K}}$. Hence $\sqrt{\chi_I} = \chi_{\sqrt{I}} = \chi_J$. Since J is a prime ideal, χ_J is an IFPI and so χ_I is an IF primary ideal of L. Also, because $b, c \notin I$, we have $\mu_{\chi_I}(b \wedge c) = 1$, but $\mu_{\chi_I}(b) \vee \mu_{\chi_I}(c) = 0$. Similarly, $\nu_{\chi_I}(b \wedge c) = 0$, but $\nu_{\chi_I}(b) \wedge \nu_{\chi_I}(c) = 1$. Thus χ_I is not an IFPI of L.

Theorem 4.7. *Let Q be an IFI of L. Then Q is an IF primary ideal if and only if the level cut set $Q_{(t,s)}$, where $t, s \in [0, 1]$ such that $t + s \leq 1$ is a primary ideal of L.*

Proof. Suppose that Q is an IF primary ideal of L. Let $a, b \in L$ be such that $a \wedge b \in Q_{(t,s)}$ and $a \notin Q_{(t,s)}, b \notin \sqrt{Q_{(t,s)}}$. Then we have

$$\mu_Q(a \wedge b) > t, \nu_Q(a \wedge b) < s \text{ and } t < \mu_Q(a), s > \nu_Q(a), t < \mu_{\sqrt{Q}}(b), s > \nu_{\sqrt{Q}}(b).$$

Since Q is an IF primary ideal, we have

$$\mu_Q(a \wedge b) \leq \mu_Q(a) \vee \mu_{\sqrt{Q}}(b) \text{ and } \nu_Q(a \wedge b) \geq \nu_Q(a) \wedge \nu_{\sqrt{Q}}(b).$$

Thus, we get $t < t$ and $s > s$, which is not possible. Hence $Q_{(t,s)}$ is a primary ideal of L.

Conversely, suppose that $Q_{(t,s)}$ is a primary ideal of L. Let $a, b \in L$ be such that

$$\mu_Q(a \wedge b) \nleq \mu_Q(a) \vee \mu_{\sqrt{Q}}(b) \text{ and } \nu_Q(a \wedge b) \ngeq \nu_Q(a) \wedge \nu_{\sqrt{Q}}(b).$$

Let $\mu_Q(a \wedge b) = t, \nu_Q(a \wedge b) = s$. Then $\mu_Q(a) < t, \mu_{\sqrt{Q}}(b) < t$ and $\nu_Q(a) > s, \nu_{\sqrt{Q}}(b) > s$. Since $Q_{(t,s)}$ is a primary ideal of L, $a \wedge b \in Q_{(t,s)}$ implies that either $a \in Q_{(t,s)}$ or $b \in \sqrt{Q_{(t,s)}}$, i.e., either $\mu_Q(a) \geq t$ or $\mu_{\sqrt{Q}}(b) \geq t$ and $\nu_Q(a) \leq s$ or $\nu_{\sqrt{Q}}(b) \leq s$, a contradiction. Hence Q is an IF primary ideal of L. □

From this onwards, L will be a complemented lattice.

Definition 4.8. A proper IFI Q of a lattice L is called a primary IFI of L if for $A, B \in IFI(L)$ such that

$$A \cap B \subseteq Q \text{ implies that either } A \subseteq Q \text{ or } B \subseteq \sqrt{Q}.$$

Now we give a relationship between a PIFI and a primary IFI.

Lemma 4.9. *If Q is a PIFI of L, then Q is a primary IFI of L.*

Proof. Let Q is a PIFI of L. Let $A \cap B \subseteq Q$ for some $A, B \in IFI(L)$. Since Q is a prime IFI, either $A \subseteq Q$ or $B \subseteq Q$. Since $Q \subseteq \sqrt{Q}$ always, we get the result. □

The following result gives the existence of primary IFIs which are not PIFI.

Theorem 4.10. *Let I be a primary ideal of L, $I \neq L$. The IFS Q of L defined by*

$$\mu_Q(x) = \begin{cases} 1, & \text{if } x \in I \\ \alpha, & \text{if } x \in L - I \end{cases}; \quad \nu_Q(x) = \begin{cases} 0, & \text{if } x \in I \\ \alpha', & \text{if } x \in L - I. \end{cases}$$

where α' is the complement of α in L (i.e., $\alpha \wedge \alpha' = 0, \alpha \vee \alpha' = 1$) is an IF primary ideal of L.

Proof. Clearly, Q is an IFI of L. Since $Q \subseteq \sqrt{Q}$, we have $\mu_Q(x) \leq \mu_{\sqrt{Q}}(x)$ and $\nu_Q(x) \geq \nu_{\sqrt{Q}}(x)$ for all $x \in L$. Therefore, if $x \in I$, then $\mu_{\sqrt{Q}}(x) = 1$ and $\nu_{\sqrt{Q}}(x) = 0$ and if $x \notin I$, then $\mu_{\sqrt{Q}}(x) = t \geq \alpha$ and $\nu_{\sqrt{Q}}(x) = s \leq \alpha'$.
Let A and B be IFIs of L such that $A \cap B \subseteq Q$. Suppose that $A \nsubseteq Q$ and $B \nsubseteq \sqrt{Q}$. let $x \in L$ be such that $\mu_A(x) > \mu_Q(x), \nu_A(x) < \nu_Q(x)$. This implies that $x \in I$, for otherwise $\mu_A(x) > 1, \nu_A(x) < 0$ which is not possible.

Let $\mu_A(x) > k_1 \geq \alpha = \mu_Q(x), \nu_A(x) < l_1 \leq \alpha' = \nu_Q(x)$.
Let $y \in L$ such that $\mu_B(y) > \mu_{\sqrt{Q}}(y), \nu_B(y) < \nu_{\sqrt{Q}}(y)$.
Clearly, $y \notin \sqrt{I}$, otherwise $\mu_B(y) > \mu_{\sqrt{Q}}(y) \geq \mu_Q(y) = 1$ and $\nu_B(y) < \nu_{\sqrt{Q}}(y) \leq \mu_Q(y) = 0$, which is not possible.

Let $\mu_A(y) = k_2$ and $\nu_A(y) = l_2$. Then $k_2 > \alpha$ and $l_2 < \alpha'$. Since I is primary, $x \wedge y \notin I$
Hence $\mu_Q(x \wedge y) = \alpha, \nu_Q(x \wedge y) = \alpha'$, we get

$$\mu_{A\cap B}(x \wedge y) \geq \min\{\mu_A(x), \mu_B(y)\} = \min\{k_1, k_2\} > \alpha = \mu_Q(x \wedge y) \text{ and}$$
$$\nu_{A\cap B}(x \wedge y) \leq \max\{\nu_A(x), \nu_B(y)\} = \max\{l_1, l_2\} < \alpha' = \nu_Q(x \wedge y)$$

which is not possible. Thus Q is a primary IFI of L. □

Theorem 4.11. *If Q is a primary IFI of L, then the level cut set $Q_{(t,s)}$, where $t, s \in [0, 1]$ such that $t + s \leq 1$ is a primary ideal of L.*

Proof. Let $a, b \in L$ be such that $a \wedge b \in Q_{(t,s)}$ and $a \notin Q_{(t,s)}$. Define IFIs A, B of L as follows:

$$A(x) = \begin{cases} (t, s), & \text{if } x \leq a \\ (0, 1), & \text{if } x \nleq a \end{cases}; \quad B(x) = \begin{cases} (t, s), & \text{if } x \leq b \\ (0, 1), & \text{if } x \nleq b. \end{cases}$$

Then $A \cap B \subseteq Q$. Also, $A \nsubseteq Q$ as $a \notin Q_{(t,s)}$ implies $\mu_Q(a) < t = \mu_A(a), \nu_Q(a) > s = \nu_A(a)$. Since Q is a primary IFI, we have $B \subseteq \sqrt{Q}$. Hence $t = \mu_B(b) \leq \mu_{\sqrt{Q}}(b), s = \nu_B(b) \geq \nu_{\sqrt{Q}}(b)$ and so $b \in \sqrt{Q_{(t,s)}}$. Thus $Q_{(t,s)}$ is a primary ideal of L. □

The following example shows that the converse of Theorem (4.11) does not hold.

Example 4.12. Consider the set $\mathbb{N}$ of natural numbers. Then ($\mathbb{N}$, divisibility) form a partially ordered set and thus a lattice under the join ($\vee$) and meet ($\wedge$) operations defined as

$$a \vee b = lcm\{a, b\} \text{ and } a \wedge b = gcd\{a, b\}; \text{ for all } a, b \in \mathbb{N}.$$

Let p be any prime number. Consider $t_i, s_i \in (0, 1), 0 \leq i \leq m$ be such that $t_1 > t_2 > ...... > t_m$ and $s_1 < s_2 < ..... < s_m$ with the condition $t_i + s_i \leq 1$.

Consider the IFI Q of $\mathbb{N}$ defined as

$$Q(x) = \begin{cases} (t_0, s_0), & \text{if } x \in (p^m] \\ (t_i, s_i), & \text{if } x \in (p^{m-i}] - (p^{m-i+1}], i = 1, 2, ...m. \end{cases}$$

Then we have

$$\sqrt{Q}(x) = \begin{cases} (t_0, s_0), & \text{if } x \in (p] \\ (t_i, s_i), & \text{if } x \in \mathbb{N} - (p]. \end{cases}$$

Define IFIs A and B of $\mathbb{N}$ by

$$A(x) = \begin{cases} (\alpha, \alpha^{'}), & \text{if } x \in (p^m] \\ (0, 1), & \text{otherwise }. \end{cases}$$

and $B(x) = (t_0, s_0)$ for all $x \in \mathbb{N}$. Then

$$(A \cap B)(x) = \begin{cases} (t_0, s_0), & \text{if } x \in (p^m] \\ (0, 1), & \text{otherwise }. \end{cases}$$

Thus $A \cap B \subseteq Q \subseteq \sqrt{Q}$ and $A \nsubseteq Q$. We note that if $x \in \mathbb{N} - (p]$, then

$$\mu_Q(x) = t_m < t_0 = \mu_B(x) \text{ and } \nu_Q(x) = s_m > s_0 = \nu_B(x)$$

Thus $B \nsubseteq \sqrt{Q}$. Hence Q is not primary IFI. However, each level cut ideal $Q_{(t_i,s_i)}$ of Q is primary, $i = 1, 2, ..., m$.

Theorem 4.13. *Let Q be a non-constant IFI of a lattice L. Then $\sqrt{Q}$ is a PIFI of L if and only if $\sqrt{Q}$ is a primary IFI of L.*

Proof. Let $\sqrt{Q}$ be a PIFI of L. Let $A, B \in IFI(L)$ be such that $A \cap B \subseteq \sqrt{Q}$. As $\sqrt{Q}$ is a prime IFI of L, either $A \subseteq \sqrt{Q}$ or $B \subseteq \sqrt{Q}$. Since $\sqrt{\sqrt{Q}} = \sqrt{Q}$. We conclude that $\sqrt{Q}$ is a primary IFI of L.
Conversely, suppose that $\sqrt{Q}$ is a primary IFI of L. Let $A, B \in IFI(L)$ be such that $A \cap B \subseteq \sqrt{Q}$. As $\sqrt{Q}$ is primary IFI, either $A \subseteq \sqrt{Q}$ or $B \subseteq \sqrt{\sqrt{Q}} = \sqrt{Q}$. Hence $\sqrt{Q}$ is a PIFI of L. □

Remark 4.14. From Example (3.6), we conclude that in general $\sqrt{P \times Q} \neq \sqrt{P} \times \sqrt{Q}$.

Theorem 4.15. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2.*
(i) Let P_1 be an IFI of L_1. Then $\sqrt{P_1 \times \chi_{L_2}} = \sqrt{P_1} \times \chi_{L_2}$.
(ii) Let P_2 be an IFI of L_2. Then $\sqrt{\chi_{L_1} \times P_2} = \chi_{L_1} \times \sqrt{P_2}$.

Proof. **(i)** Let P be an IFI of L such that $P_1 \times \chi_{L_2} \subseteq P$. By Theorem (3.5), $P = Q_1 \times Q_2$ for some IFIs Q_1 of L_1 and Q_2 of L_2. Then $P_1 \subseteq Q_1$ and $\chi_{L_2} \subseteq Q_2$. It follows that $Q_2 = \chi_{L_2}$. Thus $P \subseteq Q_1 \times \chi_{L_2}$. This shows that $\sqrt{P_1 \times \chi_{L_2}} = \sqrt{P_1} \times \chi_{L_2}$.
(ii) can be similarly proved. □

5 Intuitionistic fuzzy 2-absorbing ideals and 2-absorbing intuitionistic fuzzy ideals

Definition 5.1. ([9]) "Let L be a lattice with 0. An ideal I of L is called a 2-absorbing ideal, if for $a, b, c \in L$,

$$a \wedge b \wedge c \in I \text{ implies that either } a \wedge b \in I \text{ or } b \wedge c \in I \text{ or } c \wedge a \in I."$$

We extend the concept of a 2-absorbing ideals, in the context of an IFI of a lattice and prove some properties of intuitionistic fuzzy 2-absorbing ideals of a lattice.

Definition 5.2. A proper IFI A of a lattice L is called an intuitionistic fuzzy 2-absorbing ideal (IF2AI) of L, if for $a, b, c \in L$,

$$\mu_A(a\wedge b\wedge c)\leq \max\{\mu_A(a\wedge b),\mu_A(b\wedge c),\mu_A(c\wedge a)\} \text{ and}$$
$$\nu_A(a\wedge b\wedge c)\geq \min\{\nu_A(a\wedge b),\nu_A(b\wedge c),\nu_A(c\wedge a)\}.$$

Since $\mu_A(a\wedge b),\mu_A(b\wedge c),\mu_A(c\wedge a),\nu_A(a\wedge b),\nu_A(b\wedge c),\nu_A(c\wedge a)$ are all non-negative real numbers, the definition of an IF2AI is equivalent to : A is an IF2AI if and only if for all $a,b,c\in L$,

$$\mu_A(a\wedge b\wedge c)\leq \mu_A(a\wedge b)\vee\mu_A(b\wedge c)\vee\mu_A(c\wedge a)\} \text{ and}$$
$$\nu_A(a\wedge b\wedge c)\geq \nu_A(a\wedge b)\wedge\nu_A(b\wedge c)\wedge\nu_A(c\wedge a)\}.$$

Infact, A is an IF2AI if and only if for all $a,b,c\in L$,

$$\mu_A(a\wedge b\wedge c)= \mu_A(a\wedge b)\vee\mu_A(b\wedge c)\vee\mu_A(c\wedge a) \text{ and}$$
$$\nu_A(a\wedge b\wedge c)= \nu_A(a\wedge b)\wedge\nu_A(b\wedge c)\wedge\nu_A(c\wedge a).$$

Lemma 5.3. *Let I be an ideal of L. Then I is a 2-absorbing ideal of L if and only if χ_I is an IF2AI of L.*

Proof. Suppose that I is a 2-absorbing ideal of L. Let $a,b,c\in L$.
If $a\wedge b\wedge c\in I$, then as I is an 2-absorbing ideal, either $a\wedge b\in I$ or $b\wedge c\in I$ or $c\wedge a\in I$.
Thus in this case,

$$\mu_{\chi_I}(a\wedge b\wedge c)\leq \mu_{\chi_I}(a\wedge b)\vee\mu_{\chi_I}(b\wedge c)\vee\mu_{\chi_I}(c\wedge a)\} \text{ and}$$
$$\nu_{\chi_I}(a\wedge b\wedge c)\geq \nu_{\chi_I}(a\wedge b)\wedge\nu_{\chi_I}(b\wedge c)\wedge\nu_{\chi_I}(c\wedge a)\}.$$

If $a\wedge b\wedge c\notin I$, then clearly $a\wedge b\notin I$, $b\wedge c\notin I$, $c\wedge a\notin I$. Thus in this case,

$$\mu_{\chi_I}(a\wedge b\wedge c)\leq \mu_{\chi_I}(a\wedge b)\vee\mu_{\chi_I}(b\wedge c)\vee\mu_{\chi_I}(c\wedge a)\} \text{ and}$$
$$\nu_{\chi_I}(a\wedge b\wedge c)\geq \nu_{\chi_I}(a\wedge b)\wedge\nu_{\chi_I}(b\wedge c)\wedge\nu_{\chi_I}(c\wedge a)\}.$$

Hence χ_I is an IF2AI of L.

Conversely, suppose that χ_I is an IF2AI of L. Let $a,b,c\in L$ such that $a\wedge b\wedge c\in I$, but $a\wedge b\notin I$, $b\wedge c\notin I$, $c\wedge a\in I$. This implies that $\mu_A(a\wedge b\wedge c)=1,\nu_A(a\wedge b\wedge c)=0$ and $\mu_{\chi_I}(a\wedge b)=\mu_{\chi_I}(b\wedge c)=\mu_{\chi_I}(c\wedge a)=0$; $\nu_{\chi_I}(a\wedge b)=\nu_{\chi_I}(b\wedge c)=\nu_{\chi_I}(c\wedge a)=1$. Then

$$\mu_{\chi_I}(a\wedge b\wedge c)=1\nleq 0=\mu_{\chi_I}(a\wedge b)\vee\mu_{\chi_I}(b\wedge c)\vee\mu_{\chi_I}(c\wedge a)\} \text{ and}$$
$$\nu_{\chi_I}(a\wedge b\wedge c)=0\ngeq 1=\nu_{\chi_I}(a\wedge b)\wedge\nu_{\chi_I}(b\wedge c)\wedge\nu_{\chi_I}(c\wedge a)\},$$

a contradiction, as χ_I is an IF2AI of L. Therefore, either $a\wedge b\in I$ or $b\wedge c\in I$ or $c\wedge a\in I$. Hence I is a 2-absorbing ideal of L. □

Lemma 5.4. *An IFI A of L is an IF2AI if and only if each level cut set $A_{(t,s)}$ is a 2-absorbing ideal of L, where $t,s\in[0,1]$ such that $t+s\leq 1$.*

Proof. (i) Let A be an IF2AI of L. Let $a,b,c\in L$ be such that $a\wedge b\wedge c\in A_{(t,s)}$. Then $\mu_A(a\wedge b\wedge c)\geq t$ and $\nu_A(a\wedge b\wedge c)\leq s$. Since A is an IF2AI of L,

$$t\leq\mu_A(a\wedge b\wedge c)\leq \mu_A(a\wedge b)\vee\mu_A(b\wedge c)\vee\mu_A(c\wedge a)\} \text{ and}$$
$$s\geq\nu_A(a\wedge b\wedge c)\geq \nu_A(a\wedge b)\wedge\nu_A(b\wedge c)\wedge\nu_A(c\wedge a)\}.$$

Since $t,s,\mu_A(a\wedge b),\mu_A(b\wedge c),\mu_A(c\wedge a),\nu_A(a\wedge b),\nu_A(b\wedge c),\nu_A(c\wedge a)$ are all non-negative real numbers. Therefore, $\mu_A(a\wedge b)<t,\mu_A(b\wedge c)<t,\mu_A(c\wedge a)<t$ and $\nu_A(a\wedge b)>s,\nu_A(b\wedge c)>s,\nu_A(c\wedge a)>s$, then

$$\mu_A(a\wedge b\wedge c)\leq \mu_A(a\wedge b)\vee\mu_A(b\wedge c)\vee\mu_A(c\wedge a)\} \text{ and}$$
$$\nu_A(a\wedge b\wedge c)\geq \nu_A(a\wedge b)\wedge\nu_A(b\wedge c)\wedge\nu_A(c\wedge a).$$

This leads to $t<t$ and $s>s$, which is not possible. Hence $t\leq\mu_A(a\wedge b)$ or $t\leq\mu_A(b\wedge c)$ or $t\leq\mu_A(c\wedge a)$ and $s\geq\nu_A(a\wedge b)$ or $s\geq\nu_A(b\wedge c)$ or $s\geq\nu_A(c\wedge a)$. Thus either $a\wedge b\in A_{(t,s)}$ or $b\wedge c\in A_{(t,s)}$ or $c\wedge a\in A_{(t,s)}$. i.e., $A_{(t,s)}$ is a 2-absorbing ideal of L.

(ii) Let $A_{(t,s)}$ be a 2-absorbing ideal of L. Let $a,b,c\in L$ and $\mu_A(a\wedge b\wedge c)=t,\nu_A(a\wedge b\wedge c)=s$. Then $a\wedge b\wedge c\in A_{(t,s)}$. Since $A_{(t,s)}$ is a 2-absorbing ideal of L, either $a\wedge b\in A_{(t,s)}$ or $b\wedge c\in A_{(t,s)}$ or $c\wedge a\in A_{(t,s)}$. This implies that

$$t \leq \mu_A(a \wedge b \wedge c) \leq \mu_A(a \wedge b) \vee \mu_A(b \wedge c) \vee \mu_A(c \wedge a)\} \text{ and}$$
$$s \geq \nu_A(a \wedge b \wedge c) \geq \nu_A(a \wedge b) \wedge \nu_A(b \wedge c) \wedge \nu_A(c \wedge a).$$

Thus A is an IF2AI of L. □

Now we show that every IFPI of L is an IF2AI.

Lemma 5.5. *Let P be an IFPI of L. Then P is an IF2AI of L.*

Proof. Let P be an IFPI of L. Then for all $a, b \in L$, we have

$$\mu_P(a \wedge b) \leq \mu_P(a) \vee \mu_P(b) \text{ and } \nu_P(a \wedge b) \geq \nu_P(a) \wedge \nu_P(b).$$

Hence for all $a, b, c \in L$, we have

$$\mu_P(a \wedge b \wedge c) \leq \mu_P(a \wedge b) \vee \mu_P(c) \text{ and } \nu_P(a \wedge b \wedge c) \geq \nu_P(a \wedge b) \wedge \nu_P(c)$$
$$\mu_P(a \wedge b \wedge c) \leq \mu_P(b \wedge c) \vee \mu_P(a) \text{ and } \nu_P(a \wedge b \wedge c) \geq \nu_P(b \wedge c) \wedge \nu_P(a)$$
$$\mu_P(a \wedge b \wedge c) \leq \mu_P(c \wedge a) \vee \mu_P(b) \text{ and } \nu_P(a \wedge b \wedge c) \geq \nu_P(c \wedge a) \wedge \nu_P(b).$$

Hence

$$\mu_P(a \wedge b \wedge c) \leq \mu_P(a \wedge b) \vee \mu_P(b \wedge c) \vee \mu_P(c \wedge a) \vee \mu_P(a) \vee \mu_P(b) \vee \mu_P(c) \text{ and}$$
$$\nu_P(a \wedge b \wedge c) \geq \nu_P(a \wedge b) \wedge \nu_P(b \wedge c) \wedge \nu_P(c \wedge a) \wedge \nu_P(a) \wedge \nu_P(b) \wedge \nu_P(c).$$

By the definition of IFI, it follows that for any $x, y \in L$, $\mu_P(x) \leq \mu_P(x \wedge y)$ and $\nu_P(x) \geq \nu_P(x \wedge y)$. Thus we have

$$\mu_P(a \wedge b \wedge c) \leq \mu_P(a \wedge b) \vee \mu_P(b \wedge c) \vee \mu_P(c \wedge a)\} \text{ and}$$
$$\nu_P(a \wedge b \wedge c) \geq \nu_P(a \wedge b) \wedge \nu_P(b \wedge c) \wedge \nu_P(c \wedge a)\}.$$

Thus P is an IF2AI of L. □

The following example shows that the converse of Lemma (5.5) does not hold.

Example 5.6. Consider the lattice L as shown in figure 1. Let P be an IFS on L defined by

$$\mu_P(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x = b \\ 0, & \text{if } x = a, 1. \end{cases} \quad ; \quad \nu_P(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.4, & \text{if } x = b \\ 1, & \text{if } x = a, 1. \end{cases}$$

Then P is an IF2AI of L. However, P is not an IFPI of L as $1 = \mu_P(0) = \mu_P(a \wedge b) \neq 0.5 = 0 \vee 0.5 = \mu_P(a) \vee \mu_P(b)$ and $0 = \nu_P(0) = \nu_P(a \wedge b) \neq 0.4 = 1 \wedge 0.4 = \nu_P(a) \wedge \nu_P(b)$.

Lemma 5.7. *The intersection of any two distinct IFPIs of L is an IF2AI of L.*

Proof. Let P_1 and P_2 be two distinct IFPIs of L. We know that for any $a \in L$,

$$\mu_{P_1 \cap P_2}(a) = \mu_{P_1}(a) \wedge \mu_{P_2}(a) \text{ and } \nu_{P_1 \cap P_2}(a) = \nu_{P_1}(a) \vee \nu_{P_2}(a).$$

Let $a, b, c \in L$, we have

$$\mu_{P_1 \cap P_2}(a \wedge b \wedge c) = \mu_{P_1}(a \wedge b \wedge c) \wedge \mu_{P_2}(a \wedge b \wedge c) \text{ and}$$
$$\nu_{P_1 \cap P_2}(a \wedge b \wedge c) = \nu_{P_1}(a \wedge b \wedge c) \vee \nu_{P_2}(a \wedge b \wedge c).$$

Since every IFPI is an IF2AI, so we have

$$\mu_{P_1 \cap P_2}(a \wedge b \wedge c) \leq [\mu_{P_1}(a \wedge b) \vee \mu_{P_1}(b \wedge c) \vee \mu_{P_1}(c \wedge a)] \wedge [\mu_{P_2}(a \wedge b) \vee \mu_{P_2}(b \wedge c) \vee \mu_{P_2}(c \wedge a)]$$
and
$$\nu_{P_1 \cap P_2}(a \wedge b \wedge c) \geq [\nu_{P_1}(a \wedge b) \wedge \nu_{P_1}(b \wedge c) \wedge \nu_{P_1}(c \wedge a)] \vee [\nu_{P_2}(a \wedge b) \wedge \nu_{P_2}(b \wedge c) \wedge \nu_{P_2}(c \wedge a)].$$

Since $P_i, i = 1, 2$ are IFPIs, so we can write

$$\mu_{P_i}(a \wedge b) \vee \mu_{P_i}(b \wedge c) \vee \mu_{P_i}(c \wedge a) \leq \mu_{P_i}(a) \vee \mu_{P_i}(b) \vee \mu_{P_i}(c) \text{ and}$$
$$\nu_{P_i}(a \wedge b) \wedge \nu_{P_i}(b \wedge c) \wedge \nu_{P_i}(c \wedge a) \geq \nu_{P_i}(a) \wedge \nu_{P_i}(b) \wedge \nu_{P_i}(c)$$

We note that all the terms in the R.H.S. of the above inequalities belong to the distributive lattice $[0, 1]$. Hence we can write

$$\mu_{P_1\cap P_2}(a\wedge b\wedge c)\leq[\mu_{P_1}(a)\vee\mu_{P_1}(b)\vee\mu_{P_1}(c)]\wedge[\mu_{P_2}(a)\vee\mu_{P_2}(b)\vee\mu_{P_2}(c)]$$
$$=[\mu_{P_1}(a)\wedge\mu_{P_2}(a)]\vee[\mu_{P_1}(a)\wedge\mu_{P_2}(b)]\vee[\mu_{P_1}(a)\wedge\mu_{P_2}(c)]$$
$$\vee[\mu_{P_1}(b)\wedge\mu_{P_2}(a)]\vee[\mu_{P_1}(b)\wedge\mu_{P_2}(b)]\vee[\mu_{P_1}(b)\wedge\mu_{P_2}(c)]$$
$$\vee[\mu_{P_1}(c)\wedge\mu_{P_2}(a)]\vee[\mu_{P_1}(c)\wedge\mu_{P_2}(b)]\vee[\mu_{P_1}(c)\wedge\mu_{P_2}(a)].$$

i.e., $\mu_{P_1\cap P_2}(a\wedge b\wedge c)\leq[\mu_{P_1}(a)\wedge\mu_{P_2}(a)]\vee[\mu_{P_1}(a)\wedge\mu_{P_2}(b)]\vee[\mu_{P_1}(a)\wedge\mu_{P_2}(c)]$
$$\vee[\mu_{P_1}(b)\wedge\mu_{P_2}(a)]\vee[\mu_{P_1}(b)\wedge\mu_{P_2}(b)]\vee[\mu_{P_1}(b)\wedge\mu_{P_2}(c)]$$
$$\vee[\mu_{P_1}(c)\wedge\mu_{P_2}(a)]\vee[\mu_{P_1}(c)\wedge\mu_{P_2}(b)]\vee[\mu_{P_1}(c)\wedge\mu_{P_2}(a)]$$

Similarly, we can have

$$\nu_{P_1\cap P_2}(a\wedge b\wedge c)\geq[\nu_{P_1}(a)\vee\nu_{P_2}(a)]\wedge[\nu_{P_1}(a)\vee\nu_{P_2}(b)]\wedge[\nu_{P_1}(a)\vee\nu_{P_2}(c)]$$
$$\wedge[\nu_{P_1}(b)\vee\nu_{P_2}(a)]\wedge[\nu_{P_1}(b)\vee\nu_{P_2}(b)]\wedge[\nu_{P_1}(b)\vee\nu_{P_2}(c)]$$
$$\wedge[\nu_{P_1}(c)\vee\nu_{P_2}(a)]\wedge[\nu_{P_1}(c)\vee\nu_{P_2}(b)]\wedge[\nu_{P_1}(c)\vee\nu_{P_2}(a)]$$

Now, for any IFI A of L, we have $\mu_A(y)\leq\mu_A(x\wedge y)$ and $\nu_A(y)\geq\nu_A(x\wedge y)$ for all $x,y\in L$. This implies that

$$\mu_{P_1}(x)\wedge\mu_{P_2}(y)\leq\mu_{P_1}(x\wedge y)\wedge\mu_{P_2}(x\wedge y)=\mu_{P_1\cap P_2}(x\wedge y)\text{ and}$$
$$\nu_{P_1}(x)\vee\nu_{P_2}(y)\geq\nu_{P_1}(x\wedge y)\vee\nu_{P_2}(x\wedge y)=\nu_{P_1\cap P_2}(x\wedge y).$$

Using these, we get

$$\mu_{P_1\cap P_2}(a\wedge b\wedge c)\leq\mu_{P_1\cap P_2}(a\wedge b)\vee\mu_{P_1\cap P_2}(b\wedge c)\vee\mu_{P_1\cap P_2}(c\wedge a)\text{ and}$$
$$\nu_{P_1\cap P_2}(a\wedge b\wedge c)\geq\nu_{P_1\cap P_2}(a\wedge b)\wedge\nu_{P_1\cap P_2}(b\wedge c)\wedge\nu_{P_1\cap P_2}(c\wedge a).$$

Since $P_1\cap P_2$ is an IFI, for all $x,y\in L$, we have

$$\mu_{P_1\cap P_2}(x)\leq\mu_{P_1\cap P_2}(x\wedge y)\text{ and }\nu_{P_1\cap P_2}(x)\geq\nu_{P_1\cap P_2}(x\wedge y).$$

Using these, we get

$$\mu_{P_1\cap P_2}(a\wedge b\wedge c)\leq\mu_{P_1\cap P_2}(a\wedge b)\vee\mu_{P_1\cap P_2}(b\wedge c)\vee\mu_{P_1\cap P_2}(c\wedge a)\}\text{ and}$$
$$\nu_{P_1\cap P_2}(a\wedge b\wedge c)\geq\nu_{P_1\cap P_2}(a\wedge b)\wedge\nu_{P_1\cap P_2}(b\wedge c)\wedge\nu_{P_1\cap P_2}(c\wedge a)\}.$$

Thus $P_1\cap P_2$ is an IF2AI of L. □

The following example shows that the condition of “primeness" in Lemma (5.7) is necessary. This example also shows that in general the intersection of two IF2AIs need not be an IF2AI.

Example 5.8. Consider the lattice as shown in the following figure 4:
Define IFS A_1 and A_2 as follows

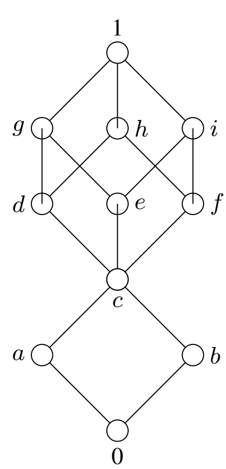

Figure 4.

$$\mu_{A_1}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x = a, c, d \\ 0.6, & \text{if } x = b \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{A_1}(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.4, & \text{if } x = a, c, d \\ 0.2, & \text{if } x = b \\ 1, & \text{otherwise} \end{cases}$$

and

$$\mu_{A_2}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.3, & \text{if } x = a, b, c, e \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{A_2}(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.6, & \text{if } x = a, b, c, e \\ 1, & \text{otherwise .} \end{cases}$$

We note that A_1 and A_2 ae IF2AIs of L. For

$$\mu_{A_1}(d \wedge e \wedge f) = \mu_{A_1}(c) \text{ and } \mu_{A_1}(d \wedge e) = \mu_{A_1}(e \wedge f) = \mu_{A_1}(f \wedge d) = \mu_{A_1}(c)$$
$$\nu_{A_1}(d \wedge e \wedge f) = \nu_{A_1}(c) \text{ and } \nu_{A_1}(d \wedge e) = \nu_{A_1}(e \wedge f) = \nu_{A_1}(f \wedge d) = \nu_{A_1}(c).$$

$$\mu_{A_1}(g \wedge h \wedge i) = \mu_{A_1}(c) = 0.5 \text{ and } \mu_{A_1}(g \wedge h) = \mu_{A_1}(d) = 0.5, \mu_{A_1}(h \wedge i) = \mu_{A_1}(f) = 0,$$
$$\mu_{A_1}(i \wedge g) = \mu_{A_1}(e) = 0.$$
$$\nu_{A_1}(g \wedge h \wedge i) = \nu_{A_1}(c) = 0.4 \text{ and } \nu_{A_1}(g \wedge h) = \nu_{A_1}(d) = 0.5, \nu_{A_1}(h \wedge i) = \nu_{A_1}(f) = 1,$$
$$\nu_{A_1}(i \wedge g) = \nu_{A_1}(e) = 1.$$

Similarly for other elements. Note that

$$\mu_{A_1 \cap A_2}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.3, & \text{if } x = a, b, c \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{A_1 \cap A_2}(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.6, & \text{if } x = a, b, c \\ 1, & \text{otherwise .} \end{cases}$$

Thus $\mu_{A_1 \cap A_2}(g \wedge h \wedge i) = \mu_{A_1 \cap A_2}(c) = 0.3$. But
$\max\{\mu_{A_1 \cap A_2}(f \wedge h), \mu_{A_1 \cap A_2}(h \wedge i), \mu_{A_1 \cap A_2}(i \wedge g)\} = \max\{\mu_{A_1 \cap A_2}(d), \mu_{A_1 \cap A_2}(f), \mu_{A_1 \cap A_2}(e)\}$
= $\max\{0, 0, 0\} = 0$. Thus

$$\mu_{A_1 \cap A_2}(g \wedge h \wedge i) = 0.3 \nleq 0 = \max\{\mu_{A_1 \cap A_2}(f \wedge h), \mu_{A_1 \cap A_2}(h \wedge i), \mu_{A_1 \cap A_2}(i \wedge g)\}.$$

Similarly, we can show that

$$\nu_{A_1 \cap A_2}(g \wedge h \wedge i) = 0.6 \ngeq 1 = \min\{\nu_{A_1 \cap A_2}(f \wedge h), \nu_{A_1 \cap A_2}(h \wedge i), \nu_{A_1 \cap A_2}(i \wedge g)\}.$$

Hence $A_1 \cap A_2$ is not an IF2AI of L.

Now we introduce the concept of a 2-absorbing intuitionistic fuzzy ideal (2-AIFI) on the lines of a prime intuitionistic fuzzy ideal (PIFI).

Definition 5.9. A proper IFI P of L is called 2-absorbing intuitionistic fuzzy ideal (2-AIFI) of L if whenever for some $A, B, C \in IFI(L)$ we have

$$A \cap B \cap C \subseteq P \text{ implies that either } A \cap B \subseteq P \text{ or } B \cap C \subseteq P \text{ or } C \cap A \subseteq P.$$

The following example shows that the concept of a "IF2AI" is different from that of a "2-AIFI".

Example 5.10. Consider the following IFIs of the Lattice L as shown in figure 1.

$$A(x) = \begin{cases} (0.80, 0.10), & \text{if } x = 0 \\ (.35, 0.50), & \text{if } x = a, 1 \\ (0.75, 0.20), & \text{if } x = b. \end{cases}; \quad B(x) = \begin{cases} (1, 0), & \text{if } x = 0 \\ (0.80, 0.15), & \text{if } x = a, 1 \\ (0.25, 0.55), & \text{if } x = b. \end{cases}$$

and

$$C(x) = \begin{cases} (1, 0), & \text{if } x = 0 \\ (0.66, 0.25), & \text{if } x = a, 1 \\ (0.75, 0.20), & \text{if } x = b. \end{cases}; \quad P(x) = \begin{cases} (0.80, 0.10), & \text{if } x = 0 \\ (0.75, 0.20), & \text{if } x = a, 1 \\ (0.80, 0.15), & \text{if } x = b. \end{cases}$$

We note that (i) P is an IF2AI and (ii) $A \cap B \cap C \subseteq P$. But $A \cap B \nsubseteq P$, $B \cap C \nsubseteq P$ and $C \cap A \nsubseteq P$. Thus P is not a 2-AIFI of L.

Lemma 5.11. *Let I be an ideal of L. If χ_I is a 2-AIFI of L, then I is a 2-AI of L.*

Proof. Suppose that χ_I is a 2-AIFI of L. Let $a \wedge b \wedge c \in I$ for some $a, b, c \in L$. Suppose that $a \wedge b \notin I$, $b \wedge c \notin I$ amd $c \wedge a \notin I$. Define IFIs

$$A(x) = \begin{cases} (1,0), & \text{if } x \in (a] \\ (0.1), & \text{otherwise} \end{cases} ; \quad B(x) = \begin{cases} (1,0), & \text{if } x \in (b] \\ (0,1), & \text{otherwise} \end{cases} ; \quad C(x) = \begin{cases} (1,0), & \text{if } x \in (c] \\ (0,1), & \text{otherwise} . \end{cases}$$

We note that

$$(A \cap B \cap C)(x) = \begin{cases} (1,0), & \text{if } x \in (a \wedge b \wedge c] \\ (0.1), & \text{otherwise} . \end{cases}$$

Thus $A \cap B \cap C \subseteq \chi_I$ but $A \cap B \nsubseteq \chi_I$, $B \cap C \nsubseteq \chi_I$ and $C \cap A \nsubseteq \chi_I$. This contradict the assumption that χ_I is a 2-AIFI of L. □

Remark 5.12. However, we are unable to prove or disprove that if I is 2-AI of L, then χ_I is 2-AIFI of L.

Lemma 5.13. *Every PIFI of a lattice L is a 2-AIFI of L.*

Proof. Let P be a PIFI of L. Suppose that $A, B, C \in IFI(L)$ and $A \cap B \cap C \subseteq P$. As P is a prime IFI of l, we have either
(1) $A \cap B \subseteq P$ or $C \subseteq P$, or
(2)$B \cap C \subseteq P$ or $A \subseteq P$, or
(3) $C \cap A \subseteq P$ or $B \subseteq P$.

Without loss of generality, suppose that $A \cap B \subseteq P$ or $C \subseteq P$.
If $A \cap B \subseteq P$, then the proof is obvious and if $C \subseteq P$, then $A \cap C \subseteq P$ and $C \cap B \subseteq P$. Thus P is a 2-AIFI of L. □

We are unable to give an example to show that the converse of Lemma (5.13) does not hold.

Proposition 5.14. *The intersection of two PIFIs of L is a 2-AIFI of L.*

Proof. Let P_1 and P_2 be two distinct PIFIs of L. Assume that A, B, C are IFIs of L such that $A \cap B \cap C \subseteq P_1 \cap P_2$ but $A \cap B \nsubseteq P_1 \cap P_2$, $B \cap C \nsubseteq P_1 \cap P_2$ and $C \cap A \nsubseteq P_1 \cap P_2$.

Clearly, $A \cap B \cap C \subseteq P_1$ and $A \cap B \cap C \subseteq P_2$. Since P_1 and P_2 are prime IFIs of L, we have (i) $A \cap B \subseteq P_1$ or $B \cap C \subseteq P_1$ or $C \cap A \subseteq P_1$ and (ii) $A \cap B \subseteq P_2$ or $B \cap C \subseteq P_2$ or $C \cap A \subseteq P_2$. We have the following cases:-
Case(1): If $A \cap B \cap C \subseteq P_1$ and $A \cap B \subseteq P_2$, then we have $A \cap B \subseteq P_1 \cap P_2$, a contradiction.
Case(2): If $C \cap A \subseteq P_1$ and $C \cap A \subseteq P_2$, we get $C \subseteq P_1 \cap P_2$ and hence $C \cap A \subseteq P_1 \cap P_2$, a contradiction.
Case(3): Let $A \cap B \cap C \subseteq P_1$ and $C \cap A \subseteq P_2$. As P_1 is a prime IFI, we get either $A \subseteq P_1$ or $B \subseteq P_1$. Hence either $A \cap C \subseteq P_1 \cap P_2$ or $B \cap C \subseteq P_1 \cap P_2$, a contradiction in either case.
Case(4): Let $C \cap A \subseteq P_1$ and $A \cap B \subseteq P_2$. As P_2 is a PIFI, we get either $A \subseteq P_2$ or $B \subseteq P_2$. Hence either $A \cap C \subseteq P_1 \cap P_2$ or $B \cap C \subseteq P_1 \cap P_2$, a contradiction in either case.

Hence at least one of the $A \cap B$ or $B \cap C$ or $C \cap A$ must be a subset of $P_1 \cap P_2$. Therefore $P_1 \cap P_2$ is a 2-AIFI of L. □

Definition 5.15. A proper IFI A of a lattice L is called an intuitionistic fuzzy 2-absorbing primary ideal (IF2API) of L, if for $a, b, c \in L$

$$\mu_A(a \wedge b \wedge c) \leq \mu_A(a \wedge b) \vee \mu_{\sqrt{A}}(b \wedge c) \vee \mu_{\sqrt{A}}(c \wedge a) \text{ and}$$
$$\nu_A(a \wedge b \wedge c) \geq \nu_A(a \wedge b) \wedge \nu_{\sqrt{A}}(b \wedge c) \wedge \nu_{\sqrt{A}}(c \wedge a).$$

Lemma 5.16. *A proper ideal I of L is a 2-absorbing primary ideal(2-API), if and only if χ_I is an IF2API of L.*

Proof. Suppose that I is a 2-absorbing prime ideal of L. Let $a, b, c \in L$.
If $a \wedge b \wedge c \in I$, then $\mu_{\chi_I}(a \wedge b \wedge c) = 1, \nu_{\chi_I}(a \wedge b \wedge c) = 0$.
As I is 2-API, we have either $a \wedge b \in I$ or $b \wedge c \in \sqrt{I}$ or $c \wedge a \in \sqrt{I}$.
Hence either $\mu_{\chi_I}(a \wedge b) = 1, \nu_{\chi_I}(a \wedge b) = 0$ or $\mu_{\sqrt{\chi_I}}(b \wedge c) = \mu_{\chi_{\sqrt{I}}}(b \wedge c) = 1, \nu_{\sqrt{\chi_I}}(b \wedge c) = \nu_{\chi_{\sqrt{I}}}(b \wedge c) = 0$ or $\mu_{\sqrt{\chi_I}}(c \wedge a) = \mu_{\chi_{\sqrt{I}}}(c \wedge a) = 1, \nu_{\sqrt{\chi_I}}(c \wedge a) = \nu_{\chi_{\sqrt{I}}}(c \wedge a) = 0$.
Thus

$$\mu_{\chi_I}(a \wedge b \wedge c) = 1 \leq 1 = \mu_{\chi_I}(a \wedge b) \vee \mu_{\chi_{\sqrt{I}}}(b \wedge c) \vee \mu_{\chi_{\sqrt{I}}}(c \wedge a) \text{ and}$$
$$\nu_{\chi_I}(a \wedge b \wedge c) = 0 \geq 0 = \nu_{\chi_I}(a \wedge b) \wedge \nu_{\chi_{\sqrt{I}}}(b \wedge c) \wedge \nu_{\chi_{\sqrt{I}}}(c \wedge a).$$

If $a \wedge b \wedge c \notin I$, then $\mu_{\chi_I}(a \wedge b \wedge c) = 0, \nu_{\chi_I}(a \wedge b \wedge c) = 1$.
Clearly, $a \wedge b \notin I$ and so $\mu_{\chi_I}(a \wedge b) = 0, \nu_{\chi_I}(a \wedge b) = 1$. Hence

$$\mu_{\chi_I}(a \wedge b \wedge c) = 0 \leq \mu_{\chi_I}(a \wedge b) \vee \mu_{\chi_{\sqrt{I}}}(b \wedge c) \vee \mu_{\chi_{\sqrt{I}}}(c \wedge a) \text{ and}$$
$$\nu_{\chi_I}(a \wedge b \wedge c) = 1 \geq \nu_{\chi_I}(a \wedge b) \wedge \nu_{\chi_{\sqrt{I}}}(b \wedge c) \wedge \nu_{\chi_{\sqrt{I}}}(c \wedge a).$$

Thus χ_I is an IF2API of L.

Conversely, suppose that χ_I is an IF2API of L.
Let $a \wedge b \wedge c \in I$. Then $\mu_{\chi_I}(a \wedge b \wedge c) = 1, \nu_{\chi_I}(a \wedge b \wedge c) = 0$.
Suppose that $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$. Since χ_I is an IF2API of L, we have

$$1 = \mu_{\chi_I}(a \wedge b \wedge c) \leq \mu_{\chi_I}(a \wedge b) \vee \mu_{\chi_{\sqrt{I}}}(b \wedge c) \vee \mu_{\chi_{\sqrt{I}}}(c \wedge a) \text{ and}$$
$$0 = \nu_{\chi_I}(a \wedge b \wedge c) \geq \nu_{\chi_I}(a \wedge b) \wedge \nu_{\chi_{\sqrt{I}}}(b \wedge c) \wedge \nu_{\chi_{\sqrt{I}}}(c \wedge a)$$

Since each of $\mu_{\chi_I}(a \wedge b), \mu_{\chi_{\sqrt{I}}}(b \wedge c), \mu_{\chi_{\sqrt{I}}}(c \wedge a)$ and $\nu_{\chi_I}(a \wedge b), \nu_{\chi_{\sqrt{I}}}(b \wedge c), \nu_{\chi_{\sqrt{I}}}(c \wedge a)$ belongs to $[0, 1]$, so atleast one of $\mu_{\chi_I}(a \wedge b), \mu_{\chi_{\sqrt{I}}}(b \wedge c), \mu_{\chi_{\sqrt{I}}}(c \wedge a)$ is 1 and atleast one of $\nu_{\chi_I}(a \wedge b), \nu_{\chi_{\sqrt{I}}}(b \wedge c), \nu_{\chi_{\sqrt{I}}}(c \wedge a)$ must be 0. This implies that either $a \wedge b \in I$ or $b \wedge c \in \sqrt{I}$ or $c \wedge a \in \sqrt{I}$. Thus I is a 2-API. □

Lemma 5.17. *Let Q is an intuitionistic fuzzy primary ideal of L, then Q is an IF2API of L*

Proof. Let Q be an IF primary ideal of L. Let $a, b, c \in L$. Then

$$\begin{aligned} \mu_Q(a \wedge b \wedge c) &= \mu_Q((a \wedge b) \wedge (b \wedge c)) \\ &\leq \mu_Q(a \wedge b) \vee \mu_{\sqrt{Q}}(b \wedge c) \\ &\leq \mu_Q(a \wedge b) \vee \mu_{\sqrt{Q}}(b \wedge c) \vee \mu_{\sqrt{Q}}(c \wedge a). \end{aligned}$$

Thus $\mu_Q(a \wedge b \wedge c) \leq \mu_Q(a \wedge b) \vee \mu_{\sqrt{Q}}(b \wedge c) \vee \mu_{\sqrt{Q}}(c \wedge a)$. Similarly, we can show that $\nu_Q(a \wedge b \wedge c) \geq \nu_Q(a \wedge b) \wedge \nu_{\sqrt{Q}}(b \wedge c) \wedge \nu_{\sqrt{Q}}(c \wedge a)$. Hence Q is an IF2API of L. □

The following example shows that an IF2API of L need not be an IF primary ideal of L.

Example 5.18. Consider the ideal $I = (0]$ of the lattice as shown in figure 5. We note that the ideal $(h] = \{x \in L : x \leq h\} = \{0, a, b, c, d, e, f, g, h\}$ and $(i] = \{0, b, c, d, g, i\}$ and the only prime ideal of L. Hence $\sqrt{I} = (h] \cap (i] = (g]$.
We note that I is a 2-absorbing primary ideal as for any $x, y, z \in L, x \wedge y \wedge z \in I$ implies that either $x \wedge y \in I$ or $y \wedge z \in \sqrt{I}$ or $z \wedge x \in \sqrt{I}$. Hence by Lemma (5.16), χ_I is an IF2API of L.
We note that $\mu_{\chi_I}(h \wedge i) = 1, \nu_{\chi_I}(h \wedge i) = 0$ but $\mu_{\chi_I}(h) = 0, \nu_{\chi_I}(h) = 1$ as well as $\mu_{\chi_{\sqrt{I}}}(i) = 0, \nu_{\chi_{\sqrt{I}}}(i) = 1$. Thus

$$\mu_{\chi_I}(h \wedge i) = 1 \nleq 0 = \mu_{\chi_I}(h) \vee \mu_{\chi_{\sqrt{I}}}(i) \text{ and } \nu_{\chi_I}(h \wedge i) = 0 \ngeq 0 = \nu_{\chi_I}(h) \wedge \nu_{\chi_{\sqrt{I}}}(i).$$

Hence χ_I is not an IF primary ideal of L.

Lemma 5.19. *If A is an IF2AI of L, then A is an IF2API of L.*

Proof. Let A be an IF2AI of L. Let $a, b, c \in L$, we have

$$\mu_A(a \wedge b \wedge c) \leq \mu_A(a \wedge b) \vee \mu_A(b \wedge c) \vee \mu_A(c \wedge a) \text{ and}$$
$$\nu_A(a \wedge b \wedge c) \geq \nu_A(a \wedge b) \wedge \nu_A(b \wedge c) \wedge \nu_A(c \wedge a).$$

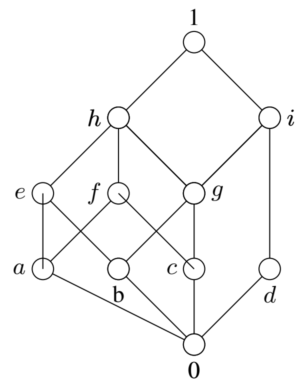

Figure 5.

Since $A \subseteq \sqrt{A}$, we get the result. □

The following example shows that an IF2API of L need not be an IF2AI.

Example 5.20. Consider the ideal $I = (0]$ of the lattice as shown in figure 6.

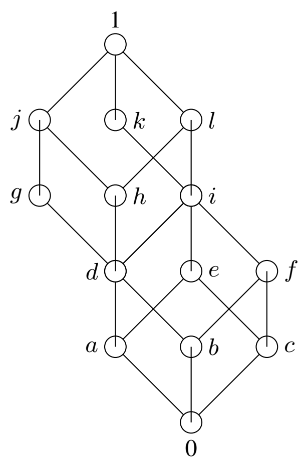

Figure 6.

Consider the ideal $I = (0]$. The only prime ideals of L are $(j]$, $(k]$, $[l]$.
We have $\sqrt{I} = (j] \cap (k] \cap [l] = (d]$. Also, $\sqrt{\chi_I} = \chi_{\sqrt{I}} = \chi_J$, where $J = (d]$.
We note that I is a 2-API of L. Hence by Lemma (5.16), χ_I is an IF2API of L.
We note that I is not a 2-AI of L, as $d \wedge e \wedge f = 0 \in I$, but $d \wedge e \notin I$, $e \wedge f \notin I$ and $d \wedge f \notin I$.
Thus we have

$$\mu_{\chi_I}(d \wedge e \wedge f) = 1 \nleq \mu_{\chi_I}(d \wedge e) \vee \mu_{\chi_I}(e \wedge f) \vee \mu_{\chi_I}(d \wedge f) \text{ and}$$
$$\nu_{\chi_I}(d \wedge e \wedge f) = 0 \ngeq \nu_{\chi_I}(d \wedge e) \wedge \nu_{\chi_I}(e \wedge f) \wedge \nu_{\chi_I}(d \wedge f).$$

Thus χ_I is not an IF2AI of L.

Lemma 5.21. *Let A be an IFI of L. If $\sqrt{A}$ is an IFPI, then A is an IF2API.*

Proof. Let A be an IFI of L. Suppose that $\sqrt{A}$ is an IFPI.
If A is not an IF2API, then there exist $a, b, c \in L$ such that

$$\mu_A(a \wedge b \wedge c) \nleq \mu_A(a \wedge b) \vee \mu_{\sqrt{A}}(b \wedge c) \vee \mu_{\sqrt{A}}(c \wedge a) \text{ and}$$
$$\nu_A(a \wedge b \wedge c) \ngeq \nu_A(a \wedge b) \wedge \nu_{\sqrt{A}}(b \wedge c) \wedge \nu_{\sqrt{A}}(c \wedge a).$$

This implies that

$$\mu_A(a\wedge b)\vee\mu_{\sqrt{A}}(b\wedge c)\vee\mu_{\sqrt{A}}(c\wedge a)<\mu_A(a\wedge b\wedge c) \text{ and}$$
$$\nu_A(a\wedge b)\wedge\nu_{\sqrt{A}}(b\wedge c)\wedge\nu_{\sqrt{A}}(c\wedge a)>\nu_A(a\wedge b\wedge c).$$

Since $\sqrt{A}$ is an IFPI, we have

$$\mu_{\sqrt{A}}(a\wedge b\wedge c)=\mu_{\sqrt{A}}(b\wedge c)\vee\mu_{\sqrt{A}}(a)=\mu_{\sqrt{A}}(a\wedge c)\vee\mu_{\sqrt{A}}(b)$$
$$\nu_{\sqrt{A}}(a\wedge b\wedge c)=\nu_{\sqrt{A}}(b\wedge c)\wedge\nu_{\sqrt{A}}(a)=\nu_{\sqrt{A}}(a\wedge c)\wedge\nu_{\sqrt{A}}(b)$$

Hence

$$\mu_{\sqrt{A}}(b\wedge c)\vee\mu_{\sqrt{A}}(a\wedge c)=\mu_{\sqrt{A}}(b\wedge c)\vee\mu_{\sqrt{A}}(a)\vee\mu_{\sqrt{A}}(c)=\mu_{\sqrt{A}}(a\wedge b\wedge c)\vee\mu_{\sqrt{A}}(c) \text{ and}$$
$$\nu_{\sqrt{A}}(b\wedge c)\wedge\nu_{\sqrt{A}}(a\wedge c)=\nu_{\sqrt{A}}(b\wedge c)\wedge\nu_{\sqrt{A}}(a)\wedge\nu_{\sqrt{A}}(c)=\nu_{\sqrt{A}}(a\wedge b\wedge c)\wedge\nu_{\sqrt{A}}(c).$$

Therefor, we get

$$\mu_A(a\wedge b)\vee\mu_{\sqrt{A}}(a\wedge b\wedge c)\vee\mu_{\sqrt{A}}(c)<\mu_A(a\wedge b\wedge c) \text{ and}$$
$$\nu_A(a\wedge b)\wedge\nu_{\sqrt{A}}(a\wedge b\wedge c)\wedge\nu_{\sqrt{A}}(c)>\nu_A(a\wedge b\wedge c).$$

This implies that $\mu_{\sqrt{A}}(a\wedge b\wedge c)<\mu_A(a\wedge b\wedge c)$ and $\nu_{\sqrt{A}}(a\wedge b\wedge c)>\nu_A(a\wedge b\wedge c)$. Which is not possible. Hence A os an IF2API. □

The following example shows that the converse of Lemma (5.21) does not hold.

Example 5.22. Consider the lattice as shown in figure 7. The only prime ideals of L containing

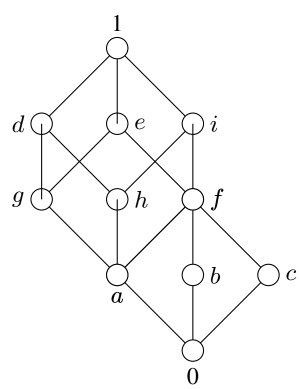

Figure 7.

the ideal $I=(c]$ are $(h]$ and $(i]$. Hence $\sqrt{I}=(h]\cap(i]=(f]$.
For any $x,y,z\in I, x\wedge y\wedge z\in I$ implies that either $x\wedge y\in I$ or $y\wedge z\in\sqrt{I}$ or $z\wedge x\in\sqrt{I}$. Hence I is 2-API and so by Lemma (5.16), χ_I is an IF2API. We note that $d\wedge e=a\in\sqrt{I}$ but $d\notin\sqrt{I}$ and $e\notin\sqrt{I}$. Thus $\sqrt{I}$ is not a prime ideal of L. Hence by Theorem (3.3). $\sqrt{\chi_I}=\chi_{\sqrt{I}}$ is not an IFPI of L.

We omit the easy proof of the following Lemma.

Lemma 5.23. *Let A be an IFI of L. Then $\sqrt{A}=\sqrt{\sqrt{A}}$.*

Theorem 5.24. *Let A be an IFI of L. Then $\sqrt{A}$ is an IFPI if and only if $\sqrt{A}$ is an IF primary ideal.*

Proof. It follows from Lemma (4.5), that if $\sqrt{A}$ is an IFPI, then $\sqrt{A}$ is an IF primary ideal. The converse follows from the definition of an IF primary ideal and by Lemma (5.23). □

The proof of the following Theorem follows from the definition of an IF2AI, an IF2API and Lemma (5.23).

Theorem 5.25. *Let A be an IFI of L. Then $\sqrt{A}$ is an IF2AI if and only if $\sqrt{A}$ is an IF2PI.*

Definition 5.26. A proper IFI Q of a lattice L is called a 2-absorbing primary intuitionistic fuzzy ideal (2-APIFI) of L, if for any $A,B,C\in IFI(L)$ such that

$A \cap B \cap C \subseteq Q$ implies that either $A \cap B \subseteq Q$ or $B \cap C \subseteq \sqrt{Q}$ or $C \cap A \subseteq \sqrt{Q}$.

Lemma 5.27. *Let I be a ideal of L. If χ_I is an 2-APIFI of L, then I is a 2-AI of L.*

Proof. Suppose that χ_I is a 2-APIFI of L. Let $a \wedge b \wedge c \in I$ for some $a, b, c \in L$.
Suppose that $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$. Then clearly, $a \notin I$ and $b, c \notin \sqrt{I}$.
Define IFIs A, B, C of L as

$$A(x) = \begin{cases} (1,0), & \text{if } x \in (a] \\ (0.1), & \text{otherwise} \end{cases} ; \quad B(x) = \begin{cases} (1,0), & \text{if } x \in (b] \\ (0,1), & \text{otherwise} \end{cases} ; \quad C(x) = \begin{cases} (1,0), & \text{if } x \in (c] \\ (0,1), & \text{otherwise} . \end{cases}$$

We note that

$$(A \cap B \cap C)(x) = \begin{cases} (1,0), & \text{if } x \in (a \wedge b \wedge c] \\ (0.1), & \text{otherwise} . \end{cases}$$

Thus $A \cap B \cap C \subseteq \chi_I$ but $A \cap B \nsubseteq \chi_I$, $B \cap C \nsubseteq \chi_{\sqrt{I}}$ and $C \cap A \nsubseteq \chi_{\sqrt{I}}$. This contradicts the assumption that χ_I is a 2-APIFI of L. □

Remark 5.28. However, we are unable to prove or disprove that if I is a 2-AI of L, then χ_I is a 2-APIFI of L.

Lemma 5.29. *If Q is a primary IFI of L, then Q is a 2-APIFI of L.*

Proof. Let Q be a primary IFI of L. Let for any $A, B, C \in IFI(L)$ such that $A \cap B \cap C \subseteq Q$. Then we have either

(i) $A \cap B \subseteq Q$ or $C \subseteq \sqrt{Q}$; or

(ii) $A \subseteq Q$ or $B \cap C \subseteq \sqrt{Q}$; or

(iii) $A \subseteq \sqrt{Q}$ or $B \cap C \subseteq Q$; or

(iv) $B \subseteq Q$ or $A \cap C \subseteq \sqrt{Q}$.

These possibilities imply that either (i) $A \cap B \subseteq Q$ or (ii) $B \cap C \subseteq \sqrt{Q}$ or (iii) $C \cap A \subseteq \sqrt{Q}$.
Hence Q is 2-APIFI of L. □

Lemma 5.30. *Let Q is a 2-AIFI of L, then Q is a 2-APIFI of L.*

Proof. Let Q is a 2-AIFI of L. Let $A, B, C \in IFI(L)$ such that $A \cap B \cap C \subseteq Q$. Then we have either $A \cap B \subseteq Q$ or $B \cap C \subseteq Q$ or $C \cap A \subseteq Q$. Since $Q \subseteq \sqrt{Q}$, we get the required result. □

Definition 5.31. Let Q be an IFI of L. If P is the only PIFI containing Q, then we say that Q is P-primary IFI of L.

Theorem 5.32. *Let Q_1, Q_2 be IFIs and P_1, P_2 be PIFIs of L. Suppose that Q_1 is a P_1- primary IFI and Q_2 is a P_2- primary IFI. Then $Q_1 \cap Q_2$ is a 2-APIFI of L.*

Proof. Since, Q_i is a P_i-primary IFI, for $i = 1, 2$. We get $\sqrt{Q_i} = P_i$.
Let $Q = Q_1 \cap Q_2$. Then $\sqrt{Q} = P_1 \cap P_2$. Now suppose that $A \cap B \cap C \subseteq Q$ for some $A, B, C \in IFI(L)$. Assume that $A \cap B \nsubseteq \sqrt{Q}$ and $B \cap C \nsubseteq \sqrt{Q}$. Then $A, B, C \nsubseteq \sqrt{Q} = P_1 \cap P_2$.
By proposition (5.14), $\sqrt{Q} = P_1 \cap P_2$ is a 2-AIFI of L. Since $A \cap B \nsubseteq \sqrt{Q}$ and $B \cap C \nsubseteq \sqrt{Q}$, we have $A \cap C \subseteq \sqrt{Q}$.

We show that $A \cap C \subseteq Q$.

Since $A \cap C \subseteq \sqrt{Q} \subseteq P_1$, we assume that $A \subseteq P_1$. As $A \nsubseteq \sqrt{Q}$ and $A \cap C \subseteq \sqrt{Q} \subseteq P_2$, we conclude that $A \nsubseteq P_2$ and $C \subseteq P_2$.
Since $C \subseteq P_2$ and $C \nsubseteq \sqrt{Q}$ we have $C \nsubseteq P_1$.
If $A \subseteq Q_1$ and $C \subseteq Q_2$, then $A \cap C \subseteq Q$ and we are done.
We may assume that $A \nsubseteq Q_1$. Since $C \subseteq P_2$ and $B \cap C \subseteq \sqrt{Q}$ which is a contradiction.
Thus, $A \subseteq Q_1$.
Since Q_2 is a P_2-primary IFI, and $C \nsubseteq Q_2$, we get $A \cap B \subseteq P_2$.
Since $A \subseteq P_1$ and $A \cap B \subseteq P_2$, we have $A \cap B \subseteq \sqrt{Q}$ which is a contradiction.
Thus, $C \subseteq Q_2$.
Hence $A \cap C \subseteq Q$. Therefore, Q is a 2-APIFI of L. □

Theorem 5.33. *Suppose that Q is a non-constant IFI of L such that $\sqrt{Q}$ is a PIFI. Then Q is a 2-APIFI of L.*

Proof. Suppose that for some $A, B, C \in IFI(L)$, $A \cap B \cap C \subseteq Q$ and $A \cap B \nsubseteq Q$.

(i) : We note that $A \cap B \cap C \subseteq Q \subseteq \sqrt{Q}$. Hence, if $A \cap B \nsubseteq Q$, then as $\sqrt{Q}$ is PIFI, we get $C \subseteq \sqrt{Q}$ and so $B \cap C \subseteq \sqrt{Q}$.

(ii) : If $A \cap B \subseteq \sqrt{Q}$, then as $\sqrt{Q}$ is PIFI, either $A \subseteq \sqrt{Q}$ or $B \subseteq \sqrt{Q}$.
Hence either $A \cap C \subseteq \sqrt{Q}$ or $C \cap B \subseteq \sqrt{Q}$. Thus, Q IS A 2-APIFI of L. □

Now we give a characterization for $\sqrt{Q}$ to be a PIFI.

Theorem 5.34. *Let Q be a non-constant IFI of a lattice L. Then $\sqrt{Q}$ is a PIFI of L if and only if $\sqrt{Q}$ is a primary IFI of L.*

Proof. Let $\sqrt{Q}$ be a PIFI of L. Let $A, B, C \in IFI(L)$ be such that $A \cap B \subseteq \sqrt{Q}$. As $\sqrt{Q}$ is a PIFI of L, either $A \subseteq \sqrt{Q}$ or $B \subseteq \sqrt{Q} = \sqrt{\sqrt{Q}}$. We conclude that $\sqrt{Q}$ is a primary IFI of L.

Conversely, suppose that $\sqrt{Q}$ is a primary IFI of L. Let $A, B, C \in IFI(L)$ be such that $A \cap B \subseteq \sqrt{Q}$. As $\sqrt{Q}$ is primary IFI of L, either $A \subseteq \sqrt{Q}$ or $B \subseteq \sqrt{\sqrt{Q}} = \sqrt{Q}$. Hence $\sqrt{Q}$ is a prime IFI of L. □

Now we prove the following characterization.

Theorem 5.35. *Let Q be a non-constant IFI of a lattice L. Then $\sqrt{Q}$ is a 2-AIFI of L if and only if $\sqrt{Q}$ is a 2-APIFI of L.*

Proof. Let $\sqrt{Q}$ be a 2-AIFI of L. Let $A, B, C \in IFI(L)$ be such that $A \cap B \subseteq \sqrt{Q}$.
As $\sqrt{Q}$ is a 2-AIFI of L, either $A \cap B \subseteq \sqrt{Q}$ or $B \cap C \subseteq \sqrt{Q}$ or $C \cap A \subseteq \sqrt{Q}$.
Using $\sqrt{Q} = \sqrt{\sqrt{Q}}$, we conclude that $\sqrt{Q}$ is a 2-APIFI of L.

Conversely, suppose that $\sqrt{Q}$ is a 2-APIFI of L. Let $A, B, C \in IFI(L)$ be such that $A \cap B \cap C \subseteq \sqrt{Q}$. As $\sqrt{Q}$ is 2-APIFI of L, either $A \cap B \subseteq \sqrt{Q}$ or $B \cap C \subseteq \sqrt{\sqrt{Q}} = \sqrt{Q}$ or $C \cap A \subseteq \sqrt{\sqrt{Q}} = \sqrt{Q}$. Hence $\sqrt{Q}$ is a 2-AIFI of L. □

Theorem 5.36. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let A_1, A_2 be an IFI of L_1 and L_2 respectively. Suppose that $\mu_{A_1}(0_1) = \mu_{A_2}(0_2) = 1$, $\nu_{A_1}(0_1) = \nu_{A_2}(0_2) = 0$, where $0_1, 0_2$ is the least element of L_1, L_2 respectively. If $A = A_1 \times A_2$ is an IF2AI of L, then A_1 is an IF2AI of L_1 and A_2 is an IF2AI of L_2.*

Proof. Let $a, b, c \in L$. Since A is an IF2AI of L, we have

$$\mu_A(a \wedge b \wedge c, 0_2) \leq \mu_A(a \wedge b, 0_2) \vee \mu_A(b \wedge c, 0_2) \vee \mu_A(c \wedge a, 0_2) \text{ and}$$
$$\nu_A(a \wedge b \wedge c, 0_2) \geq \nu_A(a \wedge b, 0_2) \wedge \nu_A(b \wedge c, 0_2) \wedge \nu_A(c \wedge a, 0_2)$$

By using the definition for $A_1 \times A_2$, we can write

$$\mu_{A_1}(a \wedge b \wedge c) \wedge \mu_{A_2}(0_2) \leq [\mu_{A_1}(a \wedge b) \wedge \mu_{A_2}(0_2)] \vee [\mu_{A_1}(b \wedge c) \wedge \mu_{A_2}(0_2)] \vee [\mu_{A_1}(c \wedge a) \wedge \mu_{A_2}(0_2)]$$
$$\nu_{A_1}(a \wedge b \wedge c) \vee \nu_{A_2}(0_2) \geq [\nu_{A_1}(a \wedge b) \vee \nu_{A_2}(0_2)] \wedge [\nu_{A_1}(b \wedge c) \vee \nu_{A_2}(0_2)] \wedge [\nu_{A_1}(c \wedge a) \vee \nu_{A_2}(0_2)]$$

By using $\mu_{A_2}(0_2) = 1, \nu_{A_2}(0_2) = 0$, we get

$$\mu_{A_1}(a \wedge b \wedge c) \leq \mu_{A_1}(a \wedge b) \vee \mu_{A_1}(b \wedge c) \vee \mu_{A_1}(c \wedge a)$$
$$\nu_{A_1}(a \wedge b \wedge c) \geq \nu_{A_1}(a \wedge b) \wedge \nu_{A_1}(b \wedge c) \wedge \nu_{A_1}(c \wedge a).$$

Thus A_1 is an IF2AI of L_1. In a same way we can show that A_2 is an IF2AI of L_2. □

By using the similar steps, we can prove the following theorem.

Theorem 5.37. *Let $L = L_1 \times L_2 \times ...... \times L_k$ be a direct product of lattices $L_1, L_2, ..., L_k$. Let $A_i (1 \leq i \leq k)$ be an IFIs of L_i respectively. Suppose that for each $i = 1, 2, .., k$, $\mu_{A_i}(0_2) = 1, \nu_{A_i}(0_2) = 0$, where 0_i is the least element of L_i. If $A = A_1 \times A_2 \times ...... \times A_k$ is an IF2AI of L, then each A_i, is an IF2AI of L_i.*

The following example shows that the converse of the Theorem 5.36 need not hold.

Example 5.38. Consider the lattices L_1, L_2 and $L = L_1 \times L_2$ as in Example 3.6. Define IFSs $A_1 \in IFS(L_1)$ and $A_2 \in IFS(L_2)$ as follows:

$$A_1(x) = \begin{cases} (1,0), & \text{if } x = 0 \\ (0.16, 0.7), & \text{if } x = a \\ (0.25, 0.5), & \text{if } x = b, 1. \end{cases} \quad ; \quad A_2(x) = \begin{cases} (1,0), & \text{if } x = 0 \\ (0,1), & \text{if } x = 1. \end{cases}$$

We note that A_1 is an IF2AI of L_1 and A_2 is an IF2AI of L_2. We consider $A \in IFS(L_1 \times L_2)$ defined by

$$\mu_A(x,y) = \mu_{A_1}(x) \wedge \mu_{A_2}(y) \text{ and } \nu_A(x,y) = \mu_{A_1}(x) \vee \nu_{A_2}(y).$$

i.e., $A = A_1 \times A_2$. It is easy to check that

$$A(x,y) = \begin{cases} (1,0), & \text{if } (x,y) = (0,0) \\ (0.25, 0.5), & \text{if } (x,y) = (b,0), (1,0) \\ (0.16, 0.7), & \text{if } (x,y) = (a,0) \\ (0,1), & \text{otherwise }. \end{cases}$$

We have

$$\mu_A[(a,1) \wedge (1,0) \wedge (b,1)] = \mu_A(0,0) = 1; \nu_A[(a,1) \wedge (1,0) \wedge (b,1)] = \nu_A(0,0) = 0$$
$$\mu_A[(a,1) \wedge (1,0)] = \mu_A(a,0) = 0.16; \nu_A[(a,1) \wedge (1,0)] = \nu_A(a,0) = 0.70.$$
$$\mu_A[(1,0) \wedge (b,1)] = \mu_A(b,0) = 0.25; \nu_A[(1,0) \wedge (b,1)] = \nu_A(b,0) = 0.50.$$
$$\mu_A[(a,1) \wedge (b,1)] = \mu_A(a \wedge b, 1) = \mu_A(0,1) = 0; \nu_A[(a,1) \wedge (b,1)] = \nu_A(a \wedge b, 1) = \nu_A(0,1) = 1.$$

Thus

$$\mu_A[(a,1) \wedge (1,0) \wedge (b,1)] = 1 \nleq 0.25 = \mu_A[(a,1) \wedge (1,0)] \vee \mu_A[(1,0) \wedge (b,1)] \vee \mu_A[(a,1) \wedge (b,1)];$$
$$\nu_A[(a,1) \wedge (1,0) \wedge (b,1)] = 0 \ngeq 0.5 = \nu_A[(a,1) \vee (1,0)] \wedge \nu_A[(1,0) \vee (b,1)] \wedge \nu_A[(a,1) \vee (b,1)].$$

Hence A is not an IF2AI of L.

Theorem 5.39. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1 and L_2. Let P_1, P_2 be IFI of L_1 and L_2 respectively. Suppose that*
(i) $\mu_{P_1}(0_1) = \mu_{P_2}(0_2) = 1, \nu_{P_1}(0_1) = \nu_{P_2}(0_2) = 0$, where $0_1, 0_2$ is the least element of L_1, L_2 respectively.
(ii) $\mu_{P_1}(1_1) = \mu_{P_2}(1_2) = 0, \nu_{P_1}(0_1) = \nu_{P_2}(0_2) = 1$, where $1_1, 1_2$ is the greatest element of L_1, L_2 respectively.

If $P = P_1 \times P_2$ is an IF2AI of L, then P_1 and P_2 are IFPI of L_1 and L_2 respectively.

Proof. Suppose that P_1 is not an IFPI of L_1, then there exists $a, b, c \in L_1$ such that

$$\mu_{P_1}(a \wedge b) \nleq \mu_{P_1}(a) \vee \mu_{P_1}(b) \text{ and } \nu_{P_1}(a \wedge b) \ngeq \nu_{P_1}(a) \wedge \nu_{P_1}(b)$$

Consider the element $x = (a, 1_2), y = (1_1, 0_2)$ and $z = (b, 1_2)$ from L. We note the following

$$\mu_P(x \wedge y \wedge z) = \mu_P(a \wedge b, 0_2) = \mu_{P_1}(a \wedge b) \vee \mu_{P_1}(0_2) = \mu_{P_1}(a \wedge b) \text{ and}$$
$$\nu_P(x \wedge y \wedge z) = \nu_P(a \wedge b, 0_2) = \nu_{P_1}(a \wedge b) \wedge \nu_{P_1}(0_2) = \nu_{P_1}(a \wedge b).$$

$$\text{Now } \mu_P(x \wedge y) = \mu_P(a, 0_2) = \mu_{P_1}(a) \wedge \mu_{P_2}(0_2) = \mu_{P_1}(a);$$
$$\nu_P(x \wedge y) = \nu_P(a, 0_2) = \nu_{P_1}(a) \vee \nu_{P_2}(0_2) = \nu_{P_1}(a) \text{ and}$$
$$\mu_P(y \wedge z) = \mu_P(b, 0_2) = \mu_{P_1}(b) \wedge \mu_{P_2}(0_2) = \mu_{P_1}(b);$$
$$\nu_P(y \wedge z) = \nu_P(b, 0_2) = \nu_{P_1}(b) \vee \nu_{P_2}(0_2) = \nu_{P_1}(b) \text{ and}$$
$$\mu_P(z \wedge x) = \mu_P(a \wedge b, 1_2) = \mu_{P_1}(a \wedge b) \wedge \mu_{P_2}(1_2) = 0;$$
$$\nu_P(z \wedge x) = \nu_P(a \wedge b, 1_2) = \nu_{P_1}(a \wedge b) \vee \nu_{P_2}(1_2) = 1.$$

Since P is an IF2AI, we have

$$\mu_P(x \wedge y \wedge z) \leq \mu_P(x \wedge y) \vee \mu_P(y \wedge z) \vee \mu_P(z \wedge x) \text{ and}$$
$$\nu_P(x \wedge y \wedge z) \geq \nu_P(x \wedge y) \wedge \nu_P(y \wedge z) \wedge \nu_P(z \wedge x).$$
$$\text{i.e., } \mu_{P_1}(a \wedge b) \leq \mu_{P_1}(a) \vee \mu_{P_1}(b) \vee 0 = \mu_{P_1}(a) \vee \mu_{P_1}(b) \text{ and}$$
$$\nu_{P_1}(a \wedge b) \geq \nu_{P_1}(a) \wedge \nu_{P_1}(b) \wedge 1 = \nu_{P_1}(a) \wedge \nu_{P_1}(b),$$

a contradiction. Hence P_1 is an IFPI of L_1. Similarly, we can show that P_1 is an IFPI of L_2. □

Theorem 5.40. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let P_1, P_2 be an IFPI of L_1 and L_2 respectively. If $P = P_1 \times P_2$, then P is an IF2AI of L.*

Proof. Let $(a, x), (b, y), (c, z) \in L$. To show that P is an IF2AI, we need to show that

$$\mu_P[(a, x) \wedge (b, y) \wedge (c, z)] \leq \mu_P[(a, x) \wedge (b, y)] \vee \mu_P[(b, y) \wedge (c, z)] \vee \mu_P[(c, z) \wedge (a, x)];$$
$$\nu_P[(a, x) \wedge (b, y) \wedge (c, z)] \geq \nu_P[(a, x) \wedge (b, y)] \wedge \nu_P[(b, y) \wedge (c, z)] \wedge \nu_P[(c, z) \wedge (a, x)].$$

i.e., to show that

$$\mu_P(a \wedge b \wedge c, x \wedge y \wedge z) \leq \mu_P(a \wedge b, x \wedge y) \vee \mu_P(b \wedge c, y \wedge z) \vee \mu_P(c \wedge a, z \wedge x);$$
$$\nu_P(a \wedge b \wedge c, x \wedge y \wedge z) \geq \nu_P(a \wedge b, x \wedge y) \wedge \nu_P(b \wedge c, y \wedge z) \wedge \nu_P(c \wedge a, z \wedge x).$$

Also, by using definition of $P_1 \times P_2$, we have

$$\mu_P(a \wedge b \wedge c, x \wedge y \wedge z) = \mu_{P_1}(a \wedge b \wedge c) \wedge \mu_{P_2}(x \wedge y \wedge z);$$
$$\nu_P(a \wedge b \wedge c, x \wedge y \wedge z) = \nu_{P_1}(a \wedge b \wedge c) \vee \nu_{P_2}(x \wedge y \wedge z).$$

As P_1 and P_2 are IFPIs of L_1 and L_2 respectively, we have

$$\mu_{P_1}(a \wedge b \wedge c) = \mu_{P_1}(a) \vee \mu_{P_1}(b) \vee \mu_{P_1}(c); \nu_{P_1}(a \wedge b \wedge c) = \nu_{P_1}(a) \wedge \nu_{P_1}(b) \wedge \nu_{P_1}(c).$$

and

$$\mu_{P_2}(x \wedge y \wedge z) = \mu_{P_2}(x) \vee \mu_{P_2}(y) \vee \mu_{P_2}(z); \nu_{P_2}(x \wedge y \wedge z) = \nu_{P_2}(x) \wedge \nu_{P_2}(y) \wedge \nu_{P_2}(z).$$

Thus, we have
$[\mu_P(a \wedge b, x \wedge y)] \vee [\mu_P(b \wedge c, y \wedge z)] \vee [\mu_P(c \wedge a, z \wedge x)]$
$= [\mu_{P_1}(a \wedge b) \wedge \mu_{P_2}(x \wedge y)] \vee [\mu_{P_1}(b \wedge c) \wedge \mu_{P_2}(y \wedge z)] \vee [\mu_{P_1}(c \wedge a) \wedge \mu_{P_2}(z \wedge x)]$.
Similarly, we have
$[\nu_P(a \wedge b, x \wedge y)] \wedge [\nu_P(b \wedge c, y \wedge z)] \wedge [\nu_P(c \wedge a, z \wedge x)]$
$= [\nu_{P_1}(a \wedge b) \vee \nu_{P_2}(x \wedge y)] \wedge [\nu_{P_1}(b \wedge c) \vee \nu_{P_2}(y \wedge z)] \wedge [\nu_{P_1}(c \wedge a) \vee \nu_{P_2}(z \wedge x)]$.
Since P_1 and P_2 are IFPIs of L_1 and L_2 respectively, we can write
$\mu_P(a \wedge b, x \wedge y) \vee \mu_P(b \wedge c, y \wedge z) \vee \mu_P(c \wedge a, z \wedge x)$
$= \{[\mu_{P_1}(a) \vee \mu_{P_2}(b)] \wedge [\mu_{P_1}(x) \vee \mu_{P_2}(y)]\} \vee \{[\mu_{P_1}(b) \vee \mu_{P_2}(c)] \wedge [\mu_{P_1}(y) \vee \mu_{P_2}(z)]\} \vee \{[\mu_{P_1}(c) \vee \mu_{P_2}(a)] \wedge [\mu_{P_1}(z) \vee \mu_{P_2}(x)]\}$
By using distributivity law, R.H.S of it can be written as
$[\mu_{P_1}(a) \vee \mu_{P_1}(b) \vee \mu_{P_1}(c)] \wedge [\mu_{P_2}(x) \vee \mu_{P_2}(y) \vee \mu_{P_2}(z)]$.
Thus, $[\mu_{P_1}(a) \vee \mu_{P_1}(b) \vee \mu_{P_1}(c)] \wedge [\mu_{P_2}(x) \vee \mu_{P_2}(y) \vee \mu_{P_2}(z)] \geq \mu_P(a \wedge b \wedge c, x \wedge y \wedge z) = \mu_{P_1}(a \wedge b \wedge c) \wedge \mu_{P_2}(x \wedge y \wedge z) = [\mu_{P_1}(a) \vee \mu_{P_1}(b) \vee \mu_{P_1}(c)] \wedge [\mu_{P_2}(x) \vee \mu_{P_2}(y) \vee \mu_{P_2}(z)]$. Which is true. Similarly, we can show that
$[\nu_{P_1}(a) \wedge \nu_{P_1}(b) \wedge \nu_{P_1}(c)] \vee [\nu_{P_2}(x) \wedge \nu_{P_2}(y) \wedge \nu_{P_2}(z)] \leq \nu_P(a \wedge b \wedge c, x \wedge y \wedge z) = \nu_{P_1}(a \wedge b \wedge c) \vee \mu_{P_2}(x \wedge y \wedge z) = [\nu_{P_1}(a) \wedge \nu_{P_1}(b) \wedge \nu_{P_1}(c)] \vee [\nu_{P_2}(x) \wedge \nu_{P_2}(y) \wedge \nu_{P_2}(z)]$. Which is also true.
Hence P is an IF2AI of L. □

Theorem 5.41. *Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let Q be an IFI of L_1. Then $Q \times \chi_{L_2}$ is a 2-AIFPI of L, if and only if Q is a 2-AIFPI of L_1.*

Proof. Suppose that $Q \times \chi_{L_2}$ is a 2-AIFPI of L.
Let $A_1, A_2, A_3 \in IFI(L_1)$ be such that $A_1 \cap A_2 \cap A_3 \subseteq Q$.
Consider $(A_1 \cap A_2 \cap A_3) \times \chi_{L_2} \subseteq Q \times \chi_{L_2}$. This implies that
$(A_1 \times \chi_{L_2}) \cap (A_2 \times \chi_{L_2}) \cap (A_3 \times \chi_{L_2}) \subseteq Q \times \chi_{L_2}$. Since $Q \times \chi_{L_2}$ is a 2-AIFPI of L, we get either $(A_1 \times \chi_{L_2}) \cap (A_2 \times \chi_{L_2}) \subseteq Q \times \chi_{L_2}$ or $(A_2 \times \chi_{L_2}) \cap (A_3 \times \chi_{L_2}) \subseteq \sqrt{Q \times \chi_{L_2}} = \sqrt{Q} \times \chi_{L_2}$ or $(A_3 \times \chi_{L_2}) \cap (A_1 \times \chi_{L_2}) \subseteq \sqrt{Q \times \chi_{L_2}} = \sqrt{Q} \times \chi_{L_2}$.
Thus $(A_1 \cap A_2) \subseteq Q$ or $(A_2 \cap A_3) \subseteq \sqrt{Q}$ or $(A_3 \cap A_1) \subseteq \sqrt{Q}$.
Hence Q is a 2-AIFPI of L_1.

The converse follows by retracing similar steps. □

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