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Existence results for a p(x)-Kirchhoff equation with a concave–convex nonlinearities

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Abstract *The existence of two nontrivial weak solutions to a class of p(x)-Kirchhoff-type equations is the subject of this paper. Our approach relies on the variable exponent theory of the Lebesque and Sobolev spaces combined with adequate variational methods, the Mountain Pass Theorem, and the Ekeland variational principle.*

1 Introduction

The purpose of this research is to investigate the existence of a weak solution to a class of nonlocal problems involving the variable exponent and concave-convex nonlinearities

$$\begin{cases} M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\left(-\Delta_{p(x)}u\right) = f_\lambda(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $f_\lambda(x,u) = \lambda\left(m_q(x)|u|^{q(x)-2}u + m_h(x)|u|^{h(x)-2}u\right)$, $\Omega \subset \mathbb{R}^N$ *($N \geq 2$) is a bounded domain with a smooth boundary,* $\lambda > 0$, *the operator* $\Delta_{p(x)}u := div\left(|\nabla u|^{p(x)-2}\nabla u\right)$ *is called the* $p(x)$*-Laplace operator,* p *is Lipschitz continuous on* $\overline{\Omega}$, *and* $q, h \in C_+(\overline{\Omega})$, $m_q, m_h > 0$ *for* $x \in \overline{\Omega}$ *such that* $m_q \in L^{\beta(x)}(\Omega)$, $\beta(x) = \frac{p(x)}{p(x)-q(x)}$, *and* $m_h \in L^{\gamma(x)}(\Omega)$, $\gamma(x) = \frac{p^*(x)}{p^*(x)-h(x)}$. *Here,*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases} \tag{1.2}$$

The problem (1.1) is a generalization of Kirchhoff's model. Kirchhoff [13] presented a model described by the equation.

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

as an expansion of the common D'Alembert's wave equation for the free elastic string vibration. The parameters in equation (1.3) have the following meanings: ρ *is the mass density,* P_0 *is the initial tension,* h *is the cross-sectional area,* E *is the material's Young modulus, and* L *is the length of the string.*

Whenever an elastic string with fixed ends is subjected to transverse vibration, its length changes over time, causing variations in the string's tension. This challenged Kirchhoff to suggest a nonlinear correction to the typical d'Alembert equation. Nash and Modeer [28], as well as Woinowsky-Krieger [27], join this correction into the Euler-Bernoulli equation for a beam (plate) with hinged ends [14].

Eventually, the problem

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u &= f(x,u), \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

received much attention specifically after the pioneer work of Lions [21], see, for instance, [3, 14, 32].

The study of Kirchhoff-type equations was previously developed to involve the p-Laplacian operator in the following problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p}|\nabla u|^p \, \mathrm{d}x\right)\Delta_p u &= f(x,u), \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{cases}$$

where M and f meet specific requirements (for more information see [7, 6, 10]).

However, many publications extended the constant case to include the $p(x)$-Laplacian operator. In their study [9], Dai et al. used variational methods to examine a nonlocal $p(x)$-Laplacian Dirichlet problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x\right)\Delta_{p(x)} u = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

and have shown the existence and multiplicity of solutions to the above $p(x)$-Kirchhoff problem, under some suitable conditions on M and f. In [4], Chen, by means of the mountain pass theorem combined with Ekeland's variational principle, obtained at least two distinct, nontrivial weak solutions for the problem (1.5) where $f(x,u) = \lambda(a(x)|u|^{\alpha(x)-2} + b(x)|u|^{\beta(x)-2})$ with α, β being two variable exponents, $\lambda > 0$ is a parameter, and M satisfies certain conditions. For more related problems, we mention [5, 12, 17, 18, 19, 20, 23, 26].

Motivated by the above works and employing the Mountain Pass Theorem and Ekeland's variational principle, this paper aims to establish the existence and multiplicity of solutions for the problem (1.1).

To state the main result, we need to make some mild assumptions about the function M and the variable exponent. The assumptions for the continuous function $M : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ are as follows:

(M_1) $M(s) = a + b \cdot s^k$, $\forall s > 0$ *with* $k > 0$.

The assumptions on the variable exponent are the following:

(E_1) We will use the notations such as p_1 and p_2 where

$$p_1 := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \le p(x) \le p_2 := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty.$$

(E_2) $1 < p_1 \le p_2 < q_1 \le q_2 < (k+1)p_1 \le (k+1)p_2^{k+1} < h_1 \le h_2 \ll p_1^*$, $\frac{(k+1)p_2^{k+1}}{p_1} < h_1$, *and* $p_2 < N$.

Remark 1.1. The paper presents results on weak solutions in a finite-dimensional domain Ω, as noted in Equation (1.2). See [2] for related references.

This paper is organized as follows. We offer some preliminary mathematical knowledge on variable exponent Lebesgue-Sobolev spaces in Section 2. In Section 3, we use the variable exponent theory of Lebesgue and Sobolev spaces combined with suitable variational methods and the Mountain Pass Theorem to show the existence of at least two nontrivial weak solutions to the problem (1.1), inspired by the idea introduced by Mashiyev, R.A, and al [25].

2 Mathematical preliminaries

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. For more information, please see [8, 24, 30].

We denote

$$C_+(\Omega) := \left\{p(x) : p(x) \in C(\overline{\Omega}), p(x) > 1, \forall x \in \overline{\Omega}\right\},$$

$$p_1 := \max\{p(x) : x \in \Omega\},\ p_2 := \min\{p(x) : x \in \Omega\},$$

$$L_+^\infty(\Omega) := \left\{p : p \in L^\infty(\Omega), ess\inf_{x\in\Omega} p(x) > 1\right\},$$

$$L^{p(x)}(\Omega) := \left\{u : u \text{ is a measurable real-valued function}, \int_\Omega |u|^{p(x)}dx < \infty\right\},$$

which is equipped with the so-called Luxembourg norm

$$\|u\|_{p(x)} = \inf\left\{\delta > 0 : \int_\Omega |\delta^{-1}u|^{p(x)} \le 1\right\},$$

then $\left(L^{p(x)}(\Omega), \|u\|_{p(x)}\right)$ *is a separable and reflexive Banach space, called generalized spaces. We define also the variable exponent Sobolev space:*

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega),$$

which makes $W^{1,p(x)}(\Omega)$ *also a separable and reflexive Banach space. The space* $W_0^{1,p(x)}(\Omega)$ *is denoted by the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,p(x)}(\Omega)$. *We point out that* $W_0^{1,p(x)}(\Omega)$ *is a separable and reflexive Banach space.*

Proposition 2.1. *([24, 30])*

- *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uvdx\right| \le \left(\frac{1}{p_1} + \frac{1}{q_1}\right)\|u\|_{p(x)}\|v\|_{q(x)} \le 2\|u\|_{p(x)}\|v\|_{q(x)}.$$

- *If* $p_1(x), p_2(x) \in C_+(\overline{\Omega})$ *and* $p_1(x) \le p_2(x)$, $\forall x \in \overline{\Omega}$, *then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is*

The next proposition illuminates the close relation between the $\|\cdot\|_{p(x)}$ *and the convex modular* $\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx$ *:*

Proposition 2.2. *([8, 24, 30]) For* $u \in L^{p(x)}(\Omega)$ *and* $p_2 < \infty$ *we have*

(i) $\|u\|_{p(x)} < 1(= 1; > 1) \Longleftrightarrow \rho_{p(x)}(u) < 1(= 1; > 1)$,

(ii) $\|u\|_{p(x)} > 1 \Longrightarrow \|u\|_{p(x)}^{p_1} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p_2}$,

(iii) $\|u\|_{p(x)} < 1 \Longrightarrow \|u\|_{p(x)}^{p_2} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p_1}$,

(iv) $\|u\|_{p(x)} = a > 0 \Longleftrightarrow \rho_{p(x)}\left(\frac{u}{a}\right) = 1$.

Proposition 2.3. *([8, 24, 30]) If* $u, u_n \in L^{p(x)}(\Omega), n = 1, 2, \ldots$, *then the following statements are equivalent*

i) $\lim_{n\to\infty} \|u_n - u\|_{p(x)} = 0$.

ii) $\lim\limits_{n\to\infty} \rho_{p(x)}(u_n - u) = 0.$

iii) $u_n \to u$ *in measure in* Ω *and* $\lim\limits_{n\to\infty} \rho_{p(x)}(u_n) = \rho_{p(x)}(u).$

Lemma 2.4. *([1]) Assume that* $r \in L^\infty_+(\Omega)$ *and* $p \in C_+(\overline{\Omega}) := \{m \in C(\overline{\Omega}) : m_1 > 1\}$. *If* $|u|^{r(x)} \in L^{p(x)}(\Omega)$*, then we have*

$$\min\left\{\|u\|^{r_1}_{r(x)p(x)}, \|u\|^{r_2}_{r(x)p(x)}\right\} \le \||u|^{r(x)}\|_{p(x)} \le \max\left\{\|u\|^{r_1}_{r(x)p(x)}, \|u\|^{r_2}_{r(x)p(x)}\right\}.$$

Remark 2.5. If $r(x) \equiv r$, $r \in \mathbb{R}$ then

$$\||u|^r\|_{p(x)} = \|u\|^r_{rp(x)}.$$

Given two Banach spaces X and Y, the symbol $X \hookrightarrow Y$ means that X is continuously embedded in Y and the symbol $X \hookrightarrow\hookrightarrow Y$ means that there is a compact embedding of X in Y.

Proposition 2.6. *([8, 24, 29, 30]) The following results hold:*

(i) Let $q, h \in C_+(\overline{\Omega})$. *If* $q(x) \le h(x)$ *for all* $x \in \overline{\Omega}$*, then* $L^{h(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

(ii) Let p *is Lipschitz continuous and* $p_2 < N$*, then for* $h \in L^\infty_+(\Omega)$ *with* $p(x) \le h(x) \le p^*(x)$ *there is a continuous embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega)$*, and also there is a constant* $C_1 > 0$ *such that* $\|u\|_{h(x)} \le C_1\|u\|_{1,p(x)}$.

(iii) Let $p, q \in C_+(\overline{\Omega})$. *If* $p(x) \le q(x) \le p^*(x), \forall x \in \overline{\Omega}$*, then* $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$.

(iv) ***(Poincaré inequality)*** *If* $p \in C_+(\overline{\Omega})$*, then there is a constant* $C_2 > 0$ *such that*

$$\|u\|_{p(x)} \le C_2\||\nabla u|\|_{p(x)}, \forall u \in W_0^{1,p(x)}(\Omega).$$

Consequently, $\|u\| := \||\nabla u|\|_{p(x)}$ *and* $\|u\|_{1,p(x)}$ *are equivalent norms on* $W_0^{1,p(x)}(\Omega)$. *In what follows,* $W_0^{1,p(x)}(\Omega)$*, with* $p \in C_+(\overline{\Omega})$*, will be considered as endowed with the norm* $\|u\|_{1,p(x)}$. *We will use* $\|u\| = \|\nabla u\|_{p(x)}$ *for* $u \in W_0^{1,p(x)}(\Omega)$ *in the following discussions.*

Lemma 2.7. ***(Palais-Smale condition [22])*** *Let* E *be a Banach space and* $I \in C^1(E, \mathbb{R})$. *If* $\{u_n\} \subset E$ *is a sequence which satisfies*

$$|I(u_n)| < M,$$
$$I'(u_n) \to 0 \text{ as } n \to \infty \text{ in } E^*,$$

where M *is a positive constant and* E^* *is the dual space of* E*, then* $\{u_n\}$ *possesses a convergent subsequence.*

Finally, we introduce the main tool utilized in this paper, which is the Mountain-Pass Theorem

Lemma 2.8. ***(Mountain-Pass Theorem [22])*** *Let* E *be a Banach space, and let* $I \in C^1(E, \mathbb{R})$ *satisfy the Palais-Smale condition. Assume that* $I(0) = 0$*, and there exists a positive real number* ρ *and* $u, v \in E$ *such that*

- $\|v\| > \rho$, $I(v) \le I(0)$.
- $\alpha = \inf\{I(u) : u \in E, \|u\| = \rho\} > 0$.

Put $G = \{g \in C([0,1], E) : g(0) = 0, g(1) = v\} \neq \emptyset$*; set* $\beta = \inf\limits_{g\in G} \sup\limits_{t\in[0,1]} I(g(t))$. *Then,* $\beta > \alpha$ *and* β *is a critical value of* I.

3 Main Results

Definition 3.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1), if it satisfies

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v dx =\lambda\left(\int_\Omega m_q|u|^{q(x)-2}uvdx + \int_\Omega m_h|u|^{h(x)-2}uvdx\right),$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

We shall look for solutions to (1.1) by finding critical points of the energy functional defined as $J_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$,

$$J_\lambda(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) - \lambda\left(\int_\Omega \frac{m_q}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{m_h}{h(x)}|u|^{h(x)}dx\right), \tag{3.1}$$

where $\widehat{M}(t) = \int_0^t M(s)ds = at + \frac{b}{k+1}t^{k+1}$. It is well known that the functional $J_\lambda \in C^1\left(W_0^{1,p(x)}(\Omega), \mathbb{R}\right)$ for any $v \in \mathbb{R}$, there holds

$$\begin{aligned}\langle J_\lambda'(u), v\rangle =&M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v dx\\ &- \lambda\left(\int_\Omega m_q|u|^{q(x)-2}uvdx + \int_\Omega m_h|u|^{h(x)-2}uvdx\right).\end{aligned} \tag{3.2}$$

The following theorem is the main result of the present paper.

Theorem 3.2. *Assume p is Lipschitz continuous, $q, h \in C_+(\overline{\Omega})$ and condition $(E2)$ is fulfilled. If*

$$\lambda \in \left(0, \min\left\{\frac{q_1 b\rho^{(k+1)p_2-q_1}}{\zeta C_1^{q_1}(k+1)p_2^{k+1}\|m_q\|_{\beta(x)}}, \frac{h_1 b\rho^{(k+1)p_2-h_1}}{\zeta C_1^{h_1}(k+1)p_2^{k+1}\|m_h\|_{\gamma(x)}}\right\}\right), \forall\zeta > 2,$$

then the problem (1.1) has at least two distinct, non-trivial weak solutions, where $\rho \in (0,1)$.

To obtain the proof of Theorem 3.2, we use Mountain Pass theorem. Therefore, we must show that J_λ satisfies Palais-smale condition in the first place.

Lemma 3.3. *Let λ satisfies the condition of Theorem 3.2. If $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ is a sequence which satisfies conditions*

$$|J_\lambda(u_n)| < M, \text{ with } M \in \mathbb{R}^+, \tag{3.3}$$

$$J_\lambda'(u_n) \to 0 \text{ as } n \to \infty \text{ in } \left(W_0^{1,p(x)}(\Omega)\right)^*, \tag{3.4}$$

then $\{u_n\}$ has a convergent subsequence.

Proof. First, we show that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Assume the contrary. Then, passing to a subsequence if necessary, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\| > 1$ for any integer n. By (3.4) we deduce that there exists $N_1 > 0$ such that for any $n > N_1$, we have

$$\|J_\lambda'(u_n)\| \le 1.$$

On the other hand, for any $n > N_1$ fixed, the application

$$W_0^{1,p(x)}(\Omega) \ni v \to \langle J_\lambda'(u_n), v\rangle$$

is linear and continuous. The above information implies

$$|\langle J_\lambda'(u_n), v\rangle| \le \|J_\lambda'(u_n)\|_{W_0^{-1,p'(x)}(\Omega)}\|v\| \le \|v\|, \forall v \in W_0^{1,p(x)}(\Omega), n > N_1.$$

Setting $v = u_n$ we have

$$\begin{aligned} -\|u_n\| \le & M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)}dx \\ & -\lambda\left(\int_\Omega m_q|u_n|^{q(x)}dx + \int_\Omega m_h|u_n|^{h(x)}dx\right) \le \|u_n\|. \end{aligned} \tag{3.5}$$

Using the assumption $\|u_n\| > 1$, relations (3.3)-(3.4), Proposition 2.1, Lemma 2.4 and Proposition 2.6 (ii) we have

$$\begin{aligned} M > & J_\lambda(u_n) - \frac{1}{h_1}\langle J_\lambda'(u_n), u_n\rangle \\ = & \widehat{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right) - \lambda\left(\int_\Omega \frac{m_q}{q(x)}|u_n|^{q(x)}dx + \int_\Omega \frac{m_h}{h(x)}|u_n|^{h(x)}dx\right) \\ & -\frac{1}{h_1}M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)}dx \\ & +\frac{\lambda}{h_1}\left(\int_\Omega m_q|u_n|^{q(x)}dx + \int_\Omega m_h|u_n|^{h(x)}dx\right) \\ \ge & \widehat{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right) - \frac{1}{h_1}M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)}dx \\ & -\lambda\left(\frac{1}{q_1}\int_\Omega m_q|u_n|^{q(x)}dx + \frac{1}{h_1}\int_\Omega m_h|u_n|^{h(x)}dx\right) \\ & +\frac{\lambda}{h_1}\left(\int_\Omega m_q|u_n|^{q(x)}dx + \int_\Omega m_h|u_n|^{h(x)}dx\right) \\ \ge & a\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right) + \frac{b}{(k+1)}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)^{k+1} \\ & -\frac{a}{h_1}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right) - \frac{b}{h_1}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx\right)^{k+1} - \frac{\lambda}{q_1}\int_\Omega m_q|u_n|^{q(x)}dx \\ & -\frac{\lambda}{h_1}\int_\Omega m_h|u_n|^{h(x)}dx + \frac{\lambda}{h_1}\int_\Omega m_q|u_n|^{q(x)}dx + \frac{\lambda}{h_1}\int_\Omega m_h|u_n|^{h(x)}dx \\ \ge & a\left(\frac{1}{p_2} - \frac{1}{h_1p_1}\right)\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right) + b\left(\frac{1}{(k+1)p_2^{k+1}} - \frac{1}{h_1p_1}\right)\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^{k+1} \\ & +\lambda\left(\frac{1}{h_1} - \frac{1}{q_1}\right)\int_\Omega m_q|u_n|^{q(x)}dx \\ \ge & a\left(\frac{1}{p_2} - \frac{1}{h_1p_1}\right)\left(\int_\Omega |\nabla u_n|^{p_1}dx\right) + b\left(\frac{1}{(k+1)p_2^{k+1}} - \frac{1}{h_1p_1}\right)\left(\int_\Omega |\nabla u_n|^{p_1}dx\right)^{k+1} \\ & +\lambda\left(\frac{1}{h_1} - \frac{1}{q_1}\right)\int_\Omega m_q|u_n|^{q(x)}dx \\ \ge & a\left(\frac{1}{p_2} - \frac{1}{h_1p_1}\right)\|u_n\|^{p_1} + b\left(\frac{1}{(k+1)p_2^{k+1}} - \frac{1}{h_1p_1}\right)\|u_n\|^{p_1(k+1)} \\ & +\lambda\left(\frac{1}{h_1} - \frac{1}{q_1}\right)C_3\|m_q\|_{\beta(x)}\|u_n\|^{q_2} \end{aligned}$$

$$\geq b\left(\frac{1}{(k+1)p_2^{k+1}}-\frac{1}{h_1p_1}\right)\|u_n\|^{p_1(k+1)}+\lambda\left(\frac{1}{h_1}-\frac{1}{q_1}\right)C_3\|m_q\|_{\beta(x)}\|u_n\|^{q_2}$$
$$\geq\left[b\left(\frac{1}{(k+1)p_2^{k+1}}-\frac{1}{h_1p_1}\right)+\lambda\left(\frac{1}{h_1}-\frac{1}{q_1}\right)C_3\|m_q\|_{\beta(x)}\|u_n\|^{q_2-p_1(k+1)}\right]\|u_n\|^{p_1(k+1)}, \tag{3.6}$$

where $C_3>0$ is a constant independent of u_n and x, for n large enough. Dividing (3.6) by $\|u_n\|^{p_1(k+1)}$ and passing to the limit as $n\to\infty$ we obtain $\frac{1}{(k+1)p_2^{k+1}}-\frac{1}{h_1p_1}<0$. Since $(E1)$–$(E2)$, this is a contradiction. It follows $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Next, we show the strong convergence of $\{u_n\}$ in $W_0^{1,p(x)}(\Omega)$. Since $\{u_n\}$ is bounded, up to a Next, we show the strong convergence of $\{u_n\}$ in $W_0^{1,p(x)}(\Omega)$. Since $\{u_n\}$ is bounded, up to a subsequence (which we still denote by $\{u_n\}$), we may assume that there exists $u\in W_0^{1,p(x)}(\Omega)$ such that

$$u_n\rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega) \text{ as } n\to\infty.$$

By Proposition 2.6 (iii) we obtain

$$u_n\to u \text{ strongly in } L^{p(x)}(\Omega) \text{ as } n\to\infty. \tag{3.7}$$

Furthermore, from [11, 16] we have

$$u_n\to u \text{ strongly in } L^{h(x)}(K) \quad \text{as } n\to\infty, \tag{3.8}$$

where K is compact subset of Ω. The above information and relation (3.4) imply

$$\langle J'_\lambda(u_n)-J'_\lambda(u),u_n-u\rangle\to 0 \text{ as } n\to\infty.$$

On the other hand, we have

$$M\left(\int_\Omega\frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega\left(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\right)\cdot(\nabla u_n-\nabla u)\,dx$$
$$=\langle J'_\lambda(u_n)-J'_\lambda(u),u_n-u\rangle-\lambda\int_\Omega m_q\left(|u_n|^{q(x)-2}u_n-|u|^{q(x)-2}u\right)(u_n-u)\,dx$$
$$-\lambda\int_\Omega m_h\left(|u_n|^{h(x)-2}u_n-|u|^{h(x)-2}u\right)(u_n-u)\,dx.$$

From Proposition 2.1, Proposition 2.3 and Lemma 2.4, we have

$$\lambda\left|\int_\Omega m_q\left(|u_n|^{q(x)-2}u_n-|u|^{q(x)-2}u\right)(u_n-u)\,dx\right|$$
$$\leq\lambda\left|\int_\Omega m_q|u_n|^{q(x)-2}u_n(u_n-u)dx\right|+\lambda\left|\int_\Omega m_q|u_1|^{q(x)-2}u\,(u_n-u)\,dx\right|$$
$$\leq C_4\|m_q\|_{\beta(x)}\||u_n|^{q(x)-1}\|_{\frac{p(x)}{q(x)-1}}\|u_n-u\|_{p(x)}+C_5\|m_q\|_{\beta(x)}\||u|^{q(x)-1}\|_{\frac{p(x)}{q(x)-1}}\|u_n-u\|_{p(x)}$$
$$\leq C_4\|m_q\|_{\beta(x)}\|u_n\|_{p(x)}^{q_2-1}\|u_n-u\|_{p(x)}+C_5\|m_q\|_{\beta(x)}\|u\|_{p(x)}^{q_2-1}\|u_n-u\|_{p(x)},$$

where $C_4,C_5>0$ and $\frac{1}{\beta(x)}+\frac{q(x)-1}{p(x)}+\frac{1}{p(x)}=1$. Similarly, we have

$$\lambda\left|\int_\Omega m_h\left(|u_n|^{h(x)-2}u_n-|u|^{h(x)-2}u\right)(u_n-u)\,dx\right|$$
$$\leq C_6\|m_h\|_{\gamma(x)}\|u_n\|_{p^*(x)}^{h_2-1}\|u_n-u\|_{p^*(x)}+C_7\|m_h\|_{\gamma(x)}\|u\|_{p^*(x)}^{h_2-1}\|u_n-u\|_{p^*(x)},$$

where $C_6,C_7>0$ and $\frac{1}{\gamma(x)}+\frac{h(x)-1}{p^*(x)}+\frac{1}{p^*(x)}=1$. Combining (3.7) and (3.8) we have

$$\|u_n-u\|_{p(x)}\to 0 \text{ as } n\to\infty,$$

and

$$\|u_n - u\|_{p^*(x)} \to 0 \text{ as } n \to \infty.$$

Therefore, from above inequalities we deduce that

$$\lim_{n\to\infty} \int_\Omega m_h \left(|u_n|^{h(x)-2}u_n - |u|^{h(x)-2}u\right)(u_n - u) = 0, \tag{3.9}$$

and

$$\lim_{n\to\infty} \int_\Omega m_q \left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right)(u_n - u)\, dx = 0, \tag{3.10}$$

thus, by (3.9) and (3.10) we obtain

$$\lim_{n\to\infty} M\left(\int_\Omega \frac{|\nabla u_n|}{p(x)}\right) \int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot(\nabla u_n - \nabla u)\, dx = 0. \tag{3.11}$$

This result and using the standard inequality in $\mathbb{R}^N$ given by

$$\left(|\varrho|^{r-2}\varrho - |\eta|^{r-2}\eta\right)\cdot(\varrho - \eta) \geq 2^{-r}|\varrho - \eta|, \forall r \geq 2, \varrho, \eta \in \mathbb{R}^N, \tag{3.12}$$

yields

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^{p(x)} dx = 0. \tag{3.13}$$

This fact and Proposition 2.3 imply $\|\nabla u_n - \nabla u\|_{p(x)} \to 0$ as $n \to \infty$. Relation (3.8) and fact that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$ authorize us to apply the conclusion of [31] in order to obtain that $u_n \to u$ strongly in $W_0^{1,p(x)}(\Omega)$. Thus, Lemma 3.3 is proved. □

Now, we show that the Mountain-Pass theorem can be applied in this case.

Lemma 3.4. *Assume $p, q, h \in C_+(\overline{\Omega})$ and condition $(E_1) - (E_2)$ is fulfilled. The following assertions hold.*

(i) There exist $\lambda > 0, \alpha > 0$ and $\rho \in (0,1)$ such that

$$J_\lambda(u) \geq \alpha, \quad \forall u \in W_0^{1,p(x)}(\Omega) \text{ with } \|u\| = \rho. \tag{3.14}$$

(ii) There exists $\omega \in W_0^{1,p(x)}(\Omega)$ such that

$$\lim_{t\to\infty} J_\lambda(t\omega) = -\infty. \tag{3.15}$$

(iii) There exists $\varphi \in W_0^{1,p(x)}(\Omega)$ such that $\varphi \geq 0, \varphi \neq 0$ and

$$J_\lambda(t\varphi) < 0, \tag{3.16}$$

for $t > 0$ small enough.

Proof. (i) Using Proposition 2.1, Proposition 2.2, Proposition 2.6 (ii) and Lemma 2.4, we deduce that for any $u \in W_0^{1,p(x)}(\Omega)$ with $\rho \in (0,1)$ we have

$$\begin{aligned} J_\lambda(u) &= a\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) + \frac{b}{(k+1)}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right)^{k+1} \\ &\quad - \lambda\int_\Omega \frac{m_q}{q(x)}|u|^{q(x)}dx - \lambda\int_\Omega \frac{m_h}{h(x)}|u|^{h(x)}dx \\ &\geq \frac{a}{p_2}\int_\Omega |\nabla u|^{p(x)}dx + \frac{b}{(k+1)p_2^{k+1}}\left(\int_\Omega |\nabla u|^{p(x)}dx\right)^{k+1} \\ &\quad - \frac{\lambda}{q_1}\int_\Omega m_q|u|^{q(x)}dx - \frac{\lambda}{h_1}\int_\Omega m_h|u|^{h(x)}dx \end{aligned}$$

$$\geq \frac{a}{p_2}\|u\|^{p_2}dx + \frac{b}{(k+1)p_2^{k+1}}\|u\|^{p_2(k+1)} - \frac{\lambda}{q_1}\int_\Omega m_q|u|^{q(x)}dx$$
$$-\frac{\lambda}{h_1}\int_\Omega m_h|u|^{h(x)}dx$$
$$\geq \frac{b}{(k+1)p_2^{k+1}}\|u\|^{p_2(k+1)} - \frac{\lambda}{q_1}\|m_q\|_{\beta(x)}\||u|^{q(x)}\|_{\frac{p(x)}{q(x)}}$$
$$-\frac{\lambda}{h_1}\|b\|_{\gamma(x)}\||u|^{h(x)}\|_{\frac{p^*(x)}{h(x)}}$$
$$\geq \frac{b}{(k+1)p_2^{k+1}}\|u\|^{p_2(k+1)} - \frac{\lambda}{q_1}\|m_q\|_{\beta(x)}\|u\|_{p(x)}^{q_1} - \frac{\lambda}{h_1}\|m_h\|_{\gamma(x)}\|u\|_{p^*(x)}^{h_1}$$
$$\geq \|u\|^{p_2(k+1)}\left[\frac{b}{(k+1)p_2^{k+1}} - C_1^{q_1}\frac{\lambda}{q_1}\|m_q\|_{\beta(x)}\|u\|^{q_1-p_2(k+1)}\right.$$
$$\left.-C_1^{h_1}\frac{\lambda}{h_1}\|m_h\|_{\gamma(x)}\|u\|^{h_1-p_2(k+1)}\right].$$

Taking

$$\lambda = \min\left\{\frac{q_1 b\rho^{(k+1)p_2-q_1}}{\zeta C_1^{q_1}(k+1)p_2^{k+1}\|m_q\|_{\beta(x)}}, \frac{h_1 b\rho^{(k+1)p_2-h_1}}{\zeta C_1^{h_1}(k+1)p_2^{k+1}\|m_h\|_{\gamma(x)}}\right\}, \quad \forall\zeta > 2,$$

we obtain

$$J_\lambda(u) \geq \frac{(\zeta-2)\rho^{(k+1)p_2}}{\zeta(k+1)p_2}, \forall u \in W_0^{1,p(x)}(\Omega) \text{ with } \|u\| = \rho.$$

Thus Lemma 3.4 (i) is proved.

(ii) Let $\omega \in C_0^\infty(\Omega)$, $\omega \geq 0$, $\omega \neq 0$ and $t > 1$. We have

$$J_\lambda(t\omega) = a\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\omega|^{p(x)}dx + \frac{b}{(k+1)}\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\omega|^{p(x)}dx\right)^{k+1}$$
$$-\lambda\int_\Omega \frac{t^{q(x)}}{q(x)}m_q|\omega|^{q(x)}dx - \lambda\int_\Omega \frac{t^{h(x)}}{h(x)}m_h|\omega|^{h(x)}dx$$
$$\leq \frac{at^{p_2}}{p_1}\int_\Omega |\nabla\omega|^{p(x)}dx + \frac{bt^{(k+1)p_2}}{(k+1)p_1^{k+1}}\left(\int_\Omega |\nabla\omega|^{p(x)}dx\right)^{k+1}$$
$$-\lambda\frac{t^{q_2}}{q_1}\int_\Omega m_q|\omega|^{q(x)}dx - \lambda\frac{t^{h_2}}{h_1}\int_\Omega m_h|\omega|^{h(x)}dx.$$
$$\leq \frac{bt^{(k+1)p_2}}{(k+1)p_1^{k+1}}\left(\int_\Omega |\nabla\omega|^{p(x)}dx\right)^{k+1}$$
$$-\lambda\frac{t^{q_2}}{q_1}\int_\Omega m_q|\omega|^{q(x)}dx - \lambda\frac{t^{h_2}}{h_1}\int_\Omega m_h|\omega|^{h(x)}dx.$$

Since $\max\{q_2, (k+1)p_2\} < h_2$ we have $J_\lambda(t\omega) \to -\infty$. Thus Lemma 3.4 (ii) is proved.

(iii) Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, $\varphi \neq 0$ and $t \in (0,1)$. We have

$$\begin{aligned}J_\lambda(t\varphi) &= a\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)}dx + \frac{b}{(k+1)}\left(\int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)}dx\right)^{k+1}\\ &- \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}m_q|\varphi|^{q(x)}dx - \lambda\int_\Omega \frac{t^{h(x)}}{h(x)}m_h|\varphi|^{h(x)}dx\\ &\leq \frac{at^{p_2}}{p_1}\int_\Omega |\nabla\varphi|^{p(x)}dx + \frac{bt^{(k+1)p_2}}{(k+1)p_1^{k+1}}\left(\int_\Omega |\nabla\varphi|^{p(x)}dx\right)^{k+1}\\ &- \lambda\frac{t^{q_1}}{q_1}\int_\Omega m_q|\varphi|^{q(x)}dx - \lambda\frac{t^{h_1}}{h_1}\int_\Omega m_h|\varphi|^{h(x)}dx.\end{aligned}$$

Since

$$\begin{aligned}&\frac{\lambda t^{q_1}}{q_1}\int_\Omega m_q|\varphi|^{q(x)}dx + \frac{\lambda t^{h_1}}{h_1}\int_\Omega m_h|\varphi|^{h(x)}dx\\ &>\frac{\lambda t^{q_1}}{h_1}\left[\int_\Omega m_q|\varphi|^{q(x)}dx + \int_\Omega m_h|\varphi|^{h(x)}dx\right],\end{aligned}$$

and also

$$\begin{aligned}&\frac{at^{p_2}}{p_1}\int_\Omega |\nabla\varphi|^{p(x)}dx + \frac{bt^{(k+1)p_2}}{(k+1)p_1^{k+1}}\left(\int_\Omega |\nabla\varphi|^{p(x)}dx\right)^{k+1}\\ &<\frac{at^{(k+1)p_2}}{p_1}\int_\Omega |\nabla\varphi|^{p(x)}dx + \frac{bt^{(k+1)p_2}}{p_1}\left(\int_\Omega |\nabla\varphi|^{p(x)}dx\right)^{k+1}\\ &<\frac{t^{(k+1)p_2}}{p_1}\left[a\int_\Omega |\nabla\varphi|^{p(x)}dx + b\left(\int_\Omega |\nabla\varphi|^{p(x)}dx\right)^{k+1}\right].\end{aligned}$$

We have

$$\begin{aligned}J_\lambda(t\varphi) \leq &\frac{t^{(k+1)p_2}}{p_1}\left[a\int_\Omega |\nabla\varphi|^{p(x)}dx + b\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^{k+1}\right]\\ &- \frac{\lambda t^{q_1}}{h_1}\left[\int_\Omega m_q|\varphi|^{q(x)}dx + \int_\Omega m_h|\varphi|^{h(x)}dx\right]\\ &<0,\end{aligned}$$

for $t < \delta^{\frac{1}{(k+1)p_2 - q_1}}$ with

$$0 < \delta < \min\left\{1, \lambda\frac{p_1\left[\int_\Omega m_q|\varphi|^{q(x)}dx + \int_\Omega m_h|\varphi|^{h(x)}dx\right]}{h_1\left[a\int_\Omega |\nabla\varphi|^{p(x)}dx + b\left(\int_\Omega |\nabla\varphi|^{p(x)}dx\right)^{k+1}\right]}\right\}.$$

Lemma 3.4 (iii) is proved.

□

Proof of Theorem 3.2 . Combining Lemma 3.3, (ii) and (iii) of Lemma 3.4 and Mountain-Pass Theorem, we deduce J_λ has at least one non-trivial critical point $J_\lambda(u_1) = \beta \geq \alpha$. Therefore u_1 is a weak solutions to problem (1.1).
Now, we prove that there exists a second weak solution $u_2 \in X$ such that $u_2 \neq u_1$. Indeed, by (3.14), the functional J_λ is bounded from below on the unit ball $\overline{B}_1(0)$. Applying the Ekeland variational principle in [15] to the functional $J_\lambda : \overline{B}_1(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B}_1(0)$ such that

$$\begin{aligned}&J_\lambda(u_\epsilon) < \inf_{u\in\overline{B}_1(0)} J_\lambda(u) + \epsilon,\\ &J_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon.\end{aligned}$$

By Lemma 3.4, we have

$$\inf_{u\in\partial B_1(0)} J_\lambda(u) \geq r > 0 \text{ and } \inf_{u\in\overline{B}_1(0)} J_\lambda(u) < 0.$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{u\in\partial B_1(0)} J_\lambda(u) - \inf_{u\in\overline{B}_1(0)} J_\lambda(u).$$

Then, $J_\lambda(u_\epsilon) < \inf_{u\in\partial B_1(0)} J_\lambda(u)$ and thus, $u_\epsilon \in B_1(0)$. So, we define the functional $I_\lambda : \overline{B}_1(0) \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + tv) - I_\lambda(u_\epsilon)}{t} \geq 0,$$

for all $t > 0$ small enough and all $v \in B_1(0)$. The above information shows that

$$\frac{J_\lambda(u_\epsilon + tv) - J_\lambda(u_\epsilon)}{t} + \epsilon\|v\| \geq 0.$$

Letting $t \to 0^+$, we deduce that

$$\langle J'_\lambda(u_\epsilon), v\rangle \geq -\epsilon\|v\|.$$

It should be noticed that $-v$ also belongs to $B_1(0)$, so replacing v by $-v$, we get

$$\langle J'_\lambda(u_\epsilon), v\rangle \leq \epsilon\|v\| \text{ or } \langle J'_\lambda(u_\epsilon), -v\rangle \geq -\epsilon\|-v\|,$$

which helps us to deduce that $\|J'_\lambda(u_\epsilon)\|_{X^*} \leq \epsilon$. Therefore, there exists a sequence $\{u_n\} \subset B_1(0)$ such that

$$J_\lambda(u_n) \to \tilde{\beta} = \inf_{u\in\overline{B}_1(0)} J_\lambda(u) < 0 \text{ and } J'_\lambda(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{3.17}$$

From Lemma 3.3, the sequence $\{u_n\}$ converges strongly to u_2 as $n \to \infty$. Moreover, since $J_\lambda \in C^1(X, \mathbb{R})$, by (3.17) it follows that $J_\lambda(u_2) = \tilde{\beta}$ and $J'_\lambda(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.1).

Finally, we point out the fact that $u_1 \neq u_2$ since $J_\lambda(u_1) = \beta > 0 > \tilde{\beta} = J_\lambda(u_2)$. Then the proof of Theorem 3.2 is complete. □

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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