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Hyperstability of Jensen functional equation with involutions in ultrametric n-Banach spaces

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Abstract *In this paper, we investigate the hyperstability of the following σ-Jensen functional equation*

$$f\big(x+\sigma(y)\big)+f\big(x+\tau(y)\big)=2f(x),$$

where $f : X \to Y$ with X is normed space, Y is ulrametric n-Banach space, and $\sigma, \tau : X \to X$ are homomorphisms. In addition, we prove some interesting corollaries corresponding to some inhomogeneous outcomes.

1 Introduction

Throughout this paper, $\mathbb{Q}$ stands for the set of all rational numbers, $\mathbb{N}$ the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N}_{m_0}$ the set of all integers greater than or equals $m_0\,(m_0 \in \mathbb{N}), \mathbb{R}_+ = [0,\infty)$ and we use the notation X_0 for the set $X \backslash \{0\}$. The famous talk of S. M. Ulam in 1940 [45] seems to be the starting point for studying the stability of functional equations, in which he discussed a number of important unsolved problems. Among these was the question of the stability of group homomorphisms.

***Ulam problem:**[45] Given a group G_1, a metric group G_2 with metric $d(\cdot\ ,\ \cdot)$ and a positive number ε, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies*

$$d\big(f(xy)\ ,\ f(x)f(y)\big) \le \varepsilon$$

for all $x, y \in G_1$, then a homomorphism $\phi : G_1 \to G_2$ exists with

$$d\big(f(x)\ ,\ \phi(x)\big) \le \delta$$

for all $x \in G_1$?

These kinds of questions serve as the foundation for the theory of stability. Under the assumption that G_1 and G_2 are Banach spaces, the case of approximately additive mappings was solved by D. H. Hyers in 1941 [34].

Hyers' [34] and Ulam [45] referred to this property as the stability of the functional equation $f(x+y) = f(x)+f(y)$. Hyers' work has initiated much of the current research in the theory of the stability of functional equations. In 1978, the theorem of Hyers was significantly generalized by Th. Rassias [42], taking into account cases where the relevant inequality is not bound. This property was called the Hyers-Ulam-Rassias stability of the additive Cauchy functional equation $f(x+y) = f(x)+f(y)$.

This terminology also applies to other functional equations. The result of Rassias [42] has been further generalized by Rassias [43], Th. Rassias and P. Šemrl [44], P. Găvruţă [32], and S. -M. Jung [36]. Simultaneously, a special kind of stability has emerged, which is called the hyperstability of functional equations. This kind states that if f satisfies a stability inequality related to *the given equation, then it is also a solution to this equation. It seems that the first hyperstability result was published in [15] and concerned ring homomorphisms. The term "hyperstability", on the other hand, appeared for the first time in [37]. Hyperstability is frequently mistaken for superstability, which also admits bounded functions. Further, J. Brzdęk and K. Ciepliński [17] introduced the following definition which describes the main ideas of such a hyperstability notion for equations in several variables ($\mathbb{R}^+$ stands for the set of all nonnegative reals and C^D denotes the family of all functions mapping a set $D \neq \phi$ into a set $C \neq \phi$).*

Definition 1.1. [17] Let S be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_+^{S^n}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ be two operators mapping a nonempty set $\mathcal{D} \subset Y^S$ into Y^{S^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \cdots, x_n) = \mathcal{F}_2\varphi(x_1, \cdots, x_n), \quad x_1, \cdots, x_n \in S, \tag{1.1}$$

is ε-hyperstable provided every $\varphi_0 \in \mathcal{D}$ that satisfies the inequality

$$d\big(\mathcal{F}_1\varphi_0(x_1, \cdots, x_n)\ ,\ \mathcal{F}_2\varphi_0(x_1, \cdots, x_n)\big) \leq \varepsilon(x_1, \cdots, x_n), \quad x_1, \cdots, x_n \in S, \tag{1.2}$$

fulfils the equation (1.1).

Brzdęk et al. [17] proved the fixed point theorem for a nonlinear operator in metric spaces and used this result to study the Hyers-Ulam stability of some functional equations in non-Archimedean metric spaces. In this work, they also obtained the fixed point result in arbitrary metric spaces as follows:

Theorem 1.2. *[17] Let X be a nonempty set, (Y, d) be a complete metric space, and $\Lambda : Y^X \to Y^X$ be a non-decreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty} \Lambda\delta_n = 0$$

for every sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in Y^X with

$$\lim_{n\to\infty} \delta_n = 0$$

Suppose that $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$d\big(\mathcal{T}\xi(x), \mathcal{T}\mu(x)\big) \leq \Lambda\big(\Delta(\xi, \mu)\big)(x), \;\; \xi, \mu \in Y^X, \;\; x \in X, \tag{1.3}$$

where $\Delta : Y^X \times Y^X :\to \mathbb{R}_+^X$ is a mapping which is defined by

$$\Delta(\xi, \mu)(x) := d\big(\xi(x), \mu(x)\big), \;\; \xi, \mu \in Y^X, \;\; x \in X. \tag{1.4}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$d\big((\mathcal{T}\varphi)(x), \varphi(x)\big) \leq \varepsilon(x) \tag{1.5}$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} \big(\Lambda^n\varepsilon\big)(x) < \infty \tag{1.6}$$

for all $x \in X$, then the limit

$$\lim_{n\to\infty} \big(\mathcal{T}^n\varphi\big)(x) \tag{1.7}$$

exists for each $x \in X$. Moreover, the function $\psi \in Y^X$ defined by

$$\psi(x) := \lim_{n\to\infty} \big(\mathcal{T}^n\varphi\big)(x) \tag{1.8}$$

is a fixed point of $\mathcal{T}$ with

$$d\big(\varphi(x)\ ,\ \psi(x)\big) \leq \varepsilon^*(x) \tag{1.9}$$

for all $x \in X$.

In 2013, Brzdęk [19] gave the fixed point result by applying Theorem 1.2 as follows:

Theorem 1.3. *[19] Let X be a nonempty set, (Y,d) be a complete metric space, $f_1,\cdots,f_r : X \to X$ and $L_1,\dots,L_r : X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ are two operators satisfying the conditions*

$$d\big(\mathcal{T}\xi(x)\ ,\ \mathcal{T}\mu(x)\big) \le \sum_{i=1}^{r} L_i(x) d\Big(\xi\big(f_i(x)\big), \mu\big(f_i(x)\big)\Big), \tag{1.10}$$

for all $\xi,\mu \in Y^X, x \in X$ and

$$\Lambda\delta(x) := \sum_{i=1}^{r} L_i(x)\delta\big(f_i(x)\big), \quad \delta \in \mathbb{R}_+^X, x \in X. \tag{1.11}$$

If there exist functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ such that

$$d\Big(\mathcal{T}\varphi(x)\ ,\ \varphi(x)\Big) \le \varepsilon(x) \tag{1.12}$$

and

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n\varepsilon)(x) < \infty \tag{1.13}$$

for all $x \in X$, then the limit (1.7) exists for each $x \in X$. Moreover, the function (1.8) is a fixed point of $\mathcal{T}$ with (1.9) for all $x \in X$.

Brzdęk [19] then used this theorem to improved, extended, and complemented several earlier classical stability results concerning the additive Cauchy equation. Many papers on the stability and hyperstability of functional equations were published thanks to this important achievement. For example, we refer to [1]-[11], [22]-[24], and [40]. Another point worth noting is that there were other versions of Theorem 1.3 in ultrametric space [5], in 2-Banach space [6], [22], and in n-Banach space [23] that helped to discuss many results on the stability of functional equations. For more details on the stability and hyperstability in 2-Banach spaces and n-Banach spaces, we refer the reader to seeing the survey [12].
M. Almahalebi et al. [9] proved a new version of Theorem 1.3 in ultrametric n-Banach space as follows:

Theorem 1.4. *Let $m \in \mathbb{N}$. Supposing that:*

(i) X is a nonempty set, $\big(Y, \|\cdot,\cdots,\cdot\|_*\big)$ is an ultrametric $(m+1)$-normed space on a non-Archimedean field and $g : X \to Y$ is a surjective mapping,

(ii) The mappings $f_1,\dots,f_r : X \to X$, $L_1,\dots,L_r : X \to \mathbb{R}_+$ and the operator $\mathcal{T} : Y^X \to Y^X$ are such that

$$\Big\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x), g(z_1), \cdots, g(z_m)\Big\|_* \le \max_{1\le i\le r}\Big\{L_i(x)\Big\|\xi\left(f_i(x)\right)-\mu\big(f_i(x)\big), g(z_1), \cdots, g(z_m)\Big\|_*\Big\} \tag{1.14}$$

for all $\xi,\mu \in Y^X$ and all $x, z_1,\cdots,z_m \in X$,

(iii) The functions $\varepsilon : X^{m+1} \to \mathbb{R}_+$ and $\varphi : X \to Y$ are such that

$$\left\|\mathcal{T}\varphi(x) - \varphi(x), g(z_1), \cdots, g(z_m)\right\|_* \le \varepsilon(x, z_1, \cdots, z_m) \tag{1.15}$$

and the operator $\Lambda : \mathbb{R}_+^{X^{m+1}} \to \mathbb{R}_+^{X^{m+1}}$ is such that

$$\lim_{n\to\infty} \Lambda^n\varepsilon(x, z_1, \cdots, z_m) = 0 \tag{1.16}$$

for all $x, z_1, \cdots, z_m \in X$, where

$$\Lambda\delta(x, z_1, \cdots, z_m) := \max_{1\le i\le r}\Big\{L_i(x)\delta\big(f_i(x), z_1, \cdots, z_m\big)\Big\}, \quad \delta \in \mathbb{R}_+^{X^{m+1}}, x, z_1, \cdots, z_m \in X. \tag{1.17}$$

Then we have:

(i) For each $x, z_1, \cdots, z_m \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x) \tag{1.18}$$

exists and the function $\psi : X \to Y$, defined in this way, is the unique fixed point of $\mathcal{T}$ with

$$\left\|\varphi(x) - \psi(x), g(z_1), \cdots, g(z_m)\right\|_* \leq \sup_{n\in\mathbb{N}_0} \left\{\Lambda^n \varepsilon(x, z_1, \cdots, z_m)\right\}, \tag{1.19}$$

(ii) If

$$\Lambda\left(\sup_{n\in\mathbb{N}_0} \left\{\Lambda^n \varepsilon(x, z_1, \cdots, z_m)\right\}\right) \leq \sup_{n\in\mathbb{N}_0} \left\{\Lambda^{n+1} \varepsilon(x, z_1, \cdots, z_m)\right\}, \tag{1.20}$$

then ψ is the unique fixed point of $\mathcal{T}$ satisfying (1.19).

Let X and E be real vector spaces. If an additive function $\sigma : X \to Y$ satisfies $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in X$, then σ is called an involution of X see [41]. For given involutions $\sigma, \tau : X \to X$, the functional equation

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x), \quad x, y \in X \tag{1.21}$$

is called a Jensen functional equation with involutions. The general solution of (1.21) has been given by B. Fadli et al. [29] when $f : X \to Y$ with X is a commutative semigroup and Y is a commutative group (2-torsion free in the first equation and uniquely 2-divisible in the second). Namely, they prove the following theorem.

Theorem 1.5. *[29] Suppose that X is a 2-torsion free. The general solution $f : X \to Y$ of the functional equation (1.21) is given by $f(x) = A(x) + a$, $x \in X$ where $A : X \to Y$ is an additive mapping such that $A \circ \tau = -A \circ \sigma$ and $a \in Y$ is a constant.*

Using Theorem 1.4 as a basic tool, we discuss some hyperstability results for the equation (1.21) in ultrametric n-Banach spaces.

2 Preliminaries

The concept of n-normed space was given by A. Misiak [38] as a generalization of the notions of classical normed space and of a 2-normed space introduced by S. Gähler [30], [31]. We need to recall some basic facts concerning n-normed spaces and some preliminary results.

Definition 2.1. [38] Let $n \in \mathbb{N}_2$, X be a real linear space with $\dim X \geq n$. An n-norm on X is a real function $\|\cdot, \cdots, \cdot\| : X^n \to [0, \infty)$ satisfies the following conditions:

(i) $\|x_1, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,

(ii) $\|x_1, \cdots, x_n\| = \|x_{i_1}, \cdots, x_{i_n}\|$ for every permutaion $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$,

(iii) $\|\alpha x_1, \cdots, x_n\| = |\alpha| \, \|x_1, \cdots, x_n\|$,

(iv) $\|x_1 + y, x_2, \cdots, x_n\| \leq \|x_1, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$

for all $\alpha \in \mathbb{R}$, and all $x, y, x_1, \cdots, x_n \in X$. The pair $(X, \|\cdot, \cdots, \cdot\|)$ is called *an n-normed space*.

We note that $\|x_1, \cdots, x_n\| \geq 0$ for all $x_1, \cdots, x_n \in X$ because

$$\begin{aligned} 2\|x_1, \cdots, x_n\| &\geq \|x_1 - x_1, \cdots, x_n\| \\ &= \|0, \cdots, x_n\| \\ &= 0. \end{aligned}$$

Example 2.2. $\mathbb{R}^n$ equipped with the function $\left\|\cdot,\cdots,\cdot\right\|_E$ defined by

$$\left\|x_1,\cdots,x_n\right\|_E = \left|\det\left(x_{ij}\right)\right| = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1},\cdots,x_{in}) \in \mathbb{R}^n$ for $i \in \{1,\cdots,n\}$, is n-normed space.

Lemma 2.3. *[24] Let $\left(X,\left\|\cdot,\cdots,\cdot\right\|\right)$ be an n-normed space. If $\left\{x_k\right\}_{k\in\mathbb{N}}$ is a convergent sequence of elements of X, then*

$$\lim_{k\to\infty}\left\|x_k,y_2,\cdots,y_n\right\| = \left\|\lim_{k\to\infty} x_k,y_2,\cdots,y_n\right\|, \quad \textit{for every } \ y_2,\cdots,y_n \in X.$$

K. Hensel [33] has presented a normed space which does not have the Archimedean property. The non-Archimedean framework is of particular relevance since the theory of non-Archimedean spaces has piqued the interest of physicists for their research, particularly in quantum physics difficulties, p-adic strings, and superstrings.

In the following, we present some basic concepts on the non-Archimedean normed spces (or more details, we refer to [27]).

Definition 2.4. [27] Let $\mathbb{K}$ be a field. A valuation on $\mathbb{K}$ is a map $\left|\cdot\right| : \mathbb{K} \to \mathbb{R}$ such that for some real number $C \geq 1$, the following hold:

(i) $\left|x\right| \geq 0$ for any $x \in \mathbb{K}$ with equality only for $x = 0$,

(ii) $\left|xy\right| = \left|x\right|\cdot\left|y\right|$ for any $x, y \in \mathbb{K}$,

(iii) For any $x \in \mathbb{K}$, if $|x| \leq 1$, then $|x+1| \leq C$.

The valuation $\left|\cdot\right|$ such that $\left|x\right| = 1$ for every non zero x and $\left|0\right| = 0$ is called *the trivial valuation*.

Definition 2.5. [27] A valuation $\left|\cdot\right|$ on $\mathbb{K}$ satisfies the ultrametric inequality if for any $x, y \in \mathbb{K}$

$$\left|x+y\right| \leq \max\left\{\left|x\right|, \left|y\right|\right\}.$$

Such valuation is called *a non-Archimedean valuation*.

Proposition 2.6. *[27] A valuation $|.|$ on $\mathbb{K}$ satisfies the ultrametric inequality if and only if one can take $C = 1$ in Definition 2.4.*

Example 2.7. (Non-Archimedean valued field)

Let p be a fixed prime number. Because of the unique fraction in $\mathbb{Z}$, every non-zero rational number x can be written as

$$x = \frac{a}{b}p^n$$

where n, a, and b are integers and $\gcd(p, ab) = 1$. We can define a valuation on $\mathbb{Q}$ as follows:

$$\left|x\right|_p = \begin{cases} p^{-n} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\left|\cdot\right|_p$ is called *the p-adic valuation*. The completion of $\mathbb{Q}$ with respect to $\left|\cdot\right|_p$ is called *the field of p-adic numbers* and is denoted $\mathbb{Q}_p$.

By the trivial valuation we mean the function $|\cdot|$ taking everything but 0 into 1 and $\left|0\right| = 0$. In any non-Archimedean field, we have $\left|1\right| = \left|-1\right| = 1$ and $\left|n\right| \leq 1$ for $n \in \mathbb{N}$.

Definition 2.8. Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $\left|\cdot\right|$. A non-Archimedean norm on X is a map $\left\|\cdot\right\|_* : X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $\|x\|_* = 0$ if and only if $x = 0$,

(ii) $\|\lambda x\|_* = |\lambda| \, \|x\|_*$, for any $x \in X$ and any $\lambda \in \mathbb{K}$,

(iii) $\|x + y\|_* \leq \max \{\|x\|_*, \, \|y\|_*\}$, for any $x, y \in X$.

Condition (3) of Definition 2.8 is referred to as the ultrametric or strong triangle inequality. The paire $(X, \|\cdot\|_*)$ is called a non-Archimedean normed space or an ultrametric normed space. For example, the paire $(\mathbb{Q}_p, |\cdot|_p)$ is a non-Archimedean normed space.

Definition 2.9. [26] Let X be a vector space with $\dim X \geq n$ over a valued field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot, \cdots, \cdot\|_* : X^n \to [0, \infty)$ is said to be *a non-Archimedean n-norm* if

(i) $\|x_1, \cdots, x_n\|_* = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,

(ii) $\|x_1, \cdots, x_n\|_* = \|x_{i_1}, \cdots, x_{i_n}\|_*$ for every permutaion $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$,

(iii) $\|\alpha x_1, \cdots, x_n\|_* = |\alpha| \, \|x_1, \cdots, x_n\|_*$,

(iv) $\|x + y, x_2, \cdots, x_n\|_* \leq \max \{\|x, x_2, \cdots, x_n\|_*, \, \|y, x_2, \cdots, x_n\|_*\}$

for all $\alpha \in \mathbb{K}$, and all $x, y, x_1, \cdots, x_n \in X$. Then $(X, \|\cdot, \cdots, \cdot\|_*)$ is called *a non-Archimedean n-normed space* or *an ultrametric n-normed space*.

Example 2.10. Let p be a fixed prime number. We defined an ultrametric n-norm on $\mathbb{Q}_p^n$ by

$$\|x_1, \cdots, x_n\|_* = \left| \det (x_{ij}) \right|_p,$$

where $x_i = (x_{i1}, \cdots, x_{in}) \in \mathbb{Q}_p^n$ for $i \in \{1, \cdots, n\}$.

If $(X, \|\cdot, \cdots, \cdot\|_*)$ is an ultrametric n-normed space, then

$$\left\| \sum_{i=1}^{k} y_i, x_2, \cdots, x_n \right\|_* \leq \max_{1 \leq i \leq k} \left\{ \|y_i, x_2, \cdots, x_n\|_* \right\}$$

for any $k \in \mathbb{N}_2, x_2, \cdots, x_n \in X$ and all $y_i \in X$ for $i \in \{1, \cdots, n\}$.

According to the conditions in Definition 2.9, we have the following lemma.

Lemma 2.11. *Let $(X, \|\cdot, \cdots, \|_*)$ be an ultrametric n-normed space. If $z_1, \cdots, z_n \in X$ are linearly independent, $x \in X$ and*

$$\|x, w_2, \cdots, w_n\|_* = 0 \quad \textit{for every } w_2, \cdots, w_n \in \{z_1, \cdots, z_n\},$$

then $x = 0$.

Lemma 2.12. *A sequence $(y_k)_{k \in \mathbb{N}}$ of elements of an ultrametric n-normed space $(X, \|\cdot, \cdots, \cdot\|_*)$ is called a Cauchy sequence if there are linear independent points $z_1, \cdots, z_n \in X$ such that*

$$\lim_{k, \ell \to \infty} \|y_k - y_\ell, w_2, \cdots, w_n\|_* = 0 \quad \textit{for every } w_2, \cdots, w_n \in \{z_1, \cdots, z_n\}.$$

Definition 2.13. A sequence $\{y_k\}_{k \in \mathbb{N}}$ is said to be *convergent* if there exists a $y \in X$ with

$$\lim_{k \to \infty} \|y_k - y, x_2, \cdots, x_n\|_* = 0 \quad \text{for every } x_2, \cdots, x_n \in X.$$

In this case, we call that $\{y_k\}_{k \in \mathbb{N}}$ converges to y or that y is the limit of $\{y_k\}_{k \in \mathbb{N}}$ and we write $\{y_k\}_{k \in \mathbb{N}} \to y$ as $k \to \infty$.

By condition (4) in Definition 2.9, we have

$$\|y_k - y_\ell, x_2, \cdots, x_n\|_* \leq \max_{\ell \leq j \leq k-1} \left\{ \|y_{j+1} - y_j, x_2, \cdots, x_n\|_* \right\}, \quad (\ell < k)$$

for all $x_2, \cdots, x_n \in X$. Therefore, a sequence $\{y_k\}_{k \in \mathbb{N}}$ is Cauchy in $(X, \|\cdot, \cdots, \cdot\|_*)$ if and only if $\{y_{k+1} - y_k\}_{k \in \mathbb{N}}$ converges to zero in an ultrametric n-normed space $(X, \|\cdot, \cdots, \cdot\|_*)$.

Definition 2.14. If every Cauchy sequence in an ultrametric n-normed space $(X, \|\cdot, \cdots, \cdot\|_*)$ converges to some $y \in X$, then $(X, \|\cdot, \cdots, \cdot\|_*)$ is said to be *complete*. Any complete ultrametric n-normed space is said to be *an ultrametric n-Banach space*.

Now we state the following results as a lemma.

Lemma 2.15. *Let $(X, \|\cdot, \cdots, \cdot\|_*)$ be an ultrametric n-normed space. Then the following conditions hold:*

(i) $\Big|\|x, x_2, \cdots, x_n\|_* - \|y, x_2, \cdots, x_n\|_*\Big| \leq \|x - y, x_2, \cdots, x_n\|_*$ *for all $x, y, x_2, \cdots, x_n \in X$,*

(ii) if $x \in X$ and $\|x, x_2, \cdots, x_n\|_* = 0$ for all $x_2, \cdots, x_n \in X$, then $x = 0$,

(iii) if $\{x_k\}_{k\in\mathbb{N}}$ is a convergent sequence of elements of X, then

$$\lim_{k\longrightarrow\infty} \|x_k, y_2, \cdots, y_n\|_* = \Big\|\lim_{k\to\infty} x_k, y_2, \cdots, y_n\Big\|_* \quad \text{for all } y_2, \cdots, y_n \in X.$$

3 Hyperstability results

In this section, we assume $p \in \mathbb{N}$, $(X, \|\cdot\|)$ is a normed space, and $(Y, \|\cdot, \cdots, \cdot\|_*)$ is an ultrametric $(p+1)$-Banach space on a non-Archimedean field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|_* : \mathbb{K} \to \mathbb{R}_+$.

Theorem 3.1. *Let $h_1, h_2 : X^{p+2} \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{U} := \Big\{n \in \mathbb{N} : \alpha_n := \max\big\{\lambda_1^\sigma(n)\lambda_2^\sigma(n), \lambda_1^{\sigma,\tau}(n)\lambda_2^{\sigma,\tau}(n)\big\} < 1\Big\},$$

where

$$\lambda_i(n) := \inf\big\{t \in \mathbb{R}_+ : h_i(nx, z_1, \cdots, z_p) \leq t\, h_i(x, z_1, \cdots, z_p)\big\},$$

$$\lambda_i^\sigma(n) := \inf\big\{t \in \mathbb{R}_+ : h_i\big(x + n\sigma(x), z_1, \cdots, z_p\big) \leq t\, h_i(x, z_1, \cdots, z_p)\big\},$$

and

$$\lambda_i^{\sigma,\tau}(n) := \inf\big\{t \in \mathbb{R}_+ : h_i\big(x + n\sigma(x) - n\tau(x), z_1, \cdots, z_p\big) \leq t\, h_i(x, z_1, \cdots, z_p)\big\}$$

for all $x, z_1, \cdots, z_p \in X$ and $n \in \mathbb{N}$, where $i = 1, 2$ such that

$$\lim_{n\to\infty} \lambda_1^\sigma(n)\lambda_2(-n) = 0.$$

Suppose that $f : X \to Y$ satisfies the inequality

$$\Big\|f\big(x + \sigma(y)\big) + f\big(x + \tau(y)\big) - 2f(x), g(z_1), \cdots, g(z_p)\Big\|_* \leq h_1(x, z_1, \cdots, z_p)h_2(y, z_1, \cdots, z_p), \tag{3.1}$$

for all $x, y, z_1 \cdots, z_p \in X_0$ where $g : X \to Y$ is a surjective mapping. Then f is a solution of (1.21) on X_0.

Proof. Replacing x by $x + m\sigma(x)$ and y by $-mx$ in (3.1) where $m \in \mathcal{U}$, we get

$$\Big\|f(x) + f\big(x + m\sigma(x) - m\tau(x)\big) - 2f\big(x + m\sigma(x)\big), g(z_1), \cdots, g(z_p)\Big\|_* \leq h_1\big(x + m\sigma(x), z_1, \cdots, z_p\big)h_2(-mx, z_1, \cdots, z_p), \tag{3.2}$$

for all $x, z_1, \cdots, z_p \in X_0$. For every $m \in \mathcal{U}$, we can define the operators $\mathcal{T}_m : Y^{X_0} \to Y^{X_0}$ and $\Lambda_m : \mathbb{R}_+^{X_0^{p+1}} \to \mathbb{R}_+^{X_0^{p+1}}$ by

$$\mathcal{T}_m\xi(x) := 2\xi\big(x + m\sigma(x)\big) - \xi\big(x + m\sigma(x) - m\tau(x)\big) \tag{3.3}$$

and

$$\Lambda_m \delta(x, z_1, \cdots, z_p) := \max\Big\{\delta\big(x + m\sigma(x), z_1, \cdots, z_p\big), \delta\big(x + m\sigma(x) - m\tau(x), z_1, \cdots, z_p\big)\Big\} \tag{3.4}$$

for all $x, z_1, \cdots, z_p \in X_0$, $\xi \in Y^{X_0}$, and $\delta \in \mathbb{R}_+^{X_0^{p+1}}$. Note that, for every $m \in \mathcal{U}$, the operator $\Lambda := \Lambda_m$ has the form given in (1.17) with $X := X_0$, $r = 2$, $L_1(x) = L_2(x) = 1$, $f_1(x) = x + m\sigma(x)$, and $f_2(x) = x + m\sigma(x) - m\tau(x)$.
Furthermore, when we put

$$\varepsilon_m(x, z_1, \cdots, z_p) := h_1\big(x + m\sigma(x), z_1, \cdots, z_p\big) h_2(-mx, z_1, \cdots, z_p),$$

the inequality (3.2) becomes

$$\Big\|\mathcal{T}_m f(x) - f(x), g(z_1), \cdots, g(z_p)\Big\|_* \leq \varepsilon_m(x, z_1, \cdots, z_p), \quad x, z_1, \cdots, z_p \in X_0, \quad m \in \mathcal{U}. \tag{3.5}$$

From here till the end of the paper, we denote by f the restriction of $f : X \to Y$ to the set $X_0 \subset X$ unless we mention otherwise. Moreover, for every $\xi, \mu \in Y^{X_0}$, we have

$$\begin{aligned}
\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), g(z_1), \cdots, g(z_p)\Big\|_* &= \Big\|2\xi\big(x + m\sigma(x)\big) - \xi\big(x + m\sigma(x) - m\tau(x)\big) \\
&\quad - 2\mu\big(x + m\sigma(x)\big) + \mu\big(x + m\sigma(x) - m\tau(x)\big), g(z_1), \cdots, g(z_p)\Big\|_* \\
&\leq \max\Big\{|2|_*\Big\|\xi\big(x + m\sigma(x)\big) - \mu\big(x + m\sigma(x)\big), g(z_1), \cdots, g(z_p)\Big\|_* \\
&\quad, \Big\|\xi\big(x + m\sigma(x) - m\tau(x)\big) - \mu\big(x + m\sigma(x) - m\tau(x)\big), g(z_1), \cdots, g(z_p)\Big\|_*\Big\} \\
&\leq \max\Big\{\Big\|\xi\big(x + m\sigma(x)\big) - \mu\big(x + m\sigma(x)\big), g(z_1), \cdots, g(z_p)\Big\|_* \\
&\quad, \Big\|\xi\big(x + m\sigma(x) - m\tau(x)\big) - \mu\big(x + m\sigma(x) - m\tau(x)\big), g(z_1), \cdots, g(z_p)\Big\|_*\Big\} \\
&= \max_{i=1,2}\Big\{L_i(x)\Big\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g(z_1), \cdots, g(z_p)\Big\|_*\Big\}, \quad x, z_1, \cdots, z_p \in X_0,\ m \in \mathcal{U}.
\end{aligned}$$

This means that (1.14) holds for $\mathcal{T} := \mathcal{T}_m$ for any $m \in \mathcal{U}$. Next, by the definitions of λ_i, λ_i^σ, and $\lambda_i^{\sigma,\tau}$, we have

$$\varepsilon_m(x, z_1, \cdots, z_p) \leq \lambda_1^\sigma(m)\lambda_2(-m)\ h_1(x, z_1, \cdots, z_p)\, h_2(x, z_1, \cdots, z_p), \quad x, z_1, \cdots, z_p \in X_0, \quad m \in \mathcal{U}, \tag{3.6}$$

therefore, by induction, we directly obtain that

$$\Lambda_m^n \varepsilon_m(x, z_1, \cdots, z_p) \leq \lambda_1^\sigma(m)\lambda_2(-m)\ \alpha_m^n h_1(x, z_1, \cdots, z_p)\, h_2(x, z_1, \cdots, z_p) \tag{3.7}$$

for all $x, z_1, \cdots, z_p \in X_0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$, where

$$\alpha_m = \max\Big\{\lambda_1^\sigma(m)\lambda_2^\sigma(m), \lambda_1^{\sigma,\tau}(m)\lambda_2^{\sigma,\tau}(m)\Big\}, \quad m \in \mathcal{U}.$$

Letting $n \to \infty$ in (3.7), we get

$$\lim_{n\to\infty} \Lambda_m^n \varepsilon_m(x, z_1, \cdots, z_p) = 0, \quad x, z_1, \cdots, z_p \in X_0,\ m \in \mathcal{U}.$$

As described above, we deduce that all assumptions of Theorem 1.4 hold. Therefore, there is, for every $m \in \mathcal{U}$, a unique fixed point $J_m : X_0 \to Y$ of the operator $\mathcal{T}_m$ defined by

$$J_m(x) := \lim_{n\to\infty} \mathcal{T}_m^n f(x), \quad x \in X_0, m \in \mathcal{U}$$

such that

$$\left\| f(x) - J_m(x), g(z_1), \cdots, g(z_p) \right\|_* \leq \sup_{n\in\mathbb{N}_0} \left\{ \Lambda_m^n \varepsilon_m(x, z_1, \cdots, z_p) \right\}, \quad x, z_1, \cdots, z_p \in X_0, m \in \mathcal{U}. \tag{3.8}$$

It means that

$$J_m(x) = 2J_m\big(x + m\sigma(x)\big) - J_m\big(x + m\sigma(x) - m\tau(x)\big), \quad x \in X_0, m \in \mathcal{U}. \tag{3.9}$$

Now, for each $m \in \mathcal{U}$ and $x, y, z_1, \cdots, z_p \in X_0$ such that $x + \sigma(y) \neq 0$ and $x + \tau(y) \neq 0$, we prove that

$$\begin{aligned} &\left\| \mathcal{T}_m^n f\big(x + \sigma(y)\big) + \mathcal{T}_m^n f\big(x + \tau(y)\big) - 2\mathcal{T}_m^n f(x), g(z_1), \cdots, g(z_p) \right\|_* \\ &\qquad\qquad \leq \alpha_m^n h_1(x, z_1, \cdots, z_p)\, h_2(y, z_1, \cdots, z_p), \end{aligned} \tag{3.10}$$

for any $n \in \mathbb{N}_0$. It is clear that if $n = 0$, then (3.10) holds by (3.1). Fix an $n \in \mathbb{N}_0$ and assume that (3.10) holds for any $m \in \mathcal{U}$ and $x, y, z_1, \cdots, z_p \in X_0$ such that $x + \sigma(y) \neq 0$ and $x + \tau(y) \neq 0$. Then, in view of (3.10), we get

$$\begin{aligned} &\left\| \mathcal{T}_m^{n+1} f\big(x + \sigma(y)\big) + \mathcal{T}_m^{n+1} f\big(x + \tau(y)\big) - 2\mathcal{T}_m^{n+1} f(x), g(z_1), \cdots, g(z_p) \right\|_* \\ &= \Bigg\| 2\mathcal{T}_m^n f\Big(\big(x + \sigma(y)\big) + m\sigma\big(x + \sigma(y)\big)\Big) - \mathcal{T}_m^n f\Big(\big(x + \sigma(y)\big) + m\sigma\big(x + \sigma(y)\big) - m\tau\big(x + \sigma(y)\big)\Big) \\ &+ 2\mathcal{T}_m^n f\Big(\big(x + \tau(y)\big) + m\sigma\big(x + \tau(y)\big)\Big) - \mathcal{T}_m^n f\Big(\big(x + \tau(y)\big) + m\sigma\big(x + \tau(y)\big) - m\tau\big(x + \tau(y)\big)\Big) \\ &- 4\mathcal{T}_m^n f\big(x + m\sigma(x)\big) + 2\mathcal{T}_m^n f\big(x + m\sigma(x) - m\tau(x)\big), g(z_1), \cdots, g(z_p) \Bigg\|_* \\ &\leq \max \Bigg\{ |2|_* \Bigg\| \mathcal{T}_m^n f\Big(\big(x + \sigma(y)\big) + m\sigma\big(x + \sigma(y)\big)\Big) + \mathcal{T}_m^n f\Big(\big(x + \tau(y)\big) + m\sigma\big(x + \tau(y)\big)\Big) \\ &\qquad - 2\mathcal{T}_m^n f\big(x + m\sigma(x)\big), g(z_1), \cdots, g(z_p) \Bigg\|_* \\ &\qquad , \Bigg\| \mathcal{T}_m^n f\Big(\big(x + \sigma(y)\big) + m\sigma\big(x + \sigma(y)\big) - m\tau\big(x + \sigma(y)\big)\Big) \\ &\qquad + \mathcal{T}_m^n f\Big(\big(x + \tau(y)\big) + m\sigma\big(x + \tau(y)\big) - m\tau\big(x + \tau(y)\big)\Big) \\ &\qquad - 2\mathcal{T}_m^n f\big(x + m\sigma(x) - m\tau(x)\big), g(z_1), \cdots, g(z_p) \Bigg\|_* \Bigg\} \end{aligned}$$

$$\leq \max\Big\{\Big\|\mathcal{T}_m^n f\Big(\big(x+\sigma(y)\big)+m\sigma\big(x+\sigma(y)\big)\Big)+\mathcal{T}_m^n f\Big(\big(x+\tau(y)\big)+m\sigma\big(x+\tau(y)\big)\Big)$$

$$-2\mathcal{T}_m^n f\big(x+m\sigma(x)\big),g(z_1),\cdots,g(z_p)\Big\|_*$$

$$,\Big\|\mathcal{T}_m^n f\Big(\big(x+\sigma(y)\big)+m\sigma\big(x+\sigma(y)\big)-m\tau\big(x+\sigma(y)\big)\Big)$$

$$+\mathcal{T}_m^n f\Big(\big(x+\tau(y)\big)+m\sigma\big(x+\tau(y)\big)-m\tau\big(x+\tau(y)\big)\Big)$$

$$-2\mathcal{T}_m^n f\big(x+m\sigma(x)-m\tau(x)\big),g(z_1),\cdots,g(z_p)\Big\|_*\Big\}$$

$$\leq \alpha_m^n \max\Big\{h_1\big(x+m\sigma(x),z_1,\cdots,z_p\big)h_2\big(y+m\sigma(y),z_1,\cdots,z_p\big)$$

$$,h_1\big(x+m\sigma(x)-m\tau(x),z_1,\cdots,z_p\big)h_2\big(y+m\sigma(y)-m\tau(y),z_1,\cdots,z_p\big)\Big\}$$

$$\leq \alpha_m^n h_1\left(x,z_1,\cdots,z_p\right)h_2\left(y,z_1,\cdots,z_p\right)\max\Big\{\lambda_1^\sigma(m)\lambda_2^\sigma(m),\lambda_1^{\sigma,\tau}(m)\lambda_2^{\sigma,\tau}(m)\Big\}$$

$$=\alpha_m^{n+1}h_1\left(x,z_1,\cdots,z_p\right)h_2\left(y,z_1,\cdots,z_p\right)$$

By mathematical induction, we deduce that (3.10) holds for any $n\in\mathbb{N}_0$. Letting $n\longrightarrow\infty$ in (3.10) and using the surjectivity of g in view of Lemma 2.11, we obtain the equality

$$J_m\big(x+\sigma(y)\big)+J_m\big(x+\tau(y)\big)=2J_m(x),\quad x,y,(x+\sigma(y)),(x+\tau(y))\in X_0,m\in\mathcal{U}. \tag{3.11}$$

In this way, we find a sequence $\{J_m\}_{m\in\mathcal{U}}$ of a Jensen functions with involutions on X_0 such that

$$\Big\|f(x)-J_m(x),g\left(z_1\right),\cdots,g\left(z_p\right)\Big\|_*\leq\sup_{n\in\mathbb{N}_0}\left\{\Lambda_m^n\varepsilon_m\left(x,z_1,\cdots,z_p\right)\right\}$$
$$\leq\lambda_1^\sigma(m)\lambda_2(-m)h_1\left(x,z_1,\cdots,z_p\right)h_2\left(x,z_1,\cdots,z_p\right)\sup_{n\in\mathbb{N}_0}\left\{\alpha_m^n\right\} \tag{3.12}$$

for all $x,z_1,\cdots,z_p\in X_0$ and $m\in\mathcal{U}$. It follows, with $m\to\infty$, that f is a solution of (1.21) on X_0. □

By similar method, we can prove the following theorem.

Theorem 3.2. *Let $h:X^{p+2}\to\mathbb{R}_+$ be a function such that*

$$\mathcal{U}:=\Big\{n\in\mathbb{N}:\alpha_n:=\max\left\{\lambda^\sigma(n),\lambda^{\sigma,\tau}(n)\right\}<1\Big\},$$

where

$$\lambda(n):=\inf\big\{t\in\mathbb{R}_+:h(nx,z_1,\cdots,z_p)\leq t\,h(x,z_1,\cdots,z_p)\big\},$$

$$\lambda^\sigma(n):=\inf\big\{t\in\mathbb{R}_+:h\big(x+n\sigma(x),z_1,\cdots,z_p\big)\leq t\,h(x,z_1,\cdots,z_p)\big\},$$

and

$$\lambda^{\sigma,\tau}(n):=\inf\big\{t\in\mathbb{R}_+:h\big(x+n\sigma(x)-n\tau(x),z_1,\cdots,z_p\big)\leq t\,h(x,z_1,\cdots,z_p)\big\}$$

for all $x,,z_1,\cdots,z_p\in X$ and $n\in\mathbb{N}$, such that

$$\lim_{n\to\infty}\lambda^\sigma(n)+\lambda(-n)=0.$$

Suppose that $f : X \to Y$ satisfies the inequality

$$\left\|f\big(x+\sigma(y)\big)+f\big(x+\tau(y)\big)-2f(x), g(z_1),\cdots,g(z_p)\right\|_* \le h(x,z_1,\cdots,z_p)+h(y,z_1,\cdots,z_p), \tag{3.13}$$

for all $x, y, z_1 \cdots, z_p \in X_0$ where $g : X \to Y$ is a surjective mapping. Then f is a solution of (1.21) on X_0.

As particular cases, we introduce the following two corollaries concerning the inhomogeneity of the functional equation (1.21).

Corollary 3.3. *Let h_1, h_2 be two functions as in Theorem 3.1, and let $f : X \to Y$ and $G : X^2 \to Y$ be two functions such that $G(0,0) = 0$ and satisfy the inequality*

$$\left\|f\big(x+\sigma(y)\big)+f\big(x+\tau(y)\big)-2f(x)-G(x,y), g(z_1),\cdots,g(z_p)\right\|_* \le h_1(x,z_1,\cdots,z_p)h_2(y,z_1,\cdots,z_p), \tag{3.14}$$

for all $x, y, z_1 \cdots, z_p \in X_0$ where $g : X \to Y$ is a surjective mapping. If the functional equation

$$f\big(x+\sigma(y)\big)+f\big(x+\tau(y)\big) = 2f(x)+G(x,y) \tag{3.15}$$

has a solution $f_0 : X \to Y$, then f is a solution of (3.15) on X.

Proof. Let $\psi : X \to Y$ be a function defined by $\psi(x) := f(x) - f_0(x)$ for all $x \in X$. Then for all $x, y, z_1, \cdots, z_p \in X$, we have

$$\begin{aligned}
&\left\|\psi\big(x+\sigma(y)\big)+\psi\big(x+\tau(y)\big)-2\psi(x), g(z_1),\cdots,g(z_p)\right\|_* = \Big\|f\big(x+\sigma(y)\big)+f\big(x+\tau(y)\big)-2f(x)\\
&-G(x,y)-f_0\big(x+\sigma(y)\big)-f_0\big(x+\tau(y)\big)+2f_0(x)+G(x,y), g(z_1),\cdots,g(z_p)\Big\|_*\\
&=\left\|f\big(x+\sigma(y)\big)+f\big(x+\tau(y)\big)-2f(x)-G(x,y), g(z_1),\cdots,g(z_p)\right\|_*\\
&\le h_1(x,z_1,\cdots,z_p)h_2(y,z_1,\cdots,z_p).
\end{aligned}$$

Therefore, ψ is a solution of (3.15) on X. Moreover, we have

$$\begin{aligned}
f\big(x+\sigma(y)\big)+f\big(x+\tau(y)\big)-2f(x)-G(x,y) &= \psi\big(x+\sigma(y)\big)+\psi\big(x+\tau(y)\big)-2\psi(x)\\
&\quad+f_0\big(x+\sigma(y)\big)+f_0\big(x+\tau(y)\big)-2f_0(x)-G(x,y),\\
&=0
\end{aligned}$$

for all $x, y \in X$, which means that f is a solution of (3.15) on X. □

With an analogue proof of Corollary 3.3, we can prove the following corollary.

Corollary 3.4. *Let h be a function as in Theorem 3.2, and let $f : X \to Y$ and $G : X^2 \to Y$ be two functions such that $G(0,0) = 0$ and satisfy the inequality*

$$\left\|f\big(x+\sigma(y)\big)+f\big(x+\tau(y)\big)-2f(x)-G(x,y), g(z_1),\cdots,g(z_p)\right\|_* \le h(x,z_1,\cdots,z_p)+h(y,z_1,\cdots,z_p), \tag{3.16}$$

for all $x, y, z_1 \cdots, z_p \in X_0$ where $g : X \to Y$ is a surjective mapping. If the functional equation (3.15) has a solution $f_0 : X \to Y$, then f is a solution of (3.15) on X.

In the case σ and τ are homomorphisms such that $\sigma = id_X$ and $\tau = -id_X$ with X is a real or complex vector space, we get the following Cauchy-Jensen functional equation

$$f(x+y)+f(x-y) = 2f(x), \quad x, y \in X. \tag{3.17}$$

Therefore, we can derive the following corollaries as consequences of Theorems 3.1 and 3.2.

Corollary 3.5. *Let $s, t, r, \theta \in \mathbb{R}$ such that $s < 0$, $t < 0$ and $r, \theta \geq 0$. If $f : X \to Y$ satisfies*

$$\left\| f(x+y) + f(x-y) - 2f(x), g(z_1), \cdots, g(z_p) \right\|_* \leq \theta \|x\|^s \|y\|^t \prod_{i=1}^{p} \|z_i\|^r, \tag{3.18}$$

for all $x, y, z_1 \cdots, z_p \in X_0$, then $f(x) = A(x) + a$, $x \in X_0$, where $A : X_0 \to Y$ is an additive mapping and $a \in Y$ is an arbitrary constant.

Proof. It is easily seen that the functions h_1 and h_2 given by

$$h_1(x, z_1 \cdots, z_p) := \theta_1 \|x\|^s \prod_{i=1}^{p} \|z_i\|^{r_1}$$

and

$$h_2(x, z_1 \cdots, z_p) := \theta_2 \|y\|^t \prod_{i=1}^{p} \|z_i\|^{r_2},$$

for all $x, y, z_1 \cdots, z_p \in X_0$ where $\theta_1 \times \theta_2 = \theta$ and $r_1 + r_2 = r$, satisfy all the conditions in Theorem 3.1 because, for each $m \in \mathbb{N}$, we have

$$\begin{aligned}
\lambda_1(-m) &= \inf \Big\{ \beta \in \mathbb{R}_+ : h_1\big(-mx, z_1 \cdots, z_p\big) \leq \beta h_1(x, z_1 \cdots, z_p) \Big\} \\
&= \inf \Big\{ \beta \in \mathbb{R}_+ : \theta_1 \| -mx\|^s \prod_{i=1}^{p} \|z_i\|^{r_1} \leq \beta \theta_1 \|x\|^s \prod_{i=1}^{p} \|z_i\|^{r_1} \Big\} \\
&= |m|^s,
\end{aligned}$$

$$\begin{aligned}
\lambda_1^\sigma(m) &= \inf \Big\{ \beta \in \mathbb{R}_+ : h_1\big(x + m\sigma(x), z_1 \cdots, z_p\big) \leq \beta h_1(x, z_1 \cdots, z_p) \Big\} \\
&= \inf \Big\{ \beta \in \mathbb{R}_+ : \theta_1 \|(m+1)x\|^s \prod_{i=1}^{p} \|z_i\|^{r_1} \leq \beta \theta_1 \|x\|^s \prod_{i=1}^{p} \|z_i\|^{r_1} \Big\} \\
&= \big|m+1\big|^s,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_1^{\sigma,\tau}(m) &= \inf \Big\{ \beta \in \mathbb{R}_+ : h_1\big(x + m\sigma(x) - m\tau(x), z_1 \cdots, z_p\big) \leq \beta h_1(x, z_1 \cdots, z_p) \Big\} \\
&= \inf \Big\{ \beta \in \mathbb{R}_+ : \theta_1 \|(2m+1)x\|^s \prod_{i=1}^{p} \|z_i\|^{r_1} \leq \beta \theta_1 \|x\|^s \prod_{i=1}^{p} \|z_i\|^{r_1} \Big\} \\
&= \big|2m+1\big|^s,
\end{aligned}$$

also, we have $\lambda_2(-m) = |m|^t$, $\lambda_2^\sigma(m) = \big|m+1\big|^t$, and $\lambda_2^{\sigma,\tau}(m) = \big|2m+1\big|^t$ for all $m \in \mathcal{U}$. Then we obtain

$$\begin{aligned}
\alpha_m &= \max \Big\{ \lambda_1^\sigma(m)\lambda_2^\sigma(m), \lambda_1^{\sigma,\tau}(m)\lambda_2^{\sigma,\tau}(m) \Big\}, \quad m \in \mathcal{U} \\
&= \max \Big\{ \big|m+1\big|^{s+t}, \big|2m+1\big|^{s+t} \Big\} < 1, \quad m \in \mathcal{U}.
\end{aligned}$$

Since, $s + t < 0$, then we get

$$\lim_{m\to\infty} \lambda_1^\sigma(m)\lambda_2(-m) = \lim_{m\to\infty} \big|m+1\big|^s \big|m\big|^t = 0.$$

According to Theorem 3.1, we get the desired result. □

By a similar method, we have the following corollaries:

Corollary 3.6. *Let $s, r, \theta \in \mathbb{R}$ such that $s < 0$ and $r, \theta \geq 0$. If $f : X \to Y$ satisfies*

$$\left\| f(x+y) + f(x-y) - 2f(x), g(z_1), \cdots, g(z_p) \right\|_* \leq \theta \Big(\|x\|^s + \|y\|^s \Big) \prod_{i=1}^{p} \|z_i\|^r, \tag{3.19}$$

for all $x, y, z_1 \cdots, z_p \in X_0$, then $f(x) = A(x) + a$, $x \in X_0$, where $A : X_0 \to Y$ is an additive mapping and $a \in Y$ is an arbitrary constant.

Corollary 3.7. *Let $f : X \to Y$ and $G : X^2 \to Y$ be two functions such that $G(0,0) = 0$ and satisfy the inequality*

$$\left\| f(x+y) + f(x-y) - 2f(x) - G(x,y), g(z_1), \cdots, g(z_p) \right\|_* \leq \theta \|x\|^s \|y\|^t \prod_{i=1}^{p} \|z_i\|^r, \tag{3.20}$$

for all $x, y, z_1 \cdots, z_p \in X_0$ where $g : X \to Y$ is a surjective mapping, $s, t, r, \theta \in \mathbb{R}$ such that $s < 0$, $t < 0$ and $r, \theta \geq 0$. If the functional equation

$$f(x+y) + f(x-y) = 2f(x) + G(x,y) \tag{3.21}$$

has a solution $f_0 : X \to Y$, then f is a solution of (3.21) on X.

Corollary 3.8. *Let $f : X \to Y$ and $G : X^2 \to Y$ be two functions such that $G(0,0) = 0$ and satisfy the inequality*

$$\left\| f(x+y) + f(x-y) - 2f(x) - G(x,y), g(z_1), \cdots, g(z_p) \right\|_* \leq \theta \Big(\|x\|^s + \|y\|^s \Big) \prod_{i=1}^{p} \|z_i\|^r, \tag{3.22}$$

for all $x, y, z_1 \cdots, z_p \in X_0$ where $g : X \to Y$ is a surjective mapping, $s, r, \theta \in \mathbb{R}$ such that $s < 0$, and $r, \theta \geq 0$. If the functional equation (3.21) has a solution $f_0 : X \to Y$, then f is a solution of (3.21) on X.

In the following corollaries, we discuss some additional hyperstability results when X is a C^*-algebra. When we take the homomorphisms σ and τ of X as $\sigma(x) = x$ and $\tau(x) = x^*$, we conclude the following functional equation

$$f(x+y) + f\big(x+y^*\big) = 2f(x), \quad x, y \in X. \tag{3.23}$$

The solution of (3.23) is given as $f(x) = A(x) + a$, $x \in X$ where $A : X \to Y$ is an additive mapping and $a \in Y$ is an arbitrary constant such that $A\big(x^*\big) = -A(x)$.

Corollary 3.9. *Let $s, t, r, \theta \in \mathbb{R}$ such that $s < 0$, $t < 0$ and $r, \theta \geq 0$ and let Y be an ultrametric $(m+1)$-Banach space. Assume that the function $f : X \to Y$ verifies the inequality*

$$\left\| f(x+y) + f\big(x+y^*\big) - 2f(x), g(z_1), \cdots, g(z_p) \right\|_* \leq \theta \|x\|^s \|y\|^t \prod_{i=1}^{p} \|z_i\|^r, \tag{3.24}$$

for all $x, y, z_1 \cdots, z_p \in X_0$, then $f(x) = A(x) + a$, $x \in X_0$, where $A : X_0 \to Y$ is an additive mapping and $a \in Y$ is an arbitrary constant such that $A\big(x^*\big) = -A(x)$.

Proof. The proof follows from Corollary 3.5 by rewriting (3.18) as the following

$$\left\| f(x+y) + f(x-y) - 2f(x) - G(x,y), g(z_1), \cdots, g(z_p) \right\|_* \leq \theta \|x\|^s \|y\|^t \prod_{i=1}^{p} \|z_i\|^r, \tag{3.25}$$

where $G(x,y) := f(x-y) - f\big(x+y^*\big)$, $x, y \in X$. Since $G(0,0) = 0$ and the equation (3.23) has a solution on X, then according to Corollary 3.7, we conclude the desired result. □

Corollary 3.10. *Let $s, t, r, \theta \in \mathbb{R}$ such that $s < 0$ and $r, \theta \geq 0$ and let Y be an ultrametric $(m+1)$-Banach space. Assume that the function $f : X \to Y$ verifies the inequality*

$$\left\| f(x+y) + f\big(x+y^*\big) - 2f(x), g(z_1), \cdots, g(z_p) \right\|_* \leq \theta\Big(\|x\|^s + \|y\|^s \Big) \prod_{i=1}^{p} \|z_i\|^r, \quad (3.26)$$

for all $x, y, z_1 \cdots, z_p \in X_0$, then $f(x) = A(x) + a$, $x \in X_0$, where $A : X_0 \to Y$ is an additive mapping and $a \in Y$ is an arbitrary constant such that $A\big(x^*\big) = -A(x)$.

Conclusions:

Throughout this paper, we demonstrate the hyperstability of a general equation with involutoins. Additionally, we derive some colollaries as special cases and direct consequences of the main results presented in this paper. This work perhaps open new horizons for the study of this type of equations in ultarmetric n-Banach spaces.

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