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The expansion of functions by Toeplitz matrices

R. Bouifden and A. Haddi

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Corresponding Author: R. Bouifden

Abstract In this paper, we present a novel matrix method for the expansion of real functions, which essentially uses Toeplitz matrices. This approach offers several advantages over existing methods.

1 Introduction

The expansion of a real function is a technique for representing a function as an infinite sum of simple functions. It is the main tool for local approximation of a function that cannot be expressed by just elementary operators (addition, subtraction, multiplication, and division) using a polynomial. It is also a potent tool in several fields of application such as limit calculation, mathematical modeling, and the study of physical phenomena (see for example [1]).

To fully understand the meaning of a mathematical concept, historical-epistemological research is important, as Sierpinska states [2]. Before reaching the form described in the current definitions, the notion of expansion had to go through several stages taking into account the evolution of a vast conceptual field [3]. These steps are categorized by R.Kouki [4] into five non-linear steps and involve the use of various geometric, analytical, and algebraic techniques. This work was begun in the early seventeenth century by Torricelli, Roberval, Fermat, Descartes, and Isaac Barrow to solve the problems of tangents, explicitly expressed by Taylor (1715), Newton, and Leibniz using polynomials, developed by Cauchy (1823) and Abel (1826), and completed by Poincaré (1886).

At present, in order to calculate the expansion of a function, it sometimes happens that we need divisions according to the increasing powers, and approximate reasoning consists in neglecting certain powers of this limiting development. In this regard, we propose a new simple algorithmic matrix technique that exploits the fundamental properties of matrix computation.

More precisely, we associate the expansion of a function f to the order n in the neighborhood of 0 a triangular matrix of Toeplitz, whose first column is the polynomial representing the expansion of f. Then we demonstrate all the calculation rules of the limited developments.

Finally, our work gives a quick algorithm to find, according to Douady (1986) [5] the object “the expansion” of non-usual functions, whose Toeplitz matrices we have chosen as a “tool” and vice versa in some cases.

2 Triangular Toeplitz matrices

2.1 Definition of a Toeplitz matrix

A Toeplitz matrix (based on Otto Toeplitz) or diagonal-constant matrix is a matrix in which each descending diagonal from left to right is constant. As a fundamental example of the Toeplitz matrix, the $n \times n$ matrix called the "shift matrix" is defined by

$$N = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 1 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 1 & 0 \end{bmatrix} \in M_n(\mathbb{R})$$

Also, if we note the identity matrix by I_n, it's a Toeplitz matrix. Explicitly, we have:

Definition 2.1. A Toeplitz lower triangular matrix of size n is a matrix of form

$$T_n = \begin{bmatrix} t_0 & 0 & \dots & 0 \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ t_{n-1} & \dots & t_1 & t_0 \end{bmatrix}$$

In this work, it should be borne in mind that a triangular Toeplitz matrix is completely determined by its first column.

2.2 A basis for $\mathbb{R}$-algebra of Toeplitz triangular matrices

In this part we show that the family $(I_n, N, ..., N^{n-1})$ is a basis of the $\mathbb{R}$-algebra of the triangular matrices of Toeplitz.

Lemma 2.2. *Let N be the $n \times n$ "shift matrix" . We have: $\forall k \in [[1, n-1]]$ $N^k = \left(\begin{array}{c|c} 0 & 0 \\ \hline I_{n-k} & 0 \end{array}\right)$, therfore N is nilpotent of index n.*

Note $\mathbb{R}[N] = \{P\,(N)\,/P$ a polynomial $\}$ the sub commutative algebra of $(M_n(\mathbb{R}), +, ., \times)$. According to this lemma (2.2), the matrix N is nilpotent of index n.
Therefore the application $\begin{array}{c} e_N : (\mathbb{R}[X], +, ., \times) \to (M_n(\mathbb{R}), +, ., \times) \\ P \to P(N) \end{array}$ induces a morphism of $\mathbb{R}$-algebra $\begin{array}{c} \bar{e_N} : \mathbb{R}[X]/\mathbb{R}[X].X^n \to R[N] \\ \bar{P} \to P(N) \end{array}$. More precisely, we have the theorem:

Theorem 2.3. *Any lower Toeplitz triangular matrix is a polynomial in N the "shift matrix".*

Note $\mathbb{R}[N] = \mathcal{L}$. We can specify the theorem (2.3) as $\mathbb{R}[N] = \mathcal{L} = \{P\,(N) : P$ polynome $\}$. In particular $(\mathcal{L}, +, ., \times)$ is a commutative matrix algebra.
Proof. Let T be a $n \times n$ lower triangular Toeplitz matrix, where $(a_0, \dots, a_{n-1})^t$ its first column. Let $P(X) = a_0 + a_1 X + \dots + a_{n-1}X^{n-1}$, then by a simple calculation we find:

$$\begin{aligned} P(N) &= a_0 I_n + a_1 N + \dots + a_{n-1}N^{n-1} \\ &= a_0 I_n + a_1 N + a_2 \left(\begin{array}{c|c} 0 & 0 \\ \hline I_{n-2} & 0 \end{array}\right) + \dots + a_{n-2}\left(\begin{array}{c|c} 0 & 0 \\ \hline I_2 & 0 \end{array}\right) + a_{n-1}\left(\begin{array}{c|c} 0 & 0 \\ \hline I_1 & 0 \end{array}\right) \\ &= T \end{aligned}$$

Corollary 2.4. *Every triangular matrix of Toeplitz T is written in a unique way $T = a_0 I_n + a_1 N + \dots + a_{n-1}N^{n-1}$ where $(a_0, \dots, a_{n-1})^t$ its first column.*

3 Toeplitz matrix associated with expansion of a function

Let f be a function that admits in the neighborhood of 0 an expansion of order n, then there is a single polynomial P such as: $f(x) = P(x) + o(x^n)$. Let $P(x) = a_0 + a_1 x + \dots + a_n x^n$ with $a_0, a_1, \dots, a_n \in \mathbb{R}$

Definition 3.1. We call Toeplitz matrix associated with the expansion of a function f, noted by $T_{(f,n)}$, the lower triangular Toeplitz matrix of order $n+1$ defined by:

$$T_{(f,n)} = \begin{bmatrix} a_0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a_1 & a_0 & 0 & \cdot & \cdot & \cdot & \cdot \\ a_2 & a_1 & a_0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ a_{n-1} & \cdot & \cdot & \cdot & a_1 & a_0 & 0 \\ a_n & a_{n-1} & \cdot & \cdot & a_2 & a_1 & a_0 \end{bmatrix}$$

Remark 3.2. The expansion of f is represented in the first column of $T_{(f,n)}$.

Here is the matrix formulation of definition (3.1)

Proposition 3.3. *Let f be a real function such as $f(x) = P(x)+o(x^n)$, and let N the "shift matrix". So : $T_{(f,n)} = P(N)$*

For example, if f is a constant function that is 1, in the neighbourhood of 0, then: $T_{(1,n)} = I_{n+1}$.

main theorem

Theorem 3.4. *Let f and g be two real functions admitting in the neighborhood of* 0 *an expansion of order n. We have:*

(i) $T_{(f+g,n)} = T_{(f,n)} + T_{(g,n)}$

(ii) $T_{(f\times g,n)} = T_{(f,n)}T_{(g,n)}$

(iii) If $f(0) \neq 0$ *, then* $T_{(f,n)}$ *is inversible and we have* $T_{(\frac{1}{f},n)} = T^{-1}_{(f,n)}$

Proof.

(i) Evident.

(ii) Put: $f(x) = P(x)+o(x^n)$ and $g(x) = Q(x)+o(x^n)$, with P and Q two polynomials of degree $\leq n$, then Euclidean division of the polynome product PQ by X^{n+1} gives

$$PQ(X) = R(X) + X^{n+1}S(X) \text{ with } deg(R) \leq n$$

So

$$R(X) = PQ(X) - X^{n+1}S(X)$$

Since

$$(f \times g)(x) = R(x) + o(x^n)$$

Then

$$\begin{aligned} T_{(f\times g,n)} &= R(N) \\ &= PQ(N) - (X^{n+1}S)(N) \\ &= PQ(N) \quad \text{(Because N is nilpotent of index n+1)} \\ &= P(N) \times Q(N) \\ &= T_f \times T_g \end{aligned}$$

(iii) Note first that $T_{(f;n)}$ is invisible because $detT_{(f;n)} \neq 0$ is equivalent to $a_0 = f(0) \neq 0$

Now:

$$\begin{aligned} I_{n+1} &= T_{(1,n)} \\ &= T_{(f\times\frac{1}{f},n)} \\ &= T_{(f,n)}T_{(\frac{1}{f},n)} \end{aligned}$$

Hence :

$$T_{(\frac{1}{f},n)} = T^{-1}_{(f,n)}$$

4 Expansion of the inverse of a function by Toeplitz matrices

Recall that the expansion of the inverse of a function $\frac{1}{f}$ can be tedious, here is the originality of our work.

Theorem 4.1. *Let f be a function that admits an expansion in 0, such as $f(0) \neq 0$, then $\frac{1}{f(x)} = a_0 + a_1 x + \ldots + a_n x^n$ such as the n-uplet $(a_1, \ldots, a_n)$ is the only solution of the Toeplitz trivial system* $T_{(f,n)} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

5 Matrix algorithm to calculate expansion

The previous theorems give us a simple matrix algorithm to determine the expansion of a function.

Example 5.1. Let's determine the expansion of the function defined by $f(x) = \frac{1}{(1-x)cos(x)}$ in the neighborhood of 0 to the order of 4.

At the begening, we will determine the expansion of $x \to \frac{1}{cos(x)}$ by applying the rule of inverse to $x \to cos(x)$, then we deduce the expansion of f from the product rule.
We know that:

$$cos(x) = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{24}x^4 + o(x^4)$$

Then

$$T_{(cos,4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{24} & 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

Since $cos(0) = 1 \neq 0$ then $T_{(cos,4)}$ is invertible. And we have : $T^{-1}_{(cos(x),4)} = T_{(\frac{1}{cos(x)},4)}$.
Put

$$\frac{1}{cos(x)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + o(x^4)$$

Then the column $(a_0, a_1, a_2, a_3, a_4)^t$ we are looking for, can be represented as follows

$$T_{(cos(x),4)} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this simple triangular system We find $\begin{cases} a_0 = 1 \\ a_1 = 0 \\ a2 - \frac{1}{2}a_0 = 0 \\ a3 - \frac{1}{2}a_1 = 0 \\ \frac{1}{24}a_0 - \frac{1}{2}a_2 + a_4 = 0 \end{cases}$

As a result $a_0 = 1; a_1 = 0; a2 = \frac{1}{2}; a3 = 0; a_4 = \frac{5}{24}$
Hence

$$\frac{1}{cos(x)} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + o(x^4)$$

On the other hand, we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4)$$

.

Then

$$T_{(\frac{1}{1-x},4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

And we have

$$T_{(\frac{1}{(1-x)cos(x)},4)} = T_{(\frac{1}{(1-x)},4)} \times T_{(\frac{1}{cos(x)},4)}$$

As a result

$$T_{(\frac{1}{(1-x)cos(x)},4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \frac{3}{2} & 1 & 1 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & 1 & 1 & 0 \\ \frac{41}{24} & \frac{3}{2} & \frac{3}{2} & 1 & 1 \end{bmatrix}$$

Therefore

$$\frac{1}{(1-x)\cos(x)} = 1 + x + \frac{3}{2}x^2 + \frac{3}{2}x^3 + \frac{41}{24}x^4 + o(x^4)$$

Note that in this part we calculate the inverse of the Toeplitz matrices, of which there are several methods[6] [7], but our algorithm based on a trivial Toeplitz system remains the easiest in our case.

This new connection between Toeplitz matrices and the expansion of function has many advantages. In addition to being able to determine expansions by simple computation, especially expansion of the inverse of a function, it gives the possibility to compute the inverse of a particular Toeplitz matrix from the expansion of ordinary functions. Take the following example:

Example 5.2. Let $\lambda \in \mathbb{R}$. Determine the inverse of the matrix $J_\lambda = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ \lambda & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & \lambda & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \lambda & 1 & 0 \\ 0 & 0 & \cdot & 0 & \lambda & 1 \end{bmatrix}$

Let f be the function defined on $\mathbb{R}$ by $f(x) = 1 + \lambda x$,then $J_\lambda = T_{(f,n)}$, and from the (3.4) we have $J_\lambda^{-1} = T_{(f,n)}^{-1} = T_{(\frac{1}{f},n)}$.

Let $y = \lambda x$, we find

$$\begin{aligned} \frac{1}{f(x)} &= \frac{1}{1+\lambda x} \\ &= \frac{1}{1+y} \\ &= 1 - y + y^2 + \cdots + (-1)^n y^n + o(y^n) \\ &= 1 - \lambda x + \lambda^2 x^2 + \cdots + (-1)^n \lambda^n x^n + o(x^n) \end{aligned}$$

Hence

$$J_\lambda^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ -\lambda & 1 & 0 & \cdot & \cdot & \cdot \\ \lambda^2 & -\lambda & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \lambda^2 & -\lambda & 1 & 0 \\ (-1)^n\lambda^n & \cdot & \cdot & \lambda^2 & -\lambda & 1 \end{bmatrix}$$

6 power of a Toeplitz matrix by the expansion of functions

The generalization of the second and third property in (3.4) gives us an important result that allows us to calculate with a simple algorithm the relative power of some square matrices of Toeplitz.

Corollary 6.1. *Let f be a function admitting in the neighborhood of* 0 *an expansion of order* n *and* $T_{(f,n)}$ *its matrix of Toeplitz. Then* $(\forall m \in \mathbf{Z})\ :\ T^{m}_{(f,n)} = T_{(f^m,n)}$

Proof. First case: $m \in \mathbf{N}$

For $m = 0$ we have $T^0 = I_{n+1}$ and $f^0 = 1$ then $I_{n+1} = T_{(f^0,n)}$

Let $m \in \mathbf{N}$

Assume that $T^{m}_{(f,n)} = T_{(f^m,n)}$ and show that $T^{m+1}_{(f,n)} = T_{(f^{m+1},n)}$

We have

$$\begin{aligned} T^{m+1}_{(f,n)} &= T^{m}_{(f,n)} \times T_{(f,n)} \\ &= T_{(f^m,n)} \times T_{(f,n)} \\ &= T_{(f^m \times f,n)} \\ &= T_{(f^{m+1},n)} \end{aligned}$$

Second case: $m \in \mathbf{Z}^-$

There is $p \in \mathbf{N}$ such as $m = -p$

Then

$$\begin{aligned} T^{m}_{(f,n)} &= (T^{-1}_{(f,n)})^p \\ &= (T_{(\frac{1}{f},n)})^p \\ &= T_{((\frac{1}{f})^p,n)} \\ &= T_{(f^{-p},n)} \\ &= T_{(f^m,n)} \end{aligned}$$

Example 6.2. Let A be the matrix defined by $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{bmatrix}$. Let's calculate A^m for every $m \in \mathbf{Z}$.

Let the function f be defined by $f(x) = 1 + ax + bx^2$. So $A = T_{(f,2)}$

Then $A^m = T^{m}_{(f,2)} = T_{(f^m,2)}$

We have

$$\begin{aligned} f^m(x) &= (1 + ax + bx^2)^m \\ &= 1 + m(ax + bx^2) + \frac{m(m-1)}{2!}(ax + bx^2)^2 + ... + \frac{m(m-1)(m-n+1)}{n!}(ax + bx^2)^n \\ &= 1 + max + (mb + \frac{m(m-1)}{2!}a^2)x^2 + ... \end{aligned}$$

Then

$$A^m = \begin{bmatrix} 1 & 0 & 0 \\ ma & 1 & 0 \\ mb + \frac{m(m-1)}{2!}a^2 & ma & 1 \end{bmatrix}$$

Example 6.3. Let's determine the relative power of the J_λ matrix introduced in (**example 5.2**)

We have already seen that:

$$J_\lambda = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ \lambda & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & \lambda & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \lambda & 1 & 0 \\ 0 & 0 & \cdot & 0 & \lambda & 1 \end{bmatrix} = T_{(1+\lambda x,n)}$$

Then

$$J_\lambda^m = T^m_{(1+\lambda x,n)}$$
$$= T_{((1+\lambda x)^m,n)}$$

Since :

$$(1+\lambda x)^m = 1 + C_m^1\lambda x + C_m^2\lambda^2x^2 + ... + C_m^{n-1}\lambda^{n-1}x^{n-1} + C_m^n\lambda^nx^n$$

Then

$$J_\lambda^m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \lambda C_m^1 & 1 & 0 & 0 & 0 & 0 \\ \lambda^2 C_m^2 & \lambda C_m^1 & 1 & 0 & 0 & 0 \\ \lambda^3 C_m^3 & . & . & 1 & 0 & 0 \\ . & . & . & \lambda C_m^1 & 1 & 0 \\ \lambda^n C_m^n & . & \lambda^3 C_m^3 & \lambda^2 C_m^2 & \lambda C_m^1 & 1 \end{bmatrix}$$

note that if $m < 0$ we have $C_m^k = C^k_{-m+k-1}(-1)^k$.

7 Conclusion

This paper aims to present a new approach to compute the expansion of functions near zero using triangular Toeplitz matrices. Only the first column of this matrix is essential.

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Author information

R. Bouifden, Department of Mathematics, Physics and New Technologies (MPNT), Faculty of Sciences of Tetouan, Morocco.
E-mail: rafik.bouifden@etu.uae.ac.ma

A. Haddi, Department of Mathematics, Physics and New Technologies (MPNT), Faculty of Sciences of Tetouan, Morocco.
E-mail: ahaddi@uae.ac.ma

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