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# ON THE COMPLEMENT OF THE INTERSECTION OF ZERO-DIVISOR AND TOTAL GRAPHS OF POLYNOMIAL AND POWER SERIES RINGS

S. Visweswaran and Hiren D. Patel

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**Corresponding Author: S. Visweswaran**

**Abstract** The rings considered in this paper are commutative with identity which are not integral domains. Let $R$ be a ring. Let $Z(R)$ denote the set of all zero-divisors of $R$ and we denote $Z(R)\backslash\{0\}$ by $Z(R)^*$. The total zero-divisor graph of $R$ denoted by $ZT(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices $x$ and $y$ are adjacent in $ZT(R)$ if and only if $xy = 0$ and $x + y \in Z(R)$. Let $(ZT(R))^c$ denote the complement of $ZT(R)$. The aim of this paper is to study whether $(ZT(R))^c$ is connected implies that $(ZT(R[X]))^c$ (respectively, $(ZT(R[[X]]))^c$) is connected and vice versa and to compare their diameters (respectively, radii) when both the graphs are connected.

## 1 Introduction

The rings considered in this paper are commutative with identity and unless otherwise specified, they are not integral domains. Let $R$ be a ring. Let $Z(R)$ denote the set of all zero-divisors of $R$ and we denote $Z(R)\backslash\{0\}$ by $Z(R)^*$. Motivated by the results proved on the zero-divisor graphs in commutative rings (see [2]) and by the interesting theorems proved by Anderson and Badawi on the total graphs of rings (see [3, 4, 5]), the concept of the total zero-divisor graph of $R$, denoted by $ZT(R)$ was introduced and investigated by Đurić et al. in [11]. Recall that the *total zero-divisor graph* of $R$ denoted by $ZT(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices $x$ and $y$ are adjacent in $ZT(R)$ if and only if $xy = 0$ and $x+y \in Z(R)$ [11]. Several interesting results were proved on $ZT(R)$ by Đurić et al. in [11].

First, it is useful to recall the following definitions and notations from commutative ring theory before we give a brief account of the results from the literature which motivated this paper. Let $R$ be a ring. We denote the set of all prime ideals, the set of all maximal ideals, and the set of all minimal prime ideals of $R$ by $Spec(R), Max(R)$, and $Min(R)$ respectively. We denote the cardinality of a set $A$ by $|A|$. If $|Max(R)| = 1$, then $R$ is said to be *quasi-local*. A Noetherian quasi-local ring is called a *local ring*. Let $I$ be an ideal of $R$ with $I \neq R$. Recall that $\mathfrak{p} \in Spec(R)$ is said to be a *maximal N-prime of* $I$ if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z_R(\frac{R}{I}) = \{r \in R \mid rx \in I$ for some $x \in R\backslash I\}$ [17]. Thus $\mathfrak{p} \in Spec(R)$ is a maximal N-prime of $(0)$ if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z(R)$. For convenience, let us denote the set of all maximal N-primes of $(0)$ in $R$ by $MNP(R)$. Note that $S = R\backslash Z(R)$ is a multiplicatively closed subset of $R$. If $I$ is an ideal of $R$ with $I \subseteq Z(R)$, then $I \cap S = \emptyset$. Hence, by Zorn's lemma and [19, Theorem 1], it follows that there exists $\mathfrak{p} \in MNP(R)$ with $I \subseteq \mathfrak{p}$. In particular, if $x \in Z(R)$, then $Rx \cap S = \emptyset$. So, $Rx \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in MNP(R)$. This shows that if $MNP(R) = \{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$, then $Z(R) = \bigcup_{\alpha\in\Lambda} \mathfrak{p}_\alpha$. Thus $|MNP(R)| = 1$ if and only if $Z(R)$ is an ideal of $R$. We use m.c. subset to denote multiplicatively closed subset. Let $I$ be an ideal of $R$ with $I \neq R$. Recall that $\mathfrak{p} \in Spec(R)$ is said to be an *associated prime of $I$ in the sense of Bourbaki* if $\mathfrak{p} = (I :_R x)$ for some $x \in R$ [16]. In such a case, we say that $\mathfrak{p}$ is a B-prime of $I$. Recall that an ideal $I$ of $R$ is an *annihilating ideal* if there exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. We denote the set of all annihilating ideals of $R$ by $\mathbb{A}(R)$ and $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. If $\mathfrak{p} \in MNP(R)$, then it is clear that $\mathfrak{p} \in \mathbb{A}(R)$ if and only if $\mathfrak{p}$ is a B-prime of $(0)$ in $R$. We denote the nilradical of $R$ by $nil(R)$ and $R$ is said to be *reduced* if $nil(R) = (0)$. The Krull dimension of $R$ is referred to as the dimension of $R$ and is denoted by $dim R$. We denote the group of units of $R$ by $U(R)$ and we denote the set of all non-units of $R$ by $NU(R)$. If $A, B$ are sets with $A$ is a subset of $B$ and $A \neq B$, then we denote it by $A \subset B$ (or by $B \supset A$).

Recall that the *zero-divisor graph* of $R$ denoted by $\Gamma(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices $x$ and $y$ are adjacent in $\Gamma(R)$ if and only if $xy = 0$ [6]. In [8], Axtell et al. investigated the preservation of the diameter of zero-divisor graph of a commutative ring under extension to polynomial and power series rings. We denote the polynomial ring in one variable $X$ over $R$ by $R[X]$ and the power series ring in one variable $X$ over $R$ by $R[[X]]$. For $f(X) \in R[X]$, we denote the ideal of $R$ generated by the coefficients of $f(X)$ by $A_f$. A complete characterization of the diameters of $\Gamma(R)$ and $\Gamma(R[X])$ was given by Lucas in [20, Theorem 3.6]. In [20, Theorem 4.9], he gave a complete characterization of the diameters of $\Gamma(R), \Gamma(R[X])$, and $\Gamma(R[[X]])$ in the case of reduced rings. Several examples were given to illustrate the behaviour of $diam(\Gamma(R))$ and $diam(\Gamma(R[[X]]))$ in the case of non-reduced rings (see [20, Section 5]).

The graphs considered in this paper are undirected and simple. For a graph $G$, we denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. Let $G = (V, E)$ be a simple graph. Recall that the *complement* of $G$ denoted by $G^c$ is defined by setting $V(G^c) = V(G) = V$ and distinct $u, v \in V$ are joined by an edge in $G^c$ if and only if there exists no edge in $G$ joining $u$ and $v$ [9, Definition 1.2.13].

If $|Z(R)^*| \geq 2$, then in [26], it was proved that the connectedness of $(\Gamma(R))^c$ depends on the behaviour of members of $MNP(R)$ (see [26, Theorem 1.1]). Also, we determined $diam((\Gamma(R))^c)$ when $(\Gamma(R))^c$ is connected and studied whether $(\Gamma(R))^c$ is connected implies that $(\Gamma(R[X]))^c$ (respectively, $(\Gamma(R[[X]]))^c$) is connected and vice versa. It is useful to recall that the *total graph* of $R$ denoted by $T(\Gamma(R))$ is an undirected graph whose vertex set is the set of all elements of $R$ and distinct vertices $x$ and $y$ are adjacent in $T(\Gamma(R))$ if and only if $x + y \in Z(R)$ [3]. For more information on the total graphs of commutative rings, the reader is referred to [3, 4, 5, 24, 25]. For an excellent and inspiring investigation on several graphs associated with commutative rings, the reader is referred to [1]. For more detailed exposition of the total graphs of commutative rings, the graphs from total graphs, and the generalized total graphs, one can refer chapters 7 to 9 from [1]. Let $T_{Z(R)^*}(\Gamma(R))$ denote the subgraph of the total graph of $R$ induced by $Z(R)^*$. It is clear that $(ZT(R))^c = (\Gamma(R))^c \cup (T_{Z(R)^*}(\Gamma(R)))^c$. Hence, $(\Gamma(R))^c$ and $(T_{Z(R)^*}(\Gamma(R)))^c$ are spanning subgraphs of $(ZT(R))^c$. If $|Z(R)^*| \geq 2$ and $MNP(R) = \{\mathfrak{p}\}$, then it was shown in [27] that $(ZT(R))^c$ is connected if and only if $\mathfrak{p} \notin \mathbb{A}(R)$; if $|MNP(R)| \geq 2$, then $(ZT(R))^c$ is connected. In [27], we determined $diam((ZT(R))^c)$ and $r((ZT(R))^c)$ when $(ZT(R))^c$ is connected. In this paper, we determine whether $(ZT(R))^c$ is connected implies that $(ZT(R[X]))^c$ (respectively, $(ZT(R[[X]]))^c$) is connected and vice versa and compare their diameters (respectively, radii) when both the graphs are connected.

For definitions and notations from graph theory that are not mentioned in this paper, one can refer [9]. Before we give a brief account of results that are proved in this paper, it is desirable to mention the needed notations from graph theory. Let $G = (V, E)$ be a connected graph. For any distinct $u, v \in V$, we denote the distance between $u$ and $v$ in $G$ by $d_G(u, v)$ or by $d(u, v)$. The diameter of $G$ is denoted by $diam(G)$. For any $v \in V$, the eccentricity of $v$ in $G$ is denoted by $e_G(v)$ or by $e(v)$. The radius of $G$ is denoted by $r(G)$.

This paper consists of five sections including the introduction. For a ring $R$, we use $R_1$ to denote either $R[X]$ or $R[[X]]$. In Section 2, with the assumption $|Z(R)^*| \geq 2$, we provide several sufficient conditions such that both $(ZT(R))^c$ and $(ZT(R_1))^c$ are connected.

In Section 3, we consider $R$ such that $|MNP(R)| = 1$ and study whether $(ZT(R))^c$ is connected implies that $(ZT(R_1))^c$ is connected and vice versa and determine their diameters (respectively, radii) when both the graphs are connected.

In Section 4, we consider $R$ such that $|MNP(R)| \geq 2$. It is proved that both $(ZT(R))^c$ and $(ZT(R_1))^c$ are connected and we compare their diameters (respectively, radii).

In Section 5, the rings $R$ considered are von Neumann regular which are not fields. It is proved that $diam((ZT(R))^c) = diam((ZT(R_1))^c) \in \{1, 2\}$. Moreover, necessary and sufficient conditions are determined in order that $r((ZT(R))^c) = 2$.

Several examples are presented to illustrate the results proved in this paper.

## 2 Some sufficient conditions under which both $(ZT(R))^c$ and $(ZT(R[X]))^c$ (respectively, $(ZT(R[[X]]))^c$) are connected

Unless otherwise specified, the rings considered in this paper are commutative with identity which are not integral domains. For a ring $R$, with $R_1 = R[X]$ or $R[[X]]$, the aim of this section is to provide some sufficient conditions under which both the graphs $(ZT(R))^c$ and $(ZT(R_1))^c$ are connected and to determine the bounds for their diameters and radii in the case when both the graphs are connected. First, we state and prove some lemmas that we need for the proofs of main results. We often use in our discussion that $(ZT(R))^c$ is the union of its spanning subgraphs $(\Gamma(R))^c$ and $(T_{Z(R)^*}(\Gamma(R)))^c$.

**Lemma 2.1.** *If $R$ is a ring and $S$ is a subset of $Z(R)$ such that $S \not\subseteq ((0) :_R a) \cup ((0) :_R b)$ for some distinct $a, b \in Z(R)^*$, then there exists a path of length at most two between $a$ and $b$ in $(\Gamma(R))^c$ and hence, in $(ZT(R))^c$.*

*Proof.* Suppose that $a$ and $b$ are not adjacent in $(ZT(R))^c$. By the assumption on $S$, $S \subseteq Z(R)$ and there exists $s \in S$ with $sa \neq 0$ and $sb \neq 0$. Thus $s \in Z(R)^*$ and $a - s - b$ is a path of length two between $a$ and $b$ in $(\Gamma(R))^c$ and hence, in $(ZT(R))^c$. □

**Lemma 2.2.** *If $R$ is a ring, then for any nonzero $f(X) \in R_1$, there exists $r \in R\backslash\{0\}$ such that $((0) :_{R_1} f(X)) \cap R \subseteq ((0) :_R r)$.*

*Proof.* Let $j \geq 0$ be least with the property that the coefficient $r_j$ of $X^j$ in $f(X)$ is nonzero. Then it is clear that $((0) :_{R_1} f(X)) \cap R \subseteq ((0) :_R r_j)$. □

**Proposition 2.3.** *If $R$ is a ring and $I$ an ideal of $R$ such that $I \subseteq Z(R)$ but $I \notin \mathbb{A}(R)$, then both $(ZT(R))^c$ and $(ZT(R_1))^c$ are connected with $diam((ZT(R))^c) \leq 2$ and $diam((ZT(R_1))^c) \leq 2$.*

*Proof.* Let $a, b \in Z(R)^*$ be distinct. Since $I \notin \mathbb{A}(R)$ by hypothesis, $I \not\subseteq ((0) :_R a) \cup ((0) :_R b)$. Hence, there exists a path of length at most two between $a$ and $b$ in $(ZT(R))^c$ by Lemma 2.1. Therefore, $(ZT(R))^c$ is connected and $diam((ZT(R))^c) \leq 2$.

Let $f(X), g(X) \in Z(R_1)^*$ be distinct. Assume that they are not adjacent in $(ZT(R_1))^c$. By Lemma 2.2, there exist $r, s \in R\backslash\{0\}$ such that $((0) :_{R_1} f(X)) \cap R \subseteq ((0) :_R r)$ and $((0) :_{R_1} g(X)) \cap R \subseteq ((0) :_R s)$. As $I \notin \mathbb{A}(R)$, $I \not\subseteq ((0) :_R r) \cup ((0) :_R s)$. Hence, $I \not\subseteq J \cup K$, where $J = ((0) :_{R_1} f(X)) \cap R$ and $K = ((0) :_{R_1} g(X)) \cap R$. Therefore, there exists $a \in I\backslash\{0\} \subseteq Z(R)^* \subset Z(R_1)^*$ such that $f(X) - a - g(X)$ is a path of length two between $f(X)$ and $g(X)$ in $(\Gamma(R_1))^c$ and hence, in $(ZT(R_1))^c$. This shows that $(ZT(R_1))^c$ is connected and $diam((ZT(R_1))^c) \leq 2$. □

The following Remarks 2.4, 2.5, and Lemma 2.6 are needed in our future discussion.

**Remark 2.4.** For a ring $R$, $(ZT(R))^c = (\Gamma(R))^c$ if and only if $(T_{Z(R)^*}(\Gamma(R)))^c$ is a subgraph of $(\Gamma(R))^c$ if and only if $(C)$ is satisfied, where $(C)$ is: if $x, y \in Z(R)^*$ are such that $x + y \notin Z(R)$, then $xy \neq 0$.

**Remark 2.5.** If $R$ is a ring with $|MNP(R)| = 1$, then for any $x, y \in Z(R)$, $x + y \in Z(R)$, since $Z(R)$ is an ideal of $R$. Therefore, $(C)$ holds trivially, where $(C)$ is as in Remark 2.4. Hence, $(ZT(R))^c = (\Gamma(R))^c$.

**Lemma 2.6.** *For a simple graph $G = (V, E)$ with $|V| \geq 2$, if both $G$ and $G^c$ are connected, then $r(G^c) \geq 2$ and $r(G) \geq 2$.*

*Proof.* Since $|V| \geq 2$ and $G$ is connected by assumption, for any $v \in V$, there exists $w \in V\backslash\{v\}$ such that $v$ and $w$ are adjacent in $G$. Hence, $d(v, w) \geq 2$ in $G^c$. Thus $e(v) \geq 2$ in $G^c$ for each $v \in V$ and so, $r(G^c) \geq 2$. Similarly, it can be shown that $r(G) \geq 2$. □

**Proposition 2.7.** *For a ring $R$ with $|Z(R)^*| \geq 2$ and $MNP(R) = \{\mathfrak{p}\}$, the following statements hold:*
*(1) $(ZT(R))^c = (\Gamma(R))^c$.*
*(2) $(ZT(R))^c$ is connected if and only if $\mathfrak{p} \notin \mathbb{A}(R)$.*
*(3) If $\mathfrak{p} \notin \mathbb{A}(R)$, then $(ZT(R_1))^c$ is also connected with $diam((ZT(R))^c) = r((ZT(R))^c) = diam((ZT(R_1))^c) = 2$ and $r((ZT(R_1))^c) \leq 2$.*

*Proof.* (1) Note that $(ZT(R))^c = (\Gamma(R))^c$ by Remark 2.5.
(2) If $\mathfrak{p} \in \mathbb{A}(R)$, then $\mathfrak{p}r = (0)$ for some $r \in R\backslash\{0\}$. It is clear that $r \in Z(R)^*$. As $|Z(R)^*| \geq 2$ by hypothesis, there exists $x \in Z(R)^*$ with $x \neq r$. Since $ry = 0$ for any $y \in Z(R) = \mathfrak{p}$, there is no path in $(\Gamma(R))^c$ between $x$ and $r$. Hence, $(ZT(R))^c$ is not connected if $\mathfrak{p} \in \mathbb{A}(R)$. If $\mathfrak{p} \notin \mathbb{A}(R)$, then $(ZT(R))^c$ is connected by Proposition 2.3.
(3) If $\mathfrak{p} \notin \mathbb{A}(R)$, then by Proposition 2.3, both $(ZT(R))^c$ and $(ZT(R_1))^c$ are connected with $diam((ZT(R))^c) \leq 2$ and $diam((ZT(R_1))^c) \leq 2$. As $\Gamma(R)$ is connected by [6, Theorem 2.3], it follows from Lemma 2.6 that $r((\Gamma(R))^c) \geq 2$. Therefore, $diam((ZT(R))^c) = r((ZT(R))^c) = 2$. Let $a \in Z(R)^*$. Since $\Gamma(R)$ is connected, we can find $x \in Z(R)^*\backslash\{a\}$ with $ax = 0$. Observe that $a + x \in Z(R) \subset Z(R_1)$. Hence, $a$ and $x$ are not adjacent in $(ZT(R_1))^c$. Therefore, $diam((ZT(R_1))^c) \geq 2$ and so, $diam((ZT(R_1))^c) = 2$. It is clear that $r((ZT(R_1))^c) \leq 2$. □

**Proposition 2.8.** *If $R$ is a ring with $nil(R) \notin \mathbb{A}(R)$, then $diam((ZT(R))^c) = r((ZT(R))^c) = diam((ZT(R_1))^c) = r((ZT(R_1))^c) = 2$.*

*Proof.* Note that $nil(R) \subseteq Z(R)$. Since $nil(R) \notin \mathbb{A}(R)$ by hypothesis, $nil(R)$ is not finitely generated and so, $Z(R)^*$ is infinite. By Proposition 2.3, both $(ZT(R))^c$ and $(ZT(R_1))^c$ are connected with $diam((ZT(R))^c) \leq 2$ and $diam((ZT(R_1))^c) \leq 2$. Let $a \in Z(R)^*$. Since $\Gamma(R)$ is connected, there exists $x \in Z(R)^*\backslash\{a\}$ with $ax = 0$. If $a + x \in Z(R)$, then $a$ and $x$ are not adjacent in $(ZT(R))^c$ and so, $d(a, x) \geq 2$ in $(ZT(R))^c$. If $a + x \notin Z(R)$, then $a \notin nil(R)$ by [20, Lemma 2.3]. Since $nil(R) \notin \mathbb{A}(R)$, there exists $b \in nil(R)$ such that $bx \neq 0$. It is clear that $a \neq bx, a(bx) = 0$, and $a + bx \in Z(R)$ by [20, Lemma 2.3]. Hence, $d(a, bx) \geq 2$ in $(ZT(R))^c$. The above arguments show that $e(a) \geq 2$ in $(ZT(R))^c$ for each $a \in Z(R)^*$. Therefore, $r((ZT(R))^c) \geq 2$ and so, $diam((ZT(R))^c) = r((ZT(R))^c) = 2$.

As $nil(R_1) \cap R = nil(R)$ and $nil(R) \notin \mathbb{A}(R)$, it follows from Lemma 2.2 that $nil(R_1) \notin \mathbb{A}(R_1)$. Hence, it follows that $diam((ZT(R_1))^c) = r((ZT(R_1))^c) = 2$. □

The following lemma is needed in the proof of Proposition 2.10.

**Lemma 2.9.** *If a ring $R$ admits a nonzero element $x$ which belongs to more than one member of $MNP(R)$, then there exists $a \in Z(R)^*\backslash\{x\}$ such that $x$ and $a$ are not adjacent in $(ZT(R))^c$.*

*Proof.* By hypothesis, we can find distinct $\mathfrak{p}_1, \mathfrak{p}_2 \in MNP(R)$ such that $x \in (\bigcap_{i=1}^{2} \mathfrak{p}_i)\backslash\{0\}$. Since distinct members of $MNP(R)$ are not comparable under inclusion, there exist $a_1 \in \mathfrak{p}_1\backslash\mathfrak{p}_2$ and $a_2 \in \mathfrak{p}_2\backslash\mathfrak{p}_1$. Thus $|Z(R)^*| \geq 2$. As $\Gamma(R)$ is connected by [6, Theorem 2.3], there exists $y \in Z(R)^*\backslash\{x\}$ such that $xy = 0$. If $y \in \mathfrak{p}_i$ for some $i \in \{1, 2\}$, then $x + y \in \mathfrak{p}_i \subset Z(R)$. Hence, $x$ and $y$ are not adjacent in $(ZT(R))^c$. If $y \notin \bigcup_{i=1}^{2} \mathfrak{p}_i$, then as $a_1 y \notin \mathfrak{p}_2$ and $x \in \bigcap_{i=1}^{2} \mathfrak{p}_i$, it follows that $x \neq a_1 y$. From $x\,(a_1 y) = 0$ and $x + a_1 y \in \mathfrak{p}_1 \subset Z(R)$, we obtain that $x$ and $a_1 y$ are not adjacent in $(ZT(R))^c$. □

**Proposition 2.10.** *If $R$ is a ring such that $|MNP(R)| \geq 3$, then $(ZT(R))^c$ and $(ZT(R_1))^c$ are connected with $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 2$, $r((ZT(R))^c) \leq 2$, and $(ZT(R_1))^c$ has radius at most 2.*

*Proof.* Let $a, b \in Z(R)^*$ be distinct. Observe that there exist $\mathfrak{p}, \mathfrak{p}' \in MNP(R)$ such that $((0) :_R a) \subseteq \mathfrak{p}$ and $((0) :_R b) \subseteq \mathfrak{p}'$. As $|MNP(R)| \geq 3$ by hypothesis, there exists $\mathfrak{p}'' \in MNP(R)$ such that $\mathfrak{p}'' \not\subseteq \mathfrak{p} \cup \mathfrak{p}'$. Hence, $\mathfrak{p}'' \not\subseteq ((0) :_R a) \cup ((0) :_R b)$. It follows from Lemma 2.1 that there exists a path of length at most two between $a$ and $b$ in $(ZT(R))^c$. This shows that $(ZT(R))^c$ is connected and $diam((ZT(R))^c) \leq 2$. If $\mathfrak{p}_1, \mathfrak{p}_2$ are distinct members of $MNP(R)$, then it is clear that $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$. Hence, we obtain from Lemma 2.9 that $diam((ZT(R))^c) \geq 2$ and so, $diam((ZT(R))^c) = 2$. Hence, $r((ZT(R))^c) \leq 2$.

Note that $Z(R_1) \cap R = Z(R)$. From $|MNP(R)| \geq 3$, it follows that $|MNP(R_1)| \geq 3$. Hence, $(ZT(R_1))^c$ is connected with $diam((ZT(R_1))^c) = 2$ and $r((ZT(R_1))^c) \leq 2$. □

## 3 Some results on the connectedness of $(ZT(R))^c$ and $(ZT(R_1))^c$ in the case $|MNP(R)| = 1$

Let $R$ be a ring such that $|MNP(R)| = 1$. With $R_1 = R[X]$ or $R[[X]]$, the aim of this section is to study whether $(ZT(R))^c$ is connected implies that $(ZT(R_1))^c$ is connected and vice versa and to determine the relationship between their diameters (respectively, radii) in the case when both the graphs are connected. We begin with the following theorem.

**Theorem 3.1.** *For a ring $R$ with $|Z(R)^*| \geq 2$ and $MNP(R) = \{\mathfrak{p}\}$, the following statements are equivalent:*
(1) *$(ZT(R))^c$ is connected;*
(2) *$(ZT(R[X]))^c$ is connected.*
*Moreover, if* (1) *holds, then $diam((ZT(R))^c) = r((ZT(R))^c) = diam((ZT(R[X]))^c) = 2$ and $r((ZT(R[X]))^c) \leq 2$.*

*Proof.* (1) $\Rightarrow$ (2). Assume that $(ZT(R))^c$ is connected. Then by Proposition 2.7(2), $\mathfrak{p} \notin \mathbb{A}(R)$ and by Proposition 2.7(3), we obtain that $(ZT(R[X]))^c$ is connected with $diam((ZT(R))^c) = r((ZT(R))^c) = diam((ZT(R[X]))^c) = 2$ and $r((ZT(R[X]))^c) \leq 2$.
(2) $\Rightarrow$ (1). Assume that $(ZT(R[X]))^c$ is connected. Observe that $Z(R[X]) \subseteq Z(R)[X]$ by McCoy's Theorem [23, Theorem 2]. As $Z(R) = \mathfrak{p}$, it follows that $Z(R[X]) \subseteq \mathfrak{p}[X]$. We consider the following cases.
**Case**$(i)$. $Z(R[X]) = \mathfrak{p}[X]$.
In this case, $MNP(R[X]) = \{\mathfrak{p}[X]\}$ and hence, $\mathfrak{p}[X] \notin \mathbb{A}(R[X])$ by Proposition 2.7(2) and so, $\mathfrak{p} \notin \mathbb{A}(R)$.
**Case**$(ii)$. $Z(R[X]) \subset \mathfrak{p}[X]$.
In this case, there exists $f(X) = \sum_{i=0}^{n} p_i X^i \in \mathfrak{p}[X] \backslash Z(R[X])$. Hence, there exists no $r \in R \backslash \{0\}$ such that $(\sum_{i=0}^{n} Rp_i)r = (0)$. Therefore, $\mathfrak{p} \notin \mathbb{A}(R)$.
Thus $\mathfrak{p} \notin \mathbb{A}(R)$ and so, $(ZT(R))^c$ is connected by Proposition 2.7(2).
Assume that (1) holds. The assertion stated in the moreover part is already noted in the proof of (1) $\Rightarrow$ (2) of this theorem. □

The following lemma and Corollary 3.3 are needed for illustrating that (2) $\Rightarrow$ (1) of Theorem 3.1 can fail to hold for power series ring (see Example 3.5).

**Lemma 3.2.** *For a ring $A$ with $nil(A) \in Spec(A)$, $(ZT(A))^c = (\Gamma(A))^c$ and $(ZT(A[X]))^c = (\Gamma(A[X]))^c$.*

*Proof.* If $x, y \in Z(A)^*$ are such that $x + y \notin Z(A)$, then $x, y \notin nil(A)$ by [20, Lemma 2.3]. As $nil(A) \in Spec(A)$ by hypothesis, $xy \notin nil(A)$ and so, $xy \neq 0$. Hence, $(ZT(A))^c = (\Gamma(A))^c$ by Remark 2.4.
As $nil(A) \in Spec(A)$, $nil(A)[X] \in Spec(A[X])$ by [7, Exercise 7$(ii)$, page 55]. Hence, $nil(A[X]) \in Spec(A[X])$, since $nil(A[X]) = nil(A)[X]$ by [7, Exercise 2$(ii)$, page 11]. It now follows as in the previous paragraph that $(ZT(A[X]))^c = (\Gamma(A[X]))^c$. □

**Corollary 3.3.** *If $A$ is a ring with $nil(A) \in Spec(A)$ and is nilpotent, then $(ZT(A[[X]]))^c = (\Gamma(A[[X]]))^c$.*

*Proof.* Note that $nil(A[[X]]) \subseteq nil(A)[[X]]$ by [7, Exercise 5$(ii)$, page 11]. By hypothesis, there exists $n \in \mathbb{N}$ such that $(nil(A))^n = (0)$. If $f(X) \in nil(A)[[X]]$, then it is clear that $f(X)^n = 0$ and so, $f(X) \in nil(A[[X]])$. Therefore, $nil(A)[[X]] \subseteq nil(A[[X]])$ and so, $nil(A[[X]]) = nil(A)[[X]]$. Since $nil(A) \in Spec(A)$ by hypothesis, $nil(A)[[X]] \in Spec(A[[X]])$ by [22, see page 5]. Therefore, $nil(A[[X]]) \in Spec(A[[X]])$ and hence, $(ZT(A[[X]]))^c = (\Gamma(A[[X]]))^c$ by Lemma 3.2. □

The following example illustrates that the converse of the assertion mentioned in Remark 2.5 can fail to hold.

**Example 3.4.** Consider distinct prime numbers $p, q$ and the $\mathbb{Z}$-module $M = \frac{\mathbb{Z}}{pq\mathbb{Z}}$. If $A = \mathbb{Z}(+)M$ is the ring obtained by using Nagata's principle of idealization, then $(ZT(A))^c = (\Gamma(A))^c$ but $|MNP(A)| > 1$.

*Proof.* Note that $((0)(+)M)^2 = (0)(+)(0+pq\mathbb{Z})$ and $nil(A) = (0)(+)M$ is nilpotent. Observe that $\frac{A}{nil(A)} \cong \mathbb{Z}$ as rings and so, $nil(A) \in Spec(A)$. Hence, $(ZT(A))^c = (\Gamma(A))^c$ by Lemma 3.2. It is not hard to verify that $(p, 0+pq\mathbb{Z}), (q, 0+pq\mathbb{Z}) \in Z(A)$ but $(p+q, 0+pq\mathbb{Z}) \notin Z(A)$ and so, $Z(A)$ is not an ideal of $A$. Therefore, $|MNP(A)| > 1$. □

**Example 3.5.** If $\mathfrak{p}$ denotes the height one prime ideal of a rank 2 discrete valuation domain $(V, \mathfrak{m})$, then for any $p \in \mathfrak{p}\backslash\{0\}$, the ring $R = \frac{V}{Vp}$ is such that $|MNP(R)| = 1, (ZT(R[[X]]))^c$ is connected with $diam((ZT(R[[X]]))^c) = r((ZT(R[[X]]))^c) = 2$ but $(ZT(R))^c$ is not connected.

*Proof.* Since $(V, \mathfrak{m})$ is a rank 2 discrete valuation domain, it follows that $\mathfrak{m} \neq \mathfrak{m}^2$ and for any $m \in \mathfrak{m}\backslash\mathfrak{m}^2$, $\mathfrak{m} = Vm$. By assumption, $p \in \mathfrak{p}\backslash\{0\}$, where $\mathfrak{p}$ is the height one prime ideal of $V$. It is convenient to denote $Vp$ by $I$. Observe that $R = \frac{V}{I}$ is quasi-local with $\frac{\mathfrak{m}}{I}$ as its unique maximal ideal and $\frac{\mathfrak{m}}{I} = R(m+I)$. Since the ideals of $V$ are comparable under inclusion, the ideals of $R$ are comparable under inclusion and so, $|MNP(R)| = 1$. We prove that $MNP(R) = \{\frac{\mathfrak{m}}{I}\}$. As $m \notin I = Vp$, there exists $v \in \mathfrak{m}$ such that $p = mv$. Since $V$ is an integral domain and $m \in NU(V)$, it follows that $v \notin I$. Thus $(m+I)(v+I) = p+I = 0+I$. Therefore, $\frac{\mathfrak{m}}{I} \subseteq Z(R) \subseteq \frac{\mathfrak{m}}{I}$ and so, $Z(R) = \frac{\mathfrak{m}}{I}$. Hence, $MNP(R) = \{\frac{\mathfrak{m}}{I}\}$. Therefore, $(ZT(R))^c = (\Gamma(R))^c$ by Proposition 2.7(1). Note that for each $i \in \mathbb{N}, m^i + I \in Z(R)\backslash\{0+I\}$ and $1 - m^i + I \in U(R)$. Hence, $m^i + I \neq m^j + I$ for all distinct $i, j \in \mathbb{N}$. Therefore, $Z(R)$ is infinite. As $\frac{\mathfrak{m}}{I} \in \mathbb{A}(R)$, we obtain from Proposition 2.7(2) that $(ZT(R))^c$ is not connected.

Since $\mathfrak{p}$ is the height one prime ideal of the rank 2 discrete valuation domain $V$, it follows that $\mathfrak{p} \neq \mathfrak{p}^2$. As $\bigcap_{n=1}^{\infty} \mathfrak{p}^n \in \text{Spec}(V)$ by [13, Theorem 17.1(3)], we get that $\bigcap_{n=1}^{\infty} \mathfrak{p}^n = (0)$. Hence, $p \notin \mathfrak{p}^k$ for some $k \geq 2$ and so, $\mathfrak{p}^k \subset Vp = I$. Therefore, $(\frac{\mathfrak{p}}{I})^k = (0+I)$. This proves that $\frac{\mathfrak{p}}{I} \subseteq nil(R)$. Observe that $nil(R) \subseteq \frac{\mathfrak{p}}{I}$, since $\frac{\mathfrak{p}}{I} \in Spec(R)$ and so, $nil(R) = \frac{\mathfrak{p}}{I}$. Thus $nil(R) \in Spec(R)$ and is nilpotent. Hence, $(ZT(R[[X]]))^c = (\Gamma(R[[X]]))^c$ by Corollary 3.3. As $nil(R)$ is a divided prime ideal of $R$ and for any $i \in \mathbb{N}, m^i + I \in Z(R)\backslash nil(R)$, $f_i(X) = (m^i + I) + X \in Z(R[[X]])$ by [20, Theorem 5.7]. Hence, there exists $\mathfrak{P}_i \in MNP(R[[X]])$ such that $f_i(X) \in \mathfrak{P}_i$. We claim that $\mathfrak{P}_i \neq \mathfrak{P}_j$ for any distinct $i, j \in \mathbb{N}$. We can assume without loss of generality that $i < j$. If $\mathfrak{P}_i = \mathfrak{P}_j$, then $f_i(X) - f_j(X) = (m^i + I)(1 - m^{j-i} + I) \in \mathfrak{P}_i$. This implies that $m^i + I \in \mathfrak{P}_i$, since $1 - m^{j-i} + I \in U(R)$. Hence, $X = f_i(X) - (m^i + I) \in \mathfrak{P}_i$. This is impossible, since $X \notin Z(R[[X]])$ but any element of $\mathfrak{P}_i$ belongs to $Z(R[[X]])$. Therefore, $\mathfrak{P}_i \neq \mathfrak{P}_j$. This shows that $MNP(R[[X]])$ is infinite. Hence, $(ZT(R[[X]]))^c$ is connected with $diam((ZT(R[[X]]))^c) = 2$ by Proposition 2.10. As $ZT(R[[X]]) = \Gamma(R[[X]])$ is connected by [6, Theorem 2.3], Lemma 2.6 implies that $r((ZT(R[[X]]))^c) \geq 2$ and so, $r((ZT(R[[X]]))^c) = 2$. □

In the following proposition, we provide a sufficient condition under which (2) ⇒ (1) of Theorem 3.1 holds for power series ring.

**Proposition 3.6.** *Let $R$ be a ring such that $Z(R) = nil(R)$. If $(ZT(R[[X]]))^c$ is connected, then $(ZT(R))^c$ is connected and $diam((ZT(R))^c) = r((ZT(R))^c) = diam((ZT(R_1))^c) = r((ZT(R_1))^c) = 2$.*

*Proof.* Assume that $Z(R) = nil(R)$ and $(ZT(R[[X]]))^c$ is connected. Thus $Z(R)$ is an ideal of $R$ and it is necessarily a prime ideal of $R$. It is convenient to denote $Z(R)$ by $\mathfrak{p}$. Hence, $MNP(R) = \{\mathfrak{p}\}$. By [12, Theorem 3], it follows that $Z(R[[X]]) \subseteq Z(R)[[X]] = \mathfrak{p}[[X]]$. We consider the following cases.

**Case**$(i)$. $Z(R[[X]]) = \mathfrak{p}[[X]]$.

It follows as in Case $(i)$ of the proof of Theorem 3.1 that $\mathfrak{p}[[X]] \notin \mathbb{A}(R[[X]])$ and so, $\mathfrak{p} \notin \mathbb{A}(R)$.

**Case**$(ii)$. $Z(R[[X]]) \subset \mathfrak{p}[[X]]$.

Hence, there exists $f(X) = \sum_{i=0}^{\infty} p_i X^i \in \mathfrak{p}[[X]]\backslash Z(R[[X]])$. Therefore, $\sum_{i=0}^{\infty} Rp_i \notin \mathbb{A}(R)$ and so, $\mathfrak{p} \notin \mathbb{A}(R)$.

Thus $\mathfrak{p} \notin \mathbb{A}(R)$. Hence, by Proposition 2.7, $(ZT(R))^c = (\Gamma(R))^c$ is connected with $diam((ZT(R))^c) = r((ZT(R))^c) = 2$. As $nil(R) \notin \mathbb{A}(R)$, it follows that $diam((ZT(R_1))^c) = r((ZT(R_1))^c) = 2$ by Proposition 2.8. □

The following example provides a quasi-local non-reduced ring $R$ with $Z(R) = nil(R) \notin \mathbb{A}(R)$.

**Example 3.7.** If $(V, \mathfrak{m})$ is a rank one non-discrete valuation domain, then for any $m \in \mathfrak{m}\backslash\{0\}$, $R = \frac{V}{Vm}$ is a quasi-local non-reduced ring with $Z(R) = nil(R) \notin \mathbb{A}(R)$.

*Proof.* Since $V$ is quasi-local with $\mathfrak{m}$ as its unique maximal ideal, it follows that $R$ is quasi-local with $\frac{\mathfrak{m}}{Vm}$ as its unique maximal ideal. As $V$ is a rank one non-discrete valuation domain, $\mathfrak{m}$ is not finitely generated and so, $\mathfrak{m} = \mathfrak{m}^2$. From $Spec(V) = \{(0), \mathfrak{m}\}$, it follows that $Spec(R) = \{\frac{\mathfrak{m}}{Vm}\}$. Hence, $nil(R) = \frac{\mathfrak{m}}{Vm}$ by [7, Proposition 1.8]. Since the ideals of $R$ are comparable under inclusion, $Z(R)$ is an ideal of $R$ and so, $Z(R) \in Spec(R)$. Therefore, $Z(R) = nil(R) = \frac{\mathfrak{m}}{Vm}$. If $\frac{\mathfrak{m}}{Vm} \in \mathbb{A}(R)$, then there exists $x \in V\backslash Vm$ such that $x\mathfrak{m} \subseteq Vm$. As $x \notin Vm$, there exists $m' \in \mathfrak{m}$ such that $m = xm'$. Hence, $x\mathfrak{m} \subseteq Vxm'$. This implies that $\mathfrak{m} \subseteq Vm' \subseteq \mathfrak{m}$ and so, $\mathfrak{m} = Vm'$. This is impossible and so, $nil(R) \notin \mathbb{A}(R)$. □

In the following example, we provide a quasi-local reduced ring $R$ due to Gilmer and Heinzer (see [15, Example, page 16]) to illustrate that $r((ZT(R[[X]]))^c) < r((ZT(R))^c)$ can happen.

**Example 3.8.** Let $\{X_i\}_{i=1}^{\infty}$ be a set of indeterminates and let $D = \bigcup_{n=1}^{\infty} K[[X_1, \ldots, X_n]]$, where for each $n \in \mathbb{N}$, $K[[X_1, \ldots, X_n]]$ is the power series ring in $X_1, \ldots, X_n$ over a field $K$. If $I$ is the ideal of $D$ generated by $\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\}$, then $R = \frac{D}{I}$ is a quasi-local reduced ring with $|MNP(R)| = 1$ and has the following properties:
(1) $(ZT(R))^c = (\Gamma(R))^c$ is connected with $diam((ZT(R))^c) = r((ZT(R))^c) = 2$.
(2) $(ZT(R[X]))^c = (\Gamma(R[X]))^c$ is connected with $diam((ZT(R[X]))^c) = r((ZT(R[X]))^c) = 2$.
(3) $(ZT(R[[X]]))^c$ and $(\Gamma(R[[X]]))^c$ are both connected and both have diameter equal to 2, $r((\Gamma(R[[X]]))^c) = 2$ but $r((ZT(R[[X]]))^c) = 1$.

*Proof.* For each $i \in \mathbb{N}$, it is convenient to denote $X_i + I$ by $x_i$. It was already noted in [15, Example, page 16] that $R$ is a quasi-local reduced ring with $\mathfrak{m} = \sum_{n=1}^{\infty} Rx_n$ as its unique maximal ideal and $Min(R) = \{\mathfrak{p}_i\}_{i=1}^{\infty}$, where for each $i \in \mathbb{N}$, $\mathfrak{p}_i$ is the ideal of $R$ generated by $\{x_j \mid j \in \mathbb{N}\backslash\{i\}\}$. Since $R$ is reduced, it follows from [7, Proposition 1.8] and [19, Theorem 10] that $\bigcap_{i=1}^{\infty} \mathfrak{p}_i = (0 + I)$. It is clear that $x_i \neq 0 + I$ for each $i \in \mathbb{N}$ but for all distinct $i, j \in \mathbb{N}$, $x_ix_j = 0 + I$. If $m \in \mathfrak{m}$, then $m \in \sum_{i=1}^{n} Rx_i$ for some $n \in \mathbb{N}$. Hence, $mx_{n+1} = 0 + I$. This shows that $\mathfrak{m} \subseteq Z(R) \subseteq \mathfrak{m}$ and so, $Z(R) = \mathfrak{m}$. Therefore, $|MNP(R)| = 1$. Since $R$ is reduced, it follows that $\mathfrak{m} \notin \mathbb{A}(R)$.
(1) As $MNP(R) = \{\mathfrak{m}\}$ and $\mathfrak{m} \notin \mathbb{A}(R)$, it follows from Proposition 2.7 that $(ZT(R))^c = (\Gamma(R))^c$ is connected with $diam((ZT(R))^c) = r((ZT(R))^c) = 2$.
(2) We claim that $Z(R[X]) = \mathfrak{m}[X]$. If $f(X) \in Z(R[X])$, then $f(X) \in Z(R)[X]$ by [23, Theorem 2]. Hence, $Z(R[X]) \subseteq Z(R)[X] = \mathfrak{m}[X]$. If $f(X) \in \mathfrak{m}[X]$, then $A_f$ is a finitely generated ideal of $R$ with $A_f \subset \mathfrak{m}$. Hence, there exists $n \in \mathbb{N}$ such that $A_f \subseteq \sum_{j=1}^{n} Rx_j$. Hence, $A_fx_{n+1} \subseteq (\sum_{j=1}^{n} Rx_j)x_{n+1} = (0 + I)$ and so, $A_fx_{n+1} = (0 + I)$. As $x_{n+1} \neq 0 + I$, it follows that $f(X) \in Z(R[X])$. This shows that $\mathfrak{m}[X] \subseteq Z(R[X])$ and therefore, $Z(R[X]) = \mathfrak{m}[X]$. Thus $MNP(R[X]) = \{\mathfrak{m}[X]\}$ and as $R[X]$ is reduced, $\mathfrak{m}[X] \notin \mathbb{A}(R[X])$. It now follows as in the proof of (1) that $(ZT(R[X]))^c = (\Gamma(R[X]))^c$ is connected with $diam((ZT(R[X]))^c) = r((ZT(R[X]))^c) = 2$.
(3) As $\mathfrak{m} \notin \mathbb{A}(R)$, the proof of Proposition 2.3 shows that both $(\Gamma(R[[X]]))^c$ and $(ZT(R[[X]]))^c$ are connected with $diam((\Gamma(R[[X]]))^c) \leq 2$. By Proposition 2.7(3), $diam((ZT(R[[X]]))^c) = 2$ and so, $diam((\Gamma(R[[X]]))^c) = 2$. Since $\Gamma(R[[X]])$ is connected by [6, Theorem 2.3], we obtain from Lemma 2.6 that $r((\Gamma(R[[X]]))^c) \geq 2$ and so, $r((\Gamma(R[[X]]))^c) = 2$.

We next verify that $r((ZT(R[[X]]))^c) = 1$. We claim that $e(f(X)) = 1$ in $(ZT(R[[X]]))^c$, where $f(X) = \sum_{j=1}^{\infty} x_{j+1}X^j$. Consider any $g(X) = \sum_{k=0}^{\infty} r_kX^k \in Z(R[[X]])^*\backslash\{f(X)\}$. If $f(X)$ and $g(X)$ are not adjacent in $(ZT(R[[X]]))^c$, then $f(X)g(X) = 0 + I$ and $f(X) + g(X) \in Z(R[[X]])$. For any $j \in \mathbb{N}$, $x_{j+1} \notin \mathfrak{p}_{j+1}$ and so, $f(X) \notin \mathfrak{p}_{j+1}[[X]]$. Since $\mathfrak{p}_{j+1} \in Spec(R)$, $\mathfrak{p}_{j+1}[[X]] \in Spec(R[[X]])$ by [22, see page 5]. Hence, $f(X)g(X) = 0 + I \in \mathfrak{p}_{j+1}[[X]]$ implies that $g(X) \in \mathfrak{p}_{j+1}[[X]]$. Therefore, $g(X) \in \bigcap_{j=1}^{\infty} \mathfrak{p}_{j+1}[[X]] = (\bigcap_{j=1}^{\infty} \mathfrak{p}_{j+1})[[X]]$. Hence, $r_k \in \bigcap_{j=1}^{\infty} \mathfrak{p}_{j+1}$ for each $k \in \mathbb{N} \cup \{0\}$. As $f(X) + g(X) \in Z(R[[X]])$ by assumption, there exists $r \in R\backslash\{0+I\}$ such that $(f(X)+g(X))r = 0+I$ by [14, Proposition 3.5]. Hence, $r_0r = 0+I$ and for each $j \geq 1$, $(x_{j+1} + r_j)r = 0 + I$. This implies that $x_{j+1}r = -r_jr \in \mathfrak{p}_{j+1}$. From $x_{j+1} \notin \mathfrak{p}_{j+1}$, we obtain that $r \in \mathfrak{p}_{j+1}$. Thus $r \in \bigcap_{j=1}^{\infty} \mathfrak{p}_{j+1}$. Since $\bigcap_{j=1}^{\infty} \mathfrak{p}_j = (0 + I)$ and $r \neq 0 + I$, it follows that $r \notin \mathfrak{p}_1$. From $r_0r = 0 + I$ and $r \notin \mathfrak{p}_1$, it follows that $r_0 \in \mathfrak{p}_1$ and so, $r_0 = 0 + I$.

Since $g(X) \neq 0 + I$, it is possible to find $k \in \mathbb{N}$ least with the property that $r_k \neq 0 + I$. Thus $r_k \notin \mathfrak{p}_1$. As $(f(X) + g(X))r = 0 + I$, we obtain that $(x_{k+1} + r_k)r = 0 + I$. Hence, $r_k r = -x_{k+1} r \in \mathfrak{p}_1$. This is impossible, since $r, r_k \notin \mathfrak{p}_1$. Therefore, either $f(X)g(X) \neq 0 + I$ or $f(X) + g(X) \notin Z(R[[X]])$. Hence, $f(X)$ and $g(X)$ are adjacent in $(ZT(R[[X]]))^c$. This proves that $e(f(X)) = 1$ in $(ZT(R[[X]]))^c$ and so, $r((ZT(R[[X]]))^c) = 1$. □

## 4 Some results on the connectedness of $(ZT(R))^c$ and $(ZT(R_1))^c$ in the case $|MNP(R)| \geq 2$

Throughout this section, unless otherwise specified, we consider rings $R$ with $|MNP(R)| \geq 2$. Let $R_1 = R[X]$ or $R[[X]]$. The aim of this section is to study whether $(ZT(R))^c$ is connected implies that $(ZT(R_1))^c$ is connected and vice versa and to determine the relationship between their diameters (respectively, radii) in the case when both the graphs are connected. In view of Proposition 2.10 (where such a study is done in the case $|MNP(R)| \geq 3$), we assume that $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. If $\mathfrak{p}_i \notin \mathbb{A}(R)$ for some $i \in \{1, 2\}$, then for such a study, one can refer Proposition 2.3 and Lemma 2.9. Hence, in the discussion to follow, we assume that $\mathfrak{p}_i \in \mathbb{A}(R)$ for each $i \in \{1, 2\}$. We begin with the following lemma.

**Lemma 4.1.** *For a ring $R$ with $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$, if $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$, then $(ZT(R))^c$ is complete.*

*Proof.* Note that $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$ and $Z(R)^* = \bigcup_{i=1}^{2} V_i$, where $V_1 = \mathfrak{p}_1 \backslash \mathfrak{p}_2$ and $V_2 = \mathfrak{p}_2 \backslash \mathfrak{p}_1$. Observe that $V_i \neq \emptyset$ for each $i \in \{1, 2\}$ and $\bigcap_{i=1}^{2} V_i = \emptyset$. Let $a, b \in Z(R)^*$ be distinct. If $a, b \in V_1$, then $ab \notin \mathfrak{p}_2$ and so, $ab \neq 0$. Similarly, if $a, b \in V_2$, then $ab \notin \mathfrak{p}_1$ and so, $ab \neq 0$. If $a$ and $b$ do not belong to the same $V_i$, then without loss of generality, we can assume that $a \in V_1$ and $b \in V_2$. Then $a + b \notin \bigcup_{i=1}^{2} \mathfrak{p}_i = Z(R)$. This shows that $a$ and $b$ are adjacent in $(ZT(R))^c$ for all distinct $a, b \in Z(R)^*$ and hence, $(ZT(R))^c$ is complete. □

We use the following lemma in the proof of Theorem 4.3.

**Lemma 4.2.** *If a graph $G = (V, E)$ is complete, then for any non-empty subset $W$ of $V$, the subgraph $H$ of $G$ induced by $W$ is also complete.*

*Proof.* Let $a, b \in W$ be distinct. As $G$ is complete, $a$ and $b$ are adjacent in $G$ and so, $a$ and $b$ are adjacent in $H$. Therefore, $H$ is complete. □

**Theorem 4.3.** *For a ring $R$, the following statements are equivalent:*
(1) *$(ZT(R))^c$ is complete;*
(2) *$|MNP(R)| = 2$ and $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$, where $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$;*
(3) *$(ZT(R_1))^c$ is complete.*

*Proof.* (1) ⇒ (2). Assume that $(ZT(R))^c$ is complete. It follows from Propositions 2.7 and 2.10 that $|MNP(R)| = 2$. If $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$, then $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$ by Lemma 2.9.
(2) ⇒ (3). Assume that $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$, where $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. For $i \in \{1, 2\}$, it follows from [7, Exercise 7($ii$), page 55] (respectively, [22, see page 5]) that $\mathfrak{p}_i[X] \in Spec(R[X])$ (respectively, $\mathfrak{p}_i[[X]] \in Spec(R[[X]])$). Note that $\bigcap_{i=1}^{2} \mathfrak{p}_i[X] = (0)$ (respectively, $\bigcap_{i=1}^{2} \mathfrak{p}_i[[X]] = (0)$). Hence, it follows that $Z(R[X]) = \bigcup_{i=1}^{2} \mathfrak{p}_i[X]$ (respectively, $Z(R[[X]]) = \bigcup_{i=1}^{2} \mathfrak{p}_i[[X]]$). As $Z(R_1) \cap R = Z(R)$ and $Z(R)$ is not an ideal of $R$, we get that $Z(R_1)$ is not an ideal of $R_1$ and so, $|MNP(R_1)| \geq 2$. It follows from the above given arguments that $MNP(R[X]) = \{\mathfrak{p}_i[X] \mid i \in \{1, 2\}\}$ and $MNP(R[[X]]) = \{\mathfrak{p}_i[[X]] \mid i \in \{1, 2\}\}$. Hence, $(ZT(R_1))^c$ is complete by Lemma 4.1.
(3) ⇒ (1). Assume that $(ZT(R_1))^c$ is complete. As $(ZT(R))^c$ is the subgraph of $(ZT(R_1))^c$ induced by $Z(R)^*$, $(ZT(R))^c$ is complete by Lemma 4.2. □

The following remark and Lemma 4.5 are needed in the proof of Proposition 4.6.

**Remark 4.4.** For a ring $A$, if $\mathfrak{p} \in MNP(A)$ satisfies $\mathfrak{p} \subseteq ((0) :_A a)$ for some $a \in A \backslash \{0\}$, then $\mathfrak{p} = ((0) :_A a)$.

**Lemma 4.5.** *If $R$ is a ring with $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1,2\}\}$, $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$, $\mathfrak{p}_1 = ((0) :_R a)$, and $\mathfrak{p}_2 = ((0) :_R b)$ for some $a, b \in R\backslash\{0\}$, then $a \neq b, ab = 0$, $a + b \in Z(R)$, and either $a^2 = 0$ or $b^2 = 0$.*

*Proof.* As $\mathfrak{p}_1 \neq \mathfrak{p}_2$, it follows that $a \neq b$ and $ab = 0$ by [10, Lemma 3.6]. This implies that $a \in \mathfrak{p}_2$ and $b \in \mathfrak{p}_1$. If $x \in (\bigcap_{i=1}^{2} \mathfrak{p}_i)\backslash\{0\}$, then $(a + b)x = 0$ and so, $a + b \in Z(R)$. From $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, we get that $a + b \in \mathfrak{p}_i$ for some $i \in \{1, 2\}$. If $a + b \in \mathfrak{p}_1$, then $a \in \mathfrak{p}_1$ and so, $a^2 = 0$. If $a + b \in \mathfrak{p}_2$, then $b \in \mathfrak{p}_2$ and hence, $b^2 = 0$. □

**Proposition 4.6.** *If $R$ is a ring with $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1,2\}\}$, $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$, and $\mathfrak{p}_i \in \mathbb{A}(R)$ for each $i \in \{1, 2\}$, then $(ZT(R))^c$ is connected and $diam((ZT(R))^c) \in \{2, 3\}$.*

*Proof.* Let $a, b \in Z(R)^*$ be distinct. If $\mathfrak{p}_i \not\subseteq ((0) :_R a) \cup ((0) :_R b)$ for some $i \in \{1, 2\}$, then it follows from the proof of Lemma 2.1 that there exists a path of length at most two between $a$ and $b$ in $(\Gamma(R))^c$ and hence, in $(ZT(R))^c$. If $\mathfrak{p}_i \subseteq ((0) :_R a) \cup ((0) :_R b)$ for each $i \in \{1, 2\}$, then by Remark 4.4, one between $\mathfrak{p}_1$ and $\mathfrak{p}_2$ equals $((0) :_R a)$ and the other equals $((0) :_R b)$. Without loss of generality, we can assume that $\mathfrak{p}_1 = ((0) :_R a)$ and $\mathfrak{p}_2 = ((0) :_R b)$. Then $a \neq b$, $ab = 0$, $a + b \in Z(R)$, and either $a^2 = 0$ or $b^2 = 0$ by Lemma 4.5. Hence, $a$ and $b$ are not adjacent in $(ZT(R))^c$. It is convenient to proceed the proof with the following cases.
**Case** (1). Either $a^2 = 0$ or $b^2 = 0$ but not both.

If $a^2 = 0$ but $b^2 \neq 0$, then $a \in \bigcap_{i=1}^{2} \mathfrak{p}_i$ and $b \in \mathfrak{p}_1\backslash\mathfrak{p}_2$. If $d \in \mathfrak{p}_2\backslash\mathfrak{p}_1$, then $ad \neq 0$ and $d + b \notin Z(R)$. Hence, $a - d - b$ is a path of length two between $a$ and $b$ in $(ZT(R))^c$. If $a^2 \neq 0$ but $b^2 = 0$, then for any $c \in \mathfrak{p}_1\backslash\mathfrak{p}_2$, $a - c - b$ is a path of length two between $a$ and $b$ in $(ZT(R))^c$.
**Case** (2). $a^2 = b^2 = 0$.

Note that $a, b \in \bigcap_{i=1}^{2} \mathfrak{p}_i$. If $c \in \mathfrak{p}_1\backslash\mathfrak{p}_2$ and $d \in \mathfrak{p}_2\backslash\mathfrak{p}_1$, then $c + d \notin Z(R)$. As $ad \neq 0$ and $cb \neq 0$, $a - d - c - b$ is a path of length three between $a$ and $b$ in $(ZT(R))^c$. In this case, we verify that there exists no path of length two between $a$ and $b$ in $(ZT(R))^c$. Let $y \in Z(R)^*$ be such that $a$ and $y$ are adjacent in $(ZT(R))^c$. Note that $ay \neq 0$, since $a + y \in Z(R)$. Hence, $y \in \mathfrak{p}_2\backslash\mathfrak{p}_1$. Therefore, $yb = 0$ and as $y + b \in Z(R)$, we get that $y$ and $b$ are not adjacent in $(ZT(R))^c$. This shows that there exists no path of length two between $a$ and $b$ in $(ZT(R))^c$.

This proves that $(ZT(R))^c$ is connected and $diam((ZT(R))^c) \leq 3$. As $diam((ZT(R))^c) \geq 2$ by Lemma 2.9, $diam((ZT(R))^c) \in \{2, 3\}$. □

We deduce Corollaries 4.7 and 4.8 from the proof of the previous proposition.

**Corollary 4.7.** *If $R$ is a ring with $MNP(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0)$, $\mathfrak{p}_1 = ((0) :_R a)$ and $\mathfrak{p}_2 = ((0) :_R b)$ for some $a, b \in R\backslash\{0\}$, then the following statements are equivalent:*
(1) $diam((ZT(R))^c) = 2$*;*
(2) *Either* $a^2 = 0$ *or* $b^2 = 0$ *but not both;*
(3) $r((ZT(R))^c) = 1$.

*Proof.* By Lemma 4.5, $ab = 0$, $a + b \in Z(R)$, and either $a^2 = 0$ or $b^2 = 0$. Observe that $a$ and $b$ are not adjacent in $(ZT(R))^c$.
$(1) \Rightarrow (2)$. Assume that $diam((ZT(R))^c) = 2$. If $a^2 = 0 = b^2$, then by the proof of case (2) of Proposition 4.6, there exists no path of length two between $a$ and $b$ in $(ZT(R))^c$. Therefore, either $a^2 = 0$ or $b^2 = 0$ but not both.
$(2) \Rightarrow (3)$. Assume that either $a^2 = 0$ or $b^2 = 0$ but not both. Without loss of generality, we can assume that $a^2 = 0$ but $b^2 \neq 0$. Then by the proof of case (1) of Proposition 4.6, for any $d \in \mathfrak{p}_2\backslash\mathfrak{p}_1$, $ad \neq 0$ and $d + b \notin Z(R)$. Let $y \in Z(R)^*\backslash\{d\}$. If $dy \neq 0$, then $d$ and $y$ are adjacent in $(ZT(R))^c$. If $dy = 0$, then $by \neq 0$ and so, $y \in \mathfrak{p}_1\backslash\mathfrak{p}_2$ and so, $d + y \notin Z(R)$. Hence, $d$ and $y$ are adjacent in $(ZT(R))^c$. This shows that $e(d) = 1$ in $(ZT(R))^c$ and so, $r((ZT(R))^c) = 1$.
$(3) \Rightarrow (1)$. Assume that $r((ZT(R))^c) = 1$. Then $diam((ZT(R))^c) \leq 2$. By Proposition 4.6, $diam((ZT(R))^c) \geq 2$ and so, $diam((ZT(R))^c) = 2$. □

**Corollary 4.8.** *If $R, \mathfrak{p}_1, \mathfrak{p}_2$ and $a, b$ are as in the statement of Corollary 4.7, then the following statements are equivalent:*
(1) $diam((ZT(R))^c) = 3$*;*
(2) $a^2 = b^2 = 0$*;*
(3) $r((ZT(R))^c) = 2$.

*Proof.* By Lemma 4.5, $ab = 0$, $a + b \in Z(R)$, and either $a^2 = 0$ or $b^2 = 0$. Observe that $a$ and $b$ are not adjacent in $(ZT(R))^c$. By Proposition 4.6, $diam((ZT(R))^c) \in \{2, 3\}$.
$(1) \Rightarrow (2)$. Assume that $diam((ZT(R))^c) = 3$. Then $a^2 = b^2 = 0$ by $(2) \Rightarrow (1)$ of Corollary 4.7.
$(2) \Rightarrow (3)$. Assume that $a^2 = b^2 = 0$. Then $r((ZT(R))^c) \geq 2$ by $(3) \Rightarrow (2)$ of Corollary 4.7. If $c \in \mathfrak{p}_1 \backslash \mathfrak{p}_2$ and $d \in \mathfrak{p}_2 \backslash \mathfrak{p}_1$, then $c + d \notin Z(R)$. Let $y \in Z(R)^* \backslash \{c\}$. If $c$ and $y$ are not adjacent in $(ZT(R))^c$, then $cy = 0$ and $c + y \in Z(R)$. Note that $dy \neq 0$. Observe that $c - d - y$ is a path of length two between $c$ and $y$ in $(ZT(R))^c$. This shows that $e(c) = 2$ in $(ZT(R))^c$ and so, $r((ZT(R))^c) \leq 2$. Hence, $r((ZT(R))^c) = 2$.
$(3) \Rightarrow (1)$. Assume that $r((ZT(R))^c) = 2$. Then $diam((ZT(R))^c) \geq 3$ by $(1) \Rightarrow (3)$ of Corollary 4.7 and so, $diam((ZT(R))^c) = 3$. □

An element $e$ of a ring $R$ is said to be *idempotent* if $e = e^2$. An idempotent element $e$ of $R$ is said to be *non-trivial* if $e \notin \{0, 1\}$.

**Remark 4.9.** If $R$ is a ring which admits a non-trivial idempotent element $e$, then $e(1 - e) = 0$ and $e + 1 - e = 1 \notin Z(R)$. Therefore, $e$ and $1 - e$ are adjacent in $(ZT(R))^c$ but they are not adjacent in $(\Gamma(R))^c$. Hence, $(ZT(R))^c \neq (\Gamma(R))^c$.

We provide the following example to illustrate Corollaries 4.7 and 4.8. For any $n \in \mathbb{N} \backslash \{1\}$, we denote the ring of integers modulo $n$ by $\mathbb{Z}_n$.

**Example 4.10.** (1) If $R = \mathbb{Z}_4 \times \mathbb{Z}_2$, then $R$ satisfies the hypotheses of Corollary 4.7, the statement (2) of Corollary 4.7, and $(ZT(R))^c \neq (\Gamma(R))^c$.
(2) If $R = \mathbb{Z}_4 \times \mathbb{Z}_4$, then $R$ satisfies the hypotheses of Corollary 4.8, the statement (2) of Corollary 4.8, and $(ZT(R))^c \neq (\Gamma(R))^c$.

*Proof.* (1) As $R = \mathbb{Z}_4 \times \mathbb{Z}_2$, it follows that $Z(R) = (Z(\mathbb{Z}_4) \times \mathbb{Z}_2) \cup (\mathbb{Z}_4 \times Z(\mathbb{Z}_2)) = (2\mathbb{Z}_4 \times \mathbb{Z}_2) \cup (\mathbb{Z}_4 \times (0))$. If $\mathfrak{p}_1 = 2\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathfrak{p}_2 = \mathbb{Z}_4 \times (0)$, then $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$. Hence, $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Note that $\bigcap_{i=1}^{2} \mathfrak{p}_i = 2\mathbb{Z}_4 \times (0) \neq (0) \times (0)$. With $a = (2, 0)$ and $b = (0, 1)$, it is clear that $\mathfrak{p}_1 = ((0) \times (0) :_R a)$ and $\mathfrak{p}_2 = ((0) \times (0) :_R b)$, $a^2 = (0, 0)$ but $b^2 = b \neq (0, 0)$. As $(1, 0)$ is a non-trivial idempotent element of $R$, $(ZT(R))^c \neq (\Gamma(R))^c$ by Remark 4.9.
(2) It follows as in the proof of (1) that $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$, where $\mathfrak{p}_1 = 2\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathfrak{p}_2 = \mathbb{Z}_4 \times 2\mathbb{Z}_4$. Thus $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Note that $\bigcap_{i=1}^{2} \mathfrak{p}_i = 2\mathbb{Z}_4 \times 2\mathbb{Z}_4 \neq (0) \times (0)$. With $a = (2, 0)$ and $b = (0, 2)$, it is clear that $\mathfrak{p}_1 = ((0) \times (0) :_R a)$ and $\mathfrak{p}_2 = ((0) \times (0) :_R b)$, $a^2 = b^2 = (0, 0)$. As $(1, 0)$ is a non-trivial idempotent element of $R$, $(ZT(R))^c \neq (\Gamma(R))^c$ by Remark 4.9. □

If $|MNP(R)| = 2$, then it is clear from the above discussion that both $(ZT(R))^c$ and $(ZT(R_1))^c$ are connected and in Theorem 4.12, we determine the relationship between their diameters. We use the following lemma in its proof.

**Lemma 4.11.** *For a ring $R$ with $|MNP(R)| \geq 2$, if $a \in Z(R)^*$ satisfies $e(a) = 1$ in $(ZT(R))^c$, then $e(a) = 1$ in $(ZT(R_1))^c$.*

*Proof.* Assume that $e(a) = 1$ in $(ZT(R))^c$. Hence, $a$ cannot belong to two distinct members from $MNP(R)$ by Lemma 2.9. Thus $a \in \mathfrak{p}$ for a unique member $\mathfrak{p} \in MNP(R)$. Note that $a \in Z(R_1)^*$, since $Z(R)^* \subset Z(R_1)^*$. Let $f(X) \in Z(R_1)^* \backslash \{a\}$. If $af(X) \neq 0$, then $a$ and $f(X)$ are adjacent in $(ZT(R_1))^c$. Suppose that $af(X) = 0$. If $j \geq 0$ is least with the property that the coefficient $b_j$ of $X^j$ in $f(X)$ is nonzero, then $ab_j = 0$. Let $\mathfrak{p}' \in MNP(R) \backslash \{\mathfrak{p}\}$. (Such a $\mathfrak{p}'$ exists, since $|MNP(R)| \geq 2$ by hypothesis.) From $ab_j = 0$ and $a \notin \mathfrak{p}'$, it follows that $b_j \in \mathfrak{p}'$. It is clear that $b_j \in Z(R)^*$ and $a \neq b_j$. As $e(a) = 1$ in $(ZT(R))^c$, $a$ and $b_j$ are adjacent in $(ZT(R))^c$ and so, $a + b_j \notin Z(R)$. From $\mathfrak{p} \subset Z(R)$ and $a \in \mathfrak{p}$, we obtain that $b_j \notin \mathfrak{p}$. We claim that $a + f(X) \notin Z(R_1)$. If $a + f(X) \in Z(R_1)$, then there exists $g(X) \in R_1 \backslash \{0\}$ such that $(a + f(X))g(X) = 0$. For any non-negative integer $m$, let us denote the coefficient of $X^m$ in $g(X)$ by $c_m$. Let $k \geq 0$ be least with the property that the coefficient $c_k$ of $X^k$ in $g(X)$ is nonzero. If $j = 0$, then from $(a + b_j)c_k = 0$, we get that $a + b_j \in Z(R)$. This is a contradiction. Therefore, $j \geq 1$. Note that the coefficient of $X^m$ in $(a + f(X))g(X)$ equals 0 for all non-negative integers $m$. The coefficient of $X^k$ in $(a + f(X))g(X)$ equals $ac_k$ and so, $ac_k = 0$. As $a$

is not nilpotent, it follows that $a \neq c_k$. As $c_k \in Z(R)^*$ and $e(a) = 1$ in $(ZT(R))^c$, we get that $a$ and $c_k$ are adjacent in $(ZT(R))^c$ and so, $a + c_k \notin Z(R)$. From $\mathfrak{p} \subset Z(R)$ and $a \in \mathfrak{p}$, we obtain that $c_k \notin \mathfrak{p}$. Observe that the coefficient of $X^{k+j}$ in $(a+f(X))g(X)$ equals $ac_{k+j}+b_jc_k$. Hence, $ac_{k+j} + b_jc_k = 0$. This implies that $b_jc_k = -ac_{k+j} \in \mathfrak{p}$. This is impossible, since $b_j, c_k \notin \mathfrak{p}$. Therefore, $a + f(X) \notin Z(R_1)$. Hence, $a$ and $f(X)$ are adjacent in $(ZT(R_1))^c$. This shows that $e(a) = 1$ in $(ZT(R_1))^c$. □

**Theorem 4.12.** *For a ring $R$ with $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$, the following statements hold:*
*(1) $diam((ZT(R))^c) = 1$ if and only if $diam((ZT(R_1))^c) = 1$.*
*(2) If $diam((ZT(R))^c) = 2$, then $diam((ZT(R_1))^c) = 2$.*
*(3) If $diam((ZT(R))^c) = 3$, then $diam((ZT(R[X]))^c) = 3$.*
*(4) If $diam((ZT(R))^c) = 3$ and if $MNP(R[[X]]) = \{\mathfrak{p}_i[[X]] \mid i \in \{1, 2\}\}$, then $(ZT(R[[X]]))^c$ also has diameter 3.*
*(5) If $diam((ZT(R[X]))^c) = 2$, then $diam((ZT(R))^c) = 2$.*
*(6) If $diam((ZT(R_1))^c) = 3$, then $diam((ZT(R))^c) = 3$.*

*Proof.* (1) This is clear, since $(ZT(R))^c$ is complete if and only if $(ZT(R_1))^c$ is complete by (1) ⇔ (3) of Theorem 4.3.
(2) Assume that $diam((ZT(R))^c) = 2$.

If $\mathfrak{p}_i \notin \mathbb{A}(R)$ for some $i \in \{1, 2\}$, then $diam((ZT(R_1))^c) \leq 2$ by Proposition 2.3 and $diam((ZT(R_1))^c) \geq 2$ by Lemma 2.9, since $\bigcap_{i=1}^2 \mathfrak{p}_i \neq (0)$. Therefore, $diam((ZT(R_1))^c) = 2$.

Suppose that $\mathfrak{p}_i \in \mathbb{A}(R)$ for each $i \in \{1, 2\}$. Then there exist $a, b \in R\backslash\{0\}$ such that $\mathfrak{p}_1 = ((0) :_R a)$ and $\mathfrak{p}_2 = ((0) :_R b)$. As $diam((ZT(R))^c) = 2$, we obtain that $\bigcap_{i=1}^2 \mathfrak{p}_i \neq (0)$ by (2) ⇒ (1) of Theorem 4.3. Note that $r((ZT(R))^c) = 1$ by (1) ⇒ (3) of Corollary 4.7. Hence, $r((ZT(R_1))^c) = 1$ by Lemma 4.11 and so, $diam((ZT(R_1))^c) \leq 2$. Since $diam((ZT(R_1))^c) \geq 2$ by (3) ⇒ (1) of Theorem 4.3, it follows that $diam((ZT(R_1))^c) = 2$.
(3) and (4) Assume that $diam((ZT(R))^c) = 3$. Then $\bigcap_{i=1}^2 \mathfrak{p}_i \neq (0)$ by (2) ⇒ (1) of Theorem 4.3. Also, $\mathfrak{p}_i \in \mathbb{A}(R)$ for each $i \in \{1, 2\}$ by Proposition 2.3. Hence, there exist $a, b \in R\backslash\{0\}$ such that $\mathfrak{p}_1 = ((0) :_R a)$ and $\mathfrak{p}_2 = ((0) :_R b)$. Note that $a^2 = b^2 = 0$ by (1) ⇒ (2) of Corollary 4.8. It is clear that $\mathfrak{p}_1[X] = ((0) :_{R[X]} a)$ and $\mathfrak{p}_2[X] = ((0) :_{R[X]} b)$. Hence, $\bigcup_{i=1}^2 \mathfrak{p}_i[X] \subseteq Z(R[X])$. If $f(X) \in Z(R[X])$, then $A_f \subseteq Z(R)$ by [23, Theorem 2]. As $Z(R) = \bigcup_{i=1}^2 \mathfrak{p}_i$, it follows that $A_f \subseteq \mathfrak{p}_i$ for some $i \in \{1, 2\}$ and so, $f(X) \in \mathfrak{p}_i[X]$. This shows that $Z(R[X]) \subseteq \bigcup_{i=1}^2 \mathfrak{p}_i[X]$. Hence, $MNP(R[X]) = \{\mathfrak{p}_i[X] \mid i \in \{1, 2\}\}$. Observe that $\bigcap_{i=1}^2 \mathfrak{p}_i[X] = (\bigcap_{i=1}^2 \mathfrak{p}_i)[X] \neq (0)$, since $\bigcap_{i=1}^2 \mathfrak{p}_i \neq (0)$. Thus $R[X]$ satisfies the hypotheses and the statement (2) of Corollary 4.8. Hence, $diam((ZT(R[X]))^c) = 3$ by (2) ⇒ (1) of Corollary 4.8.

Note that $\mathfrak{p}_i[[X]] \in Spec(R[[X]])$ for each $i \in \{1, 2\}$, $\mathfrak{p}_1[[X]] = ((0) :_{R[[X]]} a)$, and $\mathfrak{p}_2[[X]] = ((0) :_{R[[X]]} b)$. From $\bigcap_{i=1}^2 \mathfrak{p}_i \neq (0)$, it follows that $\bigcap_{i=1}^2 \mathfrak{p}_i[[X]] = (\bigcap_{i=1}^2 \mathfrak{p}_i)[[X]] \neq (0)$. Assume that $MNP(R[[X]]) = \{\mathfrak{p}_i[[X]] \mid i \in \{1, 2\}\}$. Thus $R[[X]]$ satisfies the hypotheses and the statement (2) of Corollary 4.8. Hence, $diam((ZT(R[[X]]))^c) = 3$ by (2) ⇒ (1) of Corollary 4.8. This proves (3) and (4).
(5) If $diam((ZT(R[X]))^c) = 2$, then $diam((ZT(R))^c) \geq 2$ by (1) and $diam((ZT(R))^c) \leq 2$ by (3). Therefore, $diam((ZT(R))^c) = 2$. This proves (5).
(6) If $diam((ZT(R_1))^c) = 3$, then $diam((ZT(R))^c) \geq 2$ by (1) and $diam((ZT(R))^c) \geq 3$ by (2). As $diam((ZT(R))^c) \leq 3$, it follows that $diam((ZT(R))^c) = 3$. This proves (6). □

The following example illustrates that the statement (4) of Theorem 4.12 can fail to hold if the assumption $MNP(R[[X]]) = \{\mathfrak{p}_i[[X]] \mid i \in \{1, 2\}\}$ is omitted.

**Example 4.13.** If $R$ is as in Example 3.5 and if $A = R \times R$, then $diam((ZT(A))^c) = 3$ but $diam((ZT(A[[X]]))^c) = 2$.

*Proof.* We use the same notations as in the proof of Example 3.5. It is already noted in the proof of Example 3.5 that $Z(R) = \frac{\mathfrak{m}}{I}$ is the unique maximal ideal of $R$. As $Z(A) = (Z(R) \times R) \cup (R \times Z(R))$, it follows that $MNP(A) = \{\mathfrak{p}_1 = \frac{\mathfrak{m}}{I} \times R, \mathfrak{p}_2 = R \times \frac{\mathfrak{m}}{I}\}$. It is clear that $\bigcap_{i=1}^2 \mathfrak{p}_i \neq (0 + I) \times (0 + I)$. It is already verified in the proof of Example 3.5 that $\frac{\mathfrak{m}}{I} = ((0 + I) :_R v + I)$, where $v \in \mathfrak{m}$ is such that $p = mv$. Hence, $\mathfrak{p}_1 = ((0 + I) \times (0 + I) :_A (v + I, 0 + I)), \mathfrak{p}_2 = ((0+I) \times (0+I) :_A (0+I, v+I))$. It is convenient to denote $(v+I, 0+I)$ by $a$ and $(0+I, v+I)$

by $b$. Note that $a^2 = b^2 = (0+I, 0+I)$. Thus the ring $A$ satisfies the hypotheses and the statement (2) of Corollary 4.8. Hence, $diam((ZT(A))^c) = 3$ by (2) $\Rightarrow$ (1) of Corollary 4.8. It is verified in the proof of Example 3.5 that $MNP(R[[X]])$ is infinite. Therefore, $MNP(A[[X]])$ is infinite, since $A[[X]] \cong R[[X]] \times R[[X]]$ as rings. Hence, $diam((ZT(A[[X]]))^c) = 2$ follows by Proposition 2.10. This example also illustrates that the statement (5) of Theorem 4.12 can fail to hold for power series ring. □

In Example 4.15, we provide a ring $R$ to illustrate the statement (4) of Theorem 4.12. We use the following lemma in its verification.

**Lemma 4.14.** *If* $(R, \mathfrak{m})$ *is a quasi-local ring with* $\mathfrak{m}^n = (0)$ *for some* $n \geq 2$, *then* $Z(R[[X]]) = \mathfrak{m}[[X]]$.

*Proof.* By hypothesis, $R$ is quasi-local with $\mathfrak{m}$ as its unique maximal ideal and $\mathfrak{m}^n = (0)$ for some $n \geq 2$. Hence, $nil(R) = \mathfrak{m} = Z(R)$. As $(\mathfrak{m}[[X]])^n = (0)$, it follows that $\mathfrak{m}[[X]] \subseteq Z(R[[X]])$. Let $f(X) = \sum_{i=0}^{\infty} r_i X^i \in Z(R[[X]])$. Then it is clear that $r_0 \in Z(R) = \mathfrak{m}$. Let $i \geq 1$ and assume it is shown that $r_0, \ldots, r_{i-1} \in \mathfrak{m}$. As $\sum_{j=0}^{i-1} r_j X^j$ is a nilpotent element of $R[[X]]$, $\sum_{k=i}^{\infty} r_k X^k = f(X) - \sum_{j=0}^{i-1} r_j X^j \in Z(R[[X]])$ by [20, Lemma 2.3]. Hence, $r_i \in Z(R) = \mathfrak{m}$. This shows that $f(X) \in \mathfrak{m}[[X]]$. Therefore, $Z(R[[X]]) \subseteq \mathfrak{m}[[X]]$ and so, $Z(R[[X]]) = \mathfrak{m}[[X]]$. □

**Example 4.15.** The ring $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ has the following properties: $MNP(R) = \{\mathfrak{p}_1 = 2\mathbb{Z}_4 \times \mathbb{Z}_4, \mathfrak{p}_2 = \mathbb{Z}_4 \times 2\mathbb{Z}_4\}$, $MNP(R[[X]]) = \{\mathfrak{p}_i[[X]] \mid i \in \{1, 2\}\}$, $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 3$ and $r((ZT(R))^c) = r((ZT(R_1))^c) = 2$.

*Proof.* It is already noted in the proof of Example 4.10(2) that $MNP(R) = \{\mathfrak{p}_1 = 2\mathbb{Z}_4 \times \mathbb{Z}_4, \mathfrak{p}_2 = \mathbb{Z}_4 \times 2\mathbb{Z}_4\}$, $\bigcap_{i=1}^{2} \mathfrak{p}_i \neq (0) \times (0)$. $\mathfrak{p}_1 = (((0,0)) :_R a)$, $\mathfrak{p}_2 = (((0,0)) :_R b)$ with $a = (2, 0)$ and $b = (0, 2)$, and $a^2 = b^2 = (0, 0)$. Hence, $R$ satisfies the hypotheses and the statement (2) of Corollary 4.8 and hence, $diam((ZT(R))^c) = 3$ by (2) $\Rightarrow$ (1) of Corollary 4.8 and $r((ZT(R))^c) = 2$ by (2) $\Rightarrow$ (3) of Corollary 4.8. It is noted in the proof of Theorem 4.12(3) that $MNP(R[X]) = \{\mathfrak{p}_i[X] \mid i \in \{1, 2\}\}$ and $R[X]$ satisfies the hypotheses and the statement (2) of Corollary 4.8. Hence, $diam((ZT(R[X]))^c) = 3$ and $r((ZT(R[X]))^c) = 2$. Observe that $\mathfrak{p}_1[[X]] = (((0,0)) :_{R[[X]]} a)$ and $\mathfrak{p}_2[[X]] = (((0,0)) :_{R[[X]]} b)$. It is clear that the mapping $\phi : R[[X]] \to \mathbb{Z}_4[[X]] \times \mathbb{Z}_4[[X]]$ defined by $\phi(\sum_{i=0}^{\infty}(a_i, b_i)X^i) = (\sum_{i=0}^{\infty} a_i X^i, \sum_{i=0}^{\infty} b_i X^i)$ is an isomorphism of rings. Note that $\mathbb{Z}_4$ is local with $\mathfrak{m} = 2\mathbb{Z}_4$ as its unique maximal ideal and $\mathfrak{m}^2 = (0)$. Hence, $Z(\mathbb{Z}_4[[X]]) = \mathfrak{m}[[X]]$ by Lemma 4.14. Observe that $Z(\mathbb{Z}_4[[X]] \times \mathbb{Z}_4[[X]]) = (Z(\mathbb{Z}_4[[X]]) \times \mathbb{Z}_4[[X]]) \cup (\mathbb{Z}_4[[X]] \times Z(\mathbb{Z}_4[[X]]))$. Therefore, $Z(\mathbb{Z}_4[[X]] \times \mathbb{Z}_4[[X]]) = (\mathfrak{m}[[X]] \times \mathbb{Z}_4[[X]]) \cup (\mathbb{Z}_4[[X]] \times \mathfrak{m}[[X]])$. Under the isomorphism $\phi$, $\mathfrak{p}_1[[X]]$ is mapped onto $\mathfrak{m}[[X]] \times \mathbb{Z}_4[[X]]$ and $\mathfrak{p}_2[[X]]$ is mapped onto $\mathbb{Z}_4[[X]] \times \mathfrak{m}[[X]]$. As $Z(\mathbb{Z}_4[[X]] \times \mathbb{Z}_4[[X]]) = (\mathfrak{m}[[X]] \times \mathbb{Z}_4[[X]]) \cup (\mathbb{Z}_4[[X]] \times \mathfrak{m}[[X]])$, it follows that $Z(R[[X]]) = \bigcup_{i=1}^{2} \mathfrak{p}_i[[X]]$. Hence, $MNP(R[[X]]) = \{\mathfrak{p}_i[[X]] \mid i \in \{1, 2\}\}$. Thus $R[[X]]$ satisfies the hypotheses and the statement (2) of Corollary 4.8. Hence, $diam((ZT(R[[X]]))^c) = 3$ and $r((ZT(R[[X]]))^c) = 2$. □

For a ring $R$ with $|MNP(R)| \geq 2$, if $r((ZT(R))^c) = 1$, then $r((ZT(R_1))^c) = 1$ by Lemma 4.11. The following example illustrates that the converse of the above implication can fail to hold.

**Example 4.16.** With the ring $R$ as in Example 3.8, the ring $A = R \times R$ has the following properties: $|MNP(A)| = 2$, $diam((ZT(A))^c) = 2 = r((ZT(A))^c)$, $diam((ZT(A[X]))^c) = r((ZT(A[X]))^c) = 2 = diam((ZT(A[[X]]))^c)$ but $r((ZT(A[[X]]))^c) = 1$.

*Proof.* We use the same notations as in the proof of Example 3.8. It is already noted there that $R$ is a quasi-local reduced ring with $\mathfrak{m}$ as its unique maximal ideal, $Z(R) = \mathfrak{m}$, and $\mathfrak{m} \notin \mathbb{A}(R)$. As $A = R \times R$, it follows that $Z(A) = (Z(R) \times R) \cup (R \times Z(R))$, and so, $Z(A) = (\mathfrak{m} \times R) \cup (R \times \mathfrak{m})$. It is convenient to denote $\mathfrak{m} \times R$ by $\mathfrak{P}_1$ and $R \times \mathfrak{m}$ by $\mathfrak{P}_2$. It is clear that $MNP(A) = \{\mathfrak{P}_i \mid i \in \{1, 2\}\}$. Note that $\bigcap_{i=1}^{2} \mathfrak{P}_i = \mathfrak{m} \times \mathfrak{m} \neq (0 + I) \times (0 + I)$. As $\mathfrak{m} \notin \mathbb{A}(R)$, it follows that $\mathfrak{P}_i \notin \mathbb{A}(R)$ for each $i \in \{1, 2\}$. It follows from Proposition 2.3 and Lemma 2.9 that $diam((ZT(A))^c) = diam((ZT(A[X]))^c) = diam((ZT(A[[X]]))^c) = 2$.

We next verify that $r((ZT(A))^c) = 2$. It is clear that $r((ZT(A))^c) \leq 2$. Let $a \in Z(A)^*$. Either $a = (m, r)$ or $a = (r', m')$ for some $m, m' \in \mathfrak{m}$ and $r, r' \in R$. Assume that $a = (m, r)$. As $Z(R) = \mathfrak{m}$, there exists $m_1 \in \mathfrak{m}\backslash\{0 + I\}$ such that $mm_1 = 0 + I$. Since $R$ is reduced, $m \neq m_1$. The element $b = (m_1, 0 + I) \in Z(A)^*$ is such that $b \neq (0 + I, 0 + I)$, $a \neq b$, $ab = (0 + I, 0 + I)$, and $a + b = (m + m_1, r) \in Z(A)$. Hence, $d(a, b) \geq 2$ in $(ZT(A))^c$. If $a = (r', m')$, then it can be shown that there exists $b' \in Z(A)^*$ with $d(a, b') \geq 2$ in $(ZT(A))^c$. Thus for any $a \in Z(A)^*$, $e(a) \geq 2$ in $(ZT(A))^c$ and so, $r((ZT(A))^c) \geq 2$. Hence, $diam((ZT(A))^c) = r((ZT(A))^c) = 2$.

It is shown in the proof of Example 3.8(2) that $Z(R[X]) = \mathfrak{m}[X]$. Hence, $Z(R[X] \times R[X]) = (\mathfrak{m}[X] \times R[X]) \cup (R[X] \times \mathfrak{m}[X])$. It is clear that $A[X] \cong R[X] \times R[X]$ as rings. Therefore, $diam((ZT(R[X] \times R[X]))^c) = 2$. Proceeding as in the previous paragraph, it can be shown that $r((ZT(R[X] \times R[X]))^c) = 2$. Hence, $diam((ZT(A[X]))^c) = r((ZT(A[X]))^c) = 2$.

Since $A[[X]] \cong R[[X]] \times R[[X]]$ as rings and $diam((ZT(A[[X]]))^c) = 2$, it follows that $diam((ZT(R[[X]] \times R[[X]]))^c) = 2$. We next verify that $r((ZT(R[[X]] \times R[[X]]))^c) = 1$. It is shown in the proof of Example 3.8(3) that $e(f(X)) = 1$ in $(ZT(R[[X]]))^c$, where $f(X) = \sum_{j=1}^{\infty} x_{j+1}X^j$. We claim that $e((f(X), 1 + I)) = 1$ in $(ZT(R[[X]] \times R[[X]]))^c$. Let $(g(X), h(X)) \in Z(R[[X]] \times R[[X]])^*\backslash\{(f(X), 1 + I)\}$. If $(f(X), 1 + I)(g(X), h(X)) = (0 + I, 0 + I)$, then $f(X)g(X) = 0 + I$ and $h(X) = 0 + I$. As $f(X) \neq 0 + I$ and $R[[X]]$ is reduced, it follows that $g(X) \neq f(X)$. From $e(f(X)) = 1$ in $(ZT(R[[X]]))^c$, we obtain that $f(X) + g(X) \notin Z(R[[X]])$ and so, $(f(X), 1 + I) + (g(X), 0 + I) = (f(X) + g(X), 1 + I) \notin Z(R[[X]] \times R[[X]])$. This shows that $e((f(X), 1 + I)) = 1$ in $(ZT(R[[X]] \times R[[X]]))^c$. Hence, $r((ZT(R[[X]] \times R[[X]]))^c) = 1$. Therefore, $r((ZT(A[[X]]))^c) = 1$. □

## 5 Some results on $(ZT(R))^c$ and $(ZT(R_1))^c$, where $R$ is von Neumann regular

Throughout this section, unless otherwise specified, the rings $R$ considered are von Neumann regular which are not fields. We use $R_1$ to denote either $R[X]$ or $R[[X]]$. The aim of this section is to discuss some results about the relationship between $diam((ZT(R))^c)$ and $diam((ZT(R_1))^c)$ (respectively, $r((ZT(R))^c)$ and $r((ZT(R_1))^c)$).

Recall that a ring $R$ is said to be *von Neumann regular* if given $a \in R$, there exists $b$ in $R$ such that $a = a^2b$ [13, Exercise 16, page 111]. If $R$ is not a field, then for given $a \in NU(R)\backslash\{0\}$, there exists $b \in R$ with $a = a^2b$. Hence, $ab = a^2b^2$. Note that $e = ab$ satisfies $e = e^2$ and $e \notin \{0, 1\}$. Thus $R$ admits a non-trivial idempotent and so, does $R_1$. Hence, $(ZT(R))^c \neq (\Gamma(R))^c$ and $(ZT(R_1))^c \neq (\Gamma(R_1))^c$ by Remark 4.9. A ring $R$ is von Neumann regular if and only if $R$ is reduced and $dim R = 0$ by $(a) \Leftrightarrow (d)$ of [13, Exercise 16, page 111]. Thus $Spec(R) = Max(R) = Min(R)$. If $a \in NU(R)$, then $a \in Z(R)$. Therefore, $Spec(R) = MNP(R)$. As $R$ is reduced, it follows that $\bigcap_{\mathfrak{m} \in Max(R)} \mathfrak{m} = (0)$.

**Proposition 5.1.** *For a von Neumann regular ring $R$, $diam((ZT(R))^c) = diam((ZT(R_1))^c) \in \{1, 2\}$.*

*Proof.* Assume that $R$ is von Neumann regular. As $R$ is not a field, it follows that $|Max(R)| \geq 2$. If $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2\}\}$, then $\bigcap_{i=1}^{2} \mathfrak{m}_i = (0)$. Hence, $diam((ZT(R))^c) = 1$ by Lemma 4.1 and hence, $diam((ZT(R_1))^c) = 1$ by $(1) \Rightarrow (3)$ of Theorem 4.3. If $|Max(R)| \geq 3$, then $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 2$ by Proposition 2.10. □

If $R$ is von Neumann regular with $r((ZT(R))^c) = 1$, then $r((ZT(R_1))^c) = 1$ by Lemma 4.11. In the following proposition, we characterize $R$ such that $r((ZT(R))^c) = 2$.

**Proposition 5.2.** *For a von Neumann regular ring $R$, the following statements are equivalent:*
(1) $r((ZT(R))^c) = 2$*;*
(2) $e(a) = 2$ *for each* $a \in Z(R)^*$*;*
(3) *If $a$ is any nonzero non-unit of $R$, then $a$ belongs to at least two maximal ideals of $R$.*

*Proof.* (1) $\Rightarrow$ (2). Let $a \in Z(R)^*$. Observe that $e(a) \geq 2$ in $(ZT(R))^c$, since $r((ZT(R))^c) = 2$ by assumption. Note that $e(a) \leq 2$ in $(ZT(R))^c$ by Proposition 5.1. Therefore, $e(a) = 2$ in $(ZT(R))^c$.

(2) $\Rightarrow$ (3). Let $a \in NU(R)\backslash\{0\}$. As $a \in Z(R)^*$, $e(a) = 2$ in $(ZT(R))^c$ by assumption. Hence, there exists $b \in Z(R)^*\backslash\{a\}$ such that $ab = 0$ and $a + b \in Z(R)$. Since $\bigcap_{\mathfrak{m}\in Max(R)} \mathfrak{m} = (0)$, it follows that $a \notin \mathfrak{m}_1$ and $b \notin \mathfrak{m}_2$ for some $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(R)$. As $ab = 0$, it follows that $a \in \mathfrak{m}_2$ and $b \in \mathfrak{m}_1$. Thus $a \in \mathfrak{m}_2\backslash\mathfrak{m}_1$ and $b \in \mathfrak{m}_1\backslash\mathfrak{m}_2$. Hence, $a + b \notin \bigcup_{i=1}^{2} \mathfrak{m}_i$. As $a + b \in Z(R)$, there exists $\mathfrak{m}_3 \in Max(R)$ such that $a + b \in \mathfrak{m}_3$. It is clear that $\mathfrak{m}_3 \notin \{\mathfrak{m}_i \mid i \in \{1, 2\}\}$. From $ab = 0$, $a + b \in \mathfrak{m}_3$, we get that $a^2 \in \mathfrak{m}_3$, and so, $a \in \mathfrak{m}_3$. Thus $a$ belongs to at least two maximal ideals of $R$.

(3) $\Rightarrow$ (1). Assume that any nonzero non-unit of $R$ belongs to at least two maximal ideals of $R$. Let $a \in Z(R)^*$. Since $Max(R) = MNP(R)$, $e(a) \geq 2$ in $(ZT(R))^c$ by Lemma 2.9. Therefore, $r((ZT(R))^c) \geq 2$. As $r((ZT(R))^c) \leq 2$ by Proposition 5.1, we obtain that $r((ZT(R))^c) = 2$. □

The following example illustrates Propositions 5.1 and 5.2.

**Example 5.3.** (1) Let $n \geq 3$. If $F_i$ is a field for each $i \in \{1, 2, 3, \ldots, n\}$, then the ring $R = F_1 \times F_2 \times F_3 \times \cdots \times F_n$ is such that $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 2$ and $r((ZT(R))^c) = r((ZT(R_1))^c) = 1$.

(2) If $F_i$ is a field for each $i \in \mathbb{N}$, then the ring $R = \prod_{i=1}^{\infty} F_i$ is such that $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 2$ and $r((ZT(R))^c) = r((ZT(R_1))^c) = 1$.

(3) Let $L$ be the field of algebraic numbers (that is, $L$ is the algebraic closure of $\mathbb{Q}$). If $A$ is the ring of all algebraic integers, then the ring $R = \frac{A}{\sqrt{2A}}$ is such that $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 2$ and $r((ZT(R))^c) = 2$.

*Proof.* If $F$ is a field, then for any $\alpha \in F\backslash\{0\}$, $\alpha = \alpha^2\beta$ with $\beta = \alpha^{-1}$. Using this fact, it follows that the ring $R$ mentioned in (1) (respectively, in (2)) is von Neumann regular.

(1) By assumption, $n \geq 3$. Note that $|Max(R)| = n \geq 3$ and so, $|MNP(R)| \geq 3$, since $Max(R) = MNP(R)$. Hence, $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 2$ by Proposition 2.10. Observe that $\mathfrak{m}_1 = (0) \times F_2 \times F_3 \times \cdots \times F_n \in Max(R)$ and $\mathfrak{m}_1 = R(0, 1, 1, \ldots, 1)$. Note that $a = (0, 1, 1, \ldots, 1) \in Z(R)^*$ and it belongs to a unique maximal ideal of $R$. Therefore, $r((ZT(R))^c) = 1$ by (1) $\Rightarrow$ (3) of Proposition 5.2 and so, $r((ZT(R_1))^c) = 1$ by Lemma 4.11.

(2) Let $i \in \mathbb{N}$. If $f_i$ is the element of $R$ whose $i$-th coordinate is 0 and $j$-th coordinate is 1 for all $j \in \mathbb{N}\backslash\{i\}$, then $Rf_i \in Max(R)$ and $Rf_i \neq Rf_j$ for all distinct $i, j \in \mathbb{N}$. Hence, $Max(R)$ is infinite. As $Max(R) = MNP(R)$, $MNP(R)$ is infinite, and so, $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 2$ by Proposition 2.10. Note that $f_1 \in Z(R)^*$ and it belongs to only one maximal ideal $Rf_1$ of $R$. It now follows as in (1) that $r((ZT(R))^c) = r((ZT(R_1))^c) = 1$.

(3) It is known that $dim A = 1$ and each nonzero non-unit of $A$ belongs to uncountably many maximal ideals of $A$ [13, Proposition 42.8]. As $\sqrt{2A}$ is a radical ideal of $A$, it follows that $R = \frac{A}{\sqrt{2A}}$ is reduced. It is clear that $dim R = 0$. Therefore, $R$ is von Neumann regular. Observe that $Max(R)$ is uncountable, since 2 belongs to uncountably many maximal ideals of $A$. Thus $MNP(R) = Max(R)$ is uncountable and so, $diam((ZT(R))^c) = diam((ZT(R_1))^c) = 2$ by Proposition 2.10. Let $a \in A\backslash\sqrt{2A}$ be such that $a + \sqrt{2A} \in NU(R)$. Then $Aa + A2 \neq A$. Note that $Aa + A2 = Ac$ for some $c \in NU(A)$, since $A$ is a Bézout domain by [21, see page 86]. Since $c$ belongs to uncountably many maximal ideals of $A$, $a + \sqrt{2A}$ belongs to uncountably many maximal ideals of $R$. Hence, $r((ZT(R))^c) = 2$ by (3) $\Rightarrow$ (1) of Proposition 5.2. □

The proof of Proposition 5.7 needs the following lemmas.

**Lemma 5.4.** *For a ring $R$, if $\mathfrak{m} \in Max(R)$ is such that $\mathfrak{m}[X] \subseteq Z(R[X])$, then $\mathfrak{m}[X] \in MNP(R[X])$.*

*Proof.* Note that $S = R[X]\backslash(Z(R[X]))$ is a m.c. subset of $R[X]$ and by hypothesis, $\mathfrak{m}[X] \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [19, Theorem 1] that there exists $\mathfrak{P} \in MNP(R[X])$ with $\mathfrak{m}[X] \subseteq \mathfrak{P}$. We claim that $\mathfrak{P} = \mathfrak{m}[X]$. If $\mathfrak{P} \neq \mathfrak{m}[X]$, then there exists $f(X) \in \mathfrak{P}\backslash\mathfrak{m}[X]$. Let $f(X) = \sum_{i=0}^{n} r_iX^i$. Note that $r_j \notin \mathfrak{m}$ for some $j \in \{0, \ldots, n\}$, since $f(X) \notin \mathfrak{m}[X]$ by assumption. As $\mathfrak{m} \in Max(R)$ and $r_j \notin \mathfrak{m}$, there exist $m \in \mathfrak{m}$ and $r \in R$ such that $m + rr_j = 1$. Observe that $g(X) = mX^j + rf(X) \in \mathfrak{P} \subseteq Z(R[X])$. Hence, there exists $s \in R\backslash\{0\}$ such that $sg(X) = 0$ by [23, Theorem 2]. Therefore, $s$(coefficient of $X^j$ in $g(X)$) = 0. As the coefficient of $X^j$ in $g(X)$ equals $m + rr_j$, it follows that $s = s1 = s(m + rr_j) = 0$. This is a contradiction. Therefore, $\mathfrak{P} = \mathfrak{m}[X]$ and so, $\mathfrak{m}[X] \in MNP(R[X])$. □

**Lemma 5.5.** *If $R$ is a von Neumann regular ring, then each finitely generated ideal of $R$ is principal and is generated by an idempotent element of $R$.*

*Proof.* This is well known. We provide a proof for the sake of completeness. If $r$ is a nonzero non-unit of $R$, then there exist $u \in U(R)$ and an idempotent element $e$ of $R$ with $e \notin \{0, 1\}$ such that $r = ue$ by $(1) \Rightarrow (3)$ of [13, Exercise 29, page 113]. Hence, $Rr = Re$. For any idempotent elements $e, f$ of $R$, it is not hard to show that $Re + Rf = R(e + f - ef)$ and $e + f - ef$ is an idempotent element of $R$. It is now clear that if $I$ is any finitely generated ideal of $R$, then $I = Re$ for some idempotent element $e \in I$. □

**Lemma 5.6.** *If $R$ is a von Neumann regular ring, then $MNP(R[X]) = \{\mathfrak{m}[X] \mid \mathfrak{m} \in Max(R)\}$.*

*Proof.* By hypothesis, $R$ is von Neumann regular. Note that $Max(R) = MNP(R)$. Let $\mathfrak{m} \in Max(R)$. If $g(X) \in \mathfrak{m}[X] \backslash \{0\}$, then $g(X) = \sum_{j=0}^{n} r_j X^j$ with $r_j \in \mathfrak{m}$ for each $j \in \{0, \ldots, n\}$. Now, $\sum_{j=0}^{n} Rr_j = Re$ for some idempotent element $e \in \mathfrak{m}\backslash\{0\}$ by Lemma 5.5. Thus $g(X) = eg_1(X)$ for some $g_1(X) \in R[X]$. Note that $1 - e \neq 0$ and $g(X)(1 - e) = 0$. Hence, $g(X) \in Z(R[X])$. This shows that $\mathfrak{m}[X] \subseteq Z(R[X])$. Hence, $\mathfrak{m}[X] \in MNP(R[X])$ by Lemma 5.4. Therefore, $\{\mathfrak{m}[X] \mid \mathfrak{m} \in Max(R)\} \subseteq MNP(R[X])$. If $\mathfrak{P} \in MNP(R[X])$, then $\mathfrak{P} \cap R = \mathfrak{m}$ for some $\mathfrak{m} \in Max(R)$, since $dim R = 0$. This implies that $\mathfrak{m}[X] \subseteq \mathfrak{P}$ and so, $\mathfrak{P} = \mathfrak{m}[X]$, since $\mathfrak{m}[X] \in MNP(R[X])$. This proves that $MNP(R[X]) \subseteq \{\mathfrak{m}[X] \mid \mathfrak{m} \in Max(R)\}$ and hence, $MNP(R[X]) = \{\mathfrak{m}[X] \mid \mathfrak{m} \in Max(R)\}$. □

**Proposition 5.7.** *Let $R$ be a von Neumann regular ring such that $r((ZT(R))^c) = 2$. Then $r((ZT(R[X]))^c) = 2$.*

*Proof.* Assume that $r((ZT(R))^c) = 2$. Note that $r((ZT(R[X]))^c) \leq 2$ by Proposition 5.1. If $r((ZT(R[X]))^c) = 1$, then $e(f(X)) = 1$ in $(ZT(R[X]))^c$ for some $f(X) \in Z(R[X])^*$. Hence, $f(X)$ belongs to a unique member of $MNP(R[X])$ by Lemma 2.9. Therefore, $f(X) \in \mathfrak{m}[X]$ for some $\mathfrak{m} \in Max(R)$ by Lemma 5.6. For any $\mathfrak{m}' \in Max(R)\backslash\{\mathfrak{m}\}$, $f(X) \notin \mathfrak{m}'[X]$. Since $A_f$ is a finitely generated ideal of the ring $R$ with $A_f \subseteq \mathfrak{m}$, there exists an idempotent element $e \in \mathfrak{m}\backslash\{0, 1\}$ such that $A_f = Re$ by Lemma 5.5. As $f(X) \notin \bigcup_{\mathfrak{m}' \in Max(R)\backslash\{\mathfrak{m}\}} \mathfrak{m}'[X]$, it follows that $e \notin \bigcup_{\mathfrak{m}' \in Max(R)\backslash\{\mathfrak{m}\}} \mathfrak{m}'$. Thus $e$ belongs to a unique member of $Max(R)$ and this is impossible by $(1) \Rightarrow (3)$ of Proposition 5.2. Therefore, $r((ZT(R[X]))^c) = 2$. □

For the von Neumann regular ring $R$ considered in Example 5.3(3), $r((ZT(R))^c) = 2$ and so, $r((ZT(R[X]))^c) = 2$ by Proposition 5.7. We do not know whether $r((ZT(R[[X]]))^c) = 2$ or not.

For a von Neumann regular ring $R$, it is already noted in the beginning of this section that $(ZT(R))^c \neq (\Gamma(R))^c$. If $R$ is a ring with $dim R = 0$, then in Proposition 5.11, we characterize $R$ such that $(ZT(R))^c = (\Gamma(R))^c$.

For a ring $R$, recall that the *total quotient ring of* $R$ denoted by $Tot(R)$ is defined as $Tot(R) = S^{-1}R$, where $S = R\backslash Z(R)$. We first state and prove some results that are needed in the proof of Proposition 5.11.

**Lemma 5.8.** *For a ring $R$, if $x, y \in R\backslash\{0\}$ are such that $xy = 0$ and $x + y = 1$, then $x$ and $y$ are non-trivial idempotent elements of $R$.*

*Proof.* Assume that $x, y \in R\backslash\{0\}$ are such that $xy = 0$ and $x + y = 1$. It is clear that $x, y \in NU(R)$ and $x \neq y$. Note that $x = x(x + y) = x^2$ and $y = y(x + y) = y^2$. Therefore, $x$ and $y$ are non-trivial idempotent elements of $R$. □

**Proposition 5.9.** *For a ring $R$, $(ZT(R))^c = (\Gamma(R))^c$ if and only if $Tot(R)$ has no non-trivial idempotent.*

*Proof.* Assume that $(ZT(R))^c = (\Gamma(R))^c$. Let $\frac{x}{s} \in Tot(R)$ be such that $\frac{x}{s} = \frac{x^2}{s^2}$. As $s \notin Z(R)$, it follows that $x(s - x) = 0$. If $x \notin Z(R)$, then $x = s$ and so, $\frac{x}{s} = \frac{1}{1}$. If $x \in Z(R)$, then $s - x \neq 0$ and $x \neq s - x$. We claim that $x = 0$. If $x \neq 0$, then $x, s - x \in Z(R)^*$ are such that they are adjacent in $\Gamma(R)$. Hence, they are adjacent in $ZT(R)$. Therefore, $x + s - x \in Z(R)$. This is impossible, since $s \notin Z(R)$. Therefore, $x = 0$ and so, $\frac{x}{s} = \frac{0}{1}$. This proves that $Tot(R)$ has no non-trivial idempotent.

Conversely, assume that $Tot(R)$ has no non-trivial idempotent. If $x, y \in Z(R)^*$ are such that $x + y \notin Z(R)$, then $\frac{x+y}{1} \in U(Tot(R))$ and so, there exist $t, s \notin Z(R)$ such that $(\frac{x+y}{1})\frac{t}{s} = \frac{1}{1}$. Thus $\frac{xt}{s}, \frac{yt}{s} \in Tot(R)\backslash\{\frac{0}{1}\}$ are such that $\frac{xt}{s} + \frac{yt}{s} = \frac{1}{1}$. As $Tot(R)$ has no non-trivial idempotent, $\frac{xyt^2}{s^2} \neq \frac{0}{1}$ by Lemma 5.8 and so, $xy \neq 0$. Hence, $(ZT(R))^c = (\Gamma(R))^c$ by Remark 2.4. □

**Lemma 5.10.** *If $R$ is a zero-dimensional ring, then $Z(R) = NU(R)$.*

*Proof.* By hypothesis, $dim R = 0$. It is clear that $Z(R) \subseteq NU(R)$ (this is true for any ring). If $x \in NU(R)$, then $x \in \mathfrak{m}$ for some $\mathfrak{m} \in Max(R)$. Hence, $x + nil(R) \in \frac{\mathfrak{m}}{nil(R)}$. Therefore, $x + nil(R) \in NU(\frac{R}{nil(R)})$. Observe that $\frac{R}{nil(R)}$ is reduced and $dim(\frac{R}{nil(R)}) = 0$. Hence, $\frac{R}{nil(R)}$ is von Neumann regular. Therefore, $x + nil(R) \in Z(\frac{R}{nil(R)})$. So, there exists $y \in R\backslash nil(R)$ such that $xy \in nil(R)$. Let $n \in \mathbb{N}$ be such that $x^n y^n = 0$. As $y^n \neq 0$, it follows that $x^n \in Z(R)$ and so, $x \in Z(R)$. This shows that $NU(R) \subseteq Z(R)$ and hence, $Z(R) = NU(R)$. □

**Proposition 5.11.** *For a ring $R$ with $dim R = 0$, $(ZT(R))^c = (\Gamma(R))^c$ if and only if $MNP(R)$ has only one element.*

*Proof.* For any ring $R$ with $|MNP(R)| = 1$, $(ZT(R))^c = (\Gamma(R))^c$ by Remark 2.5.

Assume that $dim R = 0$ and $(ZT(R))^c = (\Gamma(R))^c$. Observe that $Spec(R) = Max(R)$. Note that $Max(R) = MNP(R)$ and $R = Tot(R)$ by Lemma 5.10. By Proposition 5.9, $R$ has no non-trivial idempotent. If $|Max(R)| \geq 2$, then the von Neumann regular ring $\frac{R}{nil(R)}$ is such that $|Max(\frac{R}{nil(R)})| \geq 2$. Hence, $\frac{R}{nil(R)}$ admits a non-trivial idempotent element, say $r + nil(R)$. Since $nil(R)$ is a nil ideal of $R$, there exists an idempotent $e$ of $R$ such that $r + nil(R) = e + nil(R)$ by [18, Proposition 7.14]. As $r + nil(R)$ is non-trivial, it follows that $e \notin \{0, 1\}$. This is impossible, since $R$ has no non-trivial idempotent element. Therefore, $|Max(R)| = 1$ and so, $|MNP(R)| = 1$. □

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## Author information

S. Visweswaran, Department of Mathematics, Saurashtra University, Rajkot, Gujarat 360005, India.
E-mail: `s_visweswaran2006@yahoo.co.in`

Hiren D. Patel, Government Polytechnic, Bhuj, Gujarat 370001, India.
E-mail: `hdp12376@gmail.com`

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