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INTRINSIC PRIME SPECTRUM OF ALMOST DISTRIBUTIVE LATTICES

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Abstract Introduced the concepts of intrinsic ideals and inlets in an Almost Distributive Lattice(ADL). Characterized the intrinsic ideals in terms of inlets. Established a set of equivalent conditions for an ideal of an ADL to become intrinsic. Finally, derived some topological properties of the prime spectrum of intrinsic ideals of an ADL.

1 Introduction

The concept of an Almost Distributive Lattice(ADL) was introduced by Swamy U.M., and Rao G.C., [8] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set of all principal ideals of an ADL forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. In [4], the concepts of $D-$filters and prime $D-$filters are introduced in an ADL and studied their properties. In [7], a relation between the $\theta-$complement and $\theta - e-$irreducible element is established with suitable examples. In [1], the properties of divisibility in a distributive lattice are studied with respect to a filter. In [3], the idea of intrinsic ideals is introduced and the prime ideals of distributive lattices are used to study specific characteristics of these ideals. In [5], the concept of $\nu-$ideals is introduced in an ADL and their properties are discussed.

In this paper, the notions of intrinsic ideals and inlets are introduced to ADLs. It is derived that the class of all intrinsic ideals forms a distributive lattice and then proved that the class of all inlets of an ADL forms a sublattice to the lattice of all intrinsic ideals. The intrinsic ideals are characterized in element wise. Some equivalent conditions are established for every ideal of an ADL to become an intrinsic ideal. A congruence is introduced on an ADL and then a set of equivalent conditions is given for the respective quotient ADL to become a Boolean algebra. Certain basic properties of prime intrinsic ideals of an ADL are investigated. A set of equivalent conditions is given for the prime spectrum of intrinsic ideals of an ADL to become a Hausdorff space. A necessary and sufficient condition is given for the prime spectrum of intrinsic ideals to become a regular space.

2 Preliminaries

In this section, we review key definitions and important results related to ADL that will be necessary for the discussions in this paper.

Definition 2.1. [8] An Almost Distributive Lattice(ADL) with zero is an algebra $(R, \vee, \wedge, 0)$ of type (2,2,0) satisfying the following properties:

(1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,
(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(3) $(x \vee y) \wedge y = y$,
(4) $(x \vee y) \wedge x = x$,
(5) $x \vee (x \wedge y) = x$,
(6) $0 \wedge x = 0$, for any $x, y, z \in R$.

If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ or equivalently, $a \vee b = b$, then $\leq$ is a partial ordering on R. An element m in R is said to be *maximal* if it is maximal with respect to the partial ordering $\leq$ on R.

Definition 2.2. [8] Let R be a non-empty set. Fix $x_0 \in R$. For any $x, y \in R$, define $x \wedge y = y, x \vee y = x$ if $x \neq x_0$, and $x_0 \wedge y = x_0$, $x_0 \vee y = y$ if $x = x_0$. Then $(R, \vee, \wedge, x_0)$ is an ADL and it is called a Discrete ADL with x_0 as its 0. Alternatively, Discrete ADL is defined as an ADL in which every non-zero element is maximal in the poset $(R, \leq)$.

Theorem 2.3. [8] *Let R be an ADL. Then for any $a, b, c \in R$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$,
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$,
(3) $a \wedge b = b \wedge a$ *whenever* $a \leq b$,
(4) $\wedge$ *is associative in* R,
(5) $a \wedge b \wedge c = b \wedge a \wedge c$,
(6) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
(7) $a \wedge a = a$ *and* $a \vee a = a$,
(8) $0 \vee a = a$ *and* $a \wedge 0 = 0$,
(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$,
(10) $a \wedge b = 0$ *if and only if* $b \wedge a = 0$.

Theorem 2.4. [8] *Let R be an ADL and $m \in R$. Then the following are equivalent:*

(1) m *is maximal in* R,
(2) $m \vee x = m$ *for all* $x \in R$,
(3) $m \wedge x = x$ *for all* $x \in R$,
(4) $(m] = R$.

Definition 2.5. [8] A nonempty subset I of R is called an *ideal* (respectively a *filter*) of R, if $a \vee b, a \wedge x \in I$ (respectively $a \wedge b, x \vee a \in I$) for all $a, b \in I$ and all $x \in R$.

Definition 2.6. [2] For any nonempty subset A of an ADL R, define $A^* = \{ x \in R \mid a \wedge x = 0 \text{ for all } a \in A \}$. Here A^* is called the annihilator of A in R.

For any $a \in R$, we have $\{a\}^* = (a]^*$, where $(a]$ is the principal ideal generated by a. An element a of an ADL R is called dense element if $(a]^* = \{0\}$ and the set D of all dense elements in ADL is a filter if D is non-empty.

Definition 2.7. [4] For any subset A of an ADL R, define $(A, D) = \{x \in R \mid a \vee x \in D \text{ for all } a \in A\}$.

For any $a \in R$, we simply represent $(\{a\}, D)$ by (a, D). Clearly, $(m, D) = R$, where m is a maximal element of R. It is also obvious that $(0, D) = D$ and $D \subseteq (x, D)$ for all $x \in R$.

Proposition 2.8. [4] *Let R be an ADL. For any $a, b, c \in R$, we have*

(1) $a \leq b$ *implies* $(a, D) \subseteq (b, D)$,
(2) $(a \wedge b, D) = (a, D) \cap (b, D)$,
(3) $((a \vee b, D), D) = ((a, D), D) \cap ((b, D), D)$,
(4) $(a, D) = R$ *if and only if* $a \in D$.

3 Intrinsic ideals of ADLs

In this section, the notion of intrinsic ideals is introduced. A characterization theorem of intrinsic ideals is given. It is proved that the class of all intrinsic ideals forms a complete distributive lattice. A set of equivalent conditions is given for an ideal of an ADL to become intrinsic. Also, the notion of inlets is introduced in an ADL. The notion of weakly quasi-complemented ADLs is introduced and then weakly quasi-complemented ADLs are characterized in terms of inlets.

Now, we begin with the following definition.

Definition 3.1. For any non-empty subset S of an ADL R, the set $S^{\perp}$ is defined as $S^{\perp} = \{a \in R \mid (a, D) \subseteq (s, D),$ for some $s \in S\}$.

Lemma 3.2. *Let S, T be two non-empty subsets of an ADL R. Then we have the following:*

(1) $S \subseteq S^{\perp}$,
(2) $S \subseteq T$ *implies* $S^{\perp} \subseteq T^{\perp}$,
(3) $S^{\perp\perp} = S^{\perp}$,
(4) $D^{\perp} = R$.

Proof. (1) and (2) are clear.
(3) By (1) and (2), we have that $S^{\perp} \subseteq S^{\perp\perp}$. Let $a \in S^{\perp\perp}$. Then there exists $s \in S^{\perp}$ such that $(a, D) \subseteq (s, D)$. Since $s \in S^{\perp}$, there exists $t \in S$ such that $(s, D) \subseteq (t, D)$. Then $(a, D) \subseteq (s, D) \subseteq (t, D)$ and $t \in S$. Therefore $a \in S^{\perp}$ and hence $S^{\perp\perp} \subseteq S^{\perp}$. Thus $S^{\perp} = S^{\perp\perp}$.
(4) Clearly, for any $d \in D$, we have that $(d, D) = R$. That implies $(a, D) \subseteq R = (d, D)$, for all $a \in R$. Therefore $a \in D^{\perp}$, for all $a \in R$. Hence $D^{\perp} = R$. □

In case of $S = \{s\}$, we simply denote $\{s\}^{\perp}$ by $(s)^{\perp}$, where $(s)^{\perp} = \{a \in R \mid (a, D) \subseteq (s, D)\}$.

Lemma 3.3. *For any elements s, t of an ADL R, we have the following:*

(1) $(0)^{\perp} = \{a \in R \mid (a, D) = D\}$,
(2) $(s)^{\perp} = ((s])^{\perp}$,
(3) $(s)^{\perp\perp} = (s)^{\perp}$,
(4) $s \leq t$ *implies* $(s)^{\perp} \subseteq (t)^{\perp}$,
(5) $s \in (t)^{\perp}$ *implies* $(s)^{\perp} \subseteq (t)^{\perp}$,
(6) $(s \wedge t)^{\perp} = (t \wedge s)^{\perp}$ *and* $(s \vee t)^{\perp} = (t \vee s)^{\perp}$,
(7) $(s)^{\perp} \cap (t)^{\perp} = (s \wedge t)^{\perp}$,
(8) $(s, D) = (t, D)$ *if and only if* $(s)^{\perp} = (t)^{\perp}$,
(9) $(s)^{\perp} = (t)^{\perp}$ *implies* $(s \wedge c)^{\perp} = (t \wedge c)^{\perp}$ *and* $(s \vee c)^{\perp} = (t \vee c)^{\perp}$, *for any* $c \in R$.

Proof. (1) It is clear.
(2) Let $a \in (s)^{\perp}$. Then $(a, D) \subseteq (s, D)$. Since $s \in (s]$, we get that $a \in ((s])^{\perp}$. Therefore $(s)^{\perp} \subseteq ((s])^{\perp}$. Let $a \in ((s])^{\perp}$. Then there exists $b \in (s]$ such that $(a, D) \subseteq (b, D)$. Since $b \in (s]$, we get $(b, D) \subseteq (s, D)$. Hence $(a, D) \subseteq (b, D) \subseteq (s, D)$. Thus $a \in (s)^{\perp}$. Therefore $((s])^{\perp} \subseteq (s)^{\perp}$, which gives that $(s)^{\perp} = ((s])^{\perp}$.
(3) It is clear.
(4) Assume that $s \leq t$. Then $(s] \subseteq (t]$. That implies $(s)^{\perp} = (s]^{\perp} \subseteq (t]^{\perp} = (t)^{\perp}$.
(5) Let $s \in (t)^{\perp}$. Then $(s] \subseteq (t)^{\perp}$. By (3), we get $(s)^{\perp} \subseteq (t)^{\perp\perp} = (t)^{\perp}$.
(6) Since $(s \wedge t, D) = (t \wedge s, D)$ and $(s \vee t, D) = (t \vee s, D)$, we get (6).
(7) For any $s, t \in R$, we have $(s \wedge t)^{\perp} \subseteq (s)^{\perp} \cap (t)^{\perp}$. Let $a \in (s)^{\perp} \cap (t)^{\perp}$. Then $(a, D) \subseteq (s, D)$ and $(a, D) \subseteq (t, D)$. That implies $(a, D) \subseteq (s, D) \cap (t, D) = (s \wedge t, D)$. Therefore $a \in (s \wedge t)^{\perp}$ and hence $(s)^{\perp} \cap (t)^{\perp} \subseteq (s \wedge t)^{\perp}$. Thus $(s)^{\perp} \cap (t)^{\perp} = (s \wedge t)^{\perp}$.
(8) Assume that $(s, D) = (t, D)$. Then clearly $(s)^{\perp} = (t)^{\perp}$. Conversely, assume that $(s)^{\perp} = (t)^{\perp}$. Since $s \in (s)^{\perp} = (t)^{\perp}$, we get $(s, D) \subseteq (t, D)$. Similarly, we can obtain that $(t, D) \subseteq (s, D)$. Therefore $(s, D) = (t, D)$.

(9) Assume that $(s)^{\perp} = (t)^{\perp}$. Then $(s, D) = (t, D)$. Let $c \in R$. Now $(s \wedge c)^{\perp} = (s)^{\perp} \cap (c)^{\perp} = (t)^{\perp} \cap (c)^{\perp} = (t \wedge c)^{\perp}$. Now $x \in (s \vee c)^{\perp} \Leftrightarrow (x, D) \subseteq (s \vee c, D) = (((s \vee c, D), D), D) = (((s, D), D) \cap ((c, D), D), D) = (((t, D), D) \cap ((c, D), D), D) = (((t \vee c, D), D), D) = (t \vee c, D) \Leftrightarrow x \in (t \vee c)^{\perp}$. Therefore $(s \vee c)^{\perp} = (t \vee c)^{\perp}$. □

Lemma 3.4. *For any ideal I of an ADL R, $I^{\perp}$ is an ideal of R containing I.*

Proof. Clearly, we we have $I \subseteq I^{\perp}$. Let $a, b \in I^{\perp}$. Then there exist $s, t \in I$ such that $(a, D) \subseteq (s, D)$ and $(b, D) \subseteq (t, D)$. Since $s, t \in I$, we have $s \vee t \in I$. Now $(a \vee b, D) = (((a \vee b, D), D), D) = ((((a, D), D) \cap ((b, D), D)), D) \subseteq ((((s, D), D) \cap ((t, D), D)), D) = (((s \vee t, D), D), D) = (s \vee t, D)$. Since $s \vee t \in I$, we get that $a \vee b \in I^{\perp}$. Let $a \in I^{\perp}$. Then there exists an element $s \in I$ such that $(a, D) \subseteq (s, D)$. Let $b \in R$ Then $(a \wedge b, D) \subseteq (a, D) \cap (b, D) \subseteq (a, D) \subseteq (s, D)$. Since $s \in I$, we get that $a \wedge b \in I^{\perp}$. Therefore $I^{\perp}$ is an ideal of R containing I. □

Lemma 3.5. *Let I and J be any two ideals of an ADL R. Then we have*

(1) $I^{\perp} \cap J^{\perp} = (I \cap J)^{\perp}$,
(2) $(I \vee J)^{\perp} = (I^{\perp} \vee J^{\perp})^{\perp}$.

Proof. (1) Clearly, we have that $(I \cap J)^{\perp} \subseteq I^{\perp} \cap J^{\perp}$. Let $a \in I^{\perp} \cap J^{\perp}$. Then there exist $s \in I$ and $t \in J$ such that $(a, D) \subseteq (s, D)$ and $(a, D) \subseteq (t, D)$. That implies $(a, D) \subseteq (s, D) \cap (t, D) = (s \wedge t, D)$. Since $s \wedge t \in I \cap J$, we get that $a \in (I \cap J)^{\perp}$. Therefore $I^{\perp} \cap J^{\perp} \subseteq (I \cap J)^{\perp}$. Hence $I^{\perp} \cap J^{\perp} = (I \cap J)^{\perp}$.
(2) Clearly, we have that $I \vee J \subseteq I^{\perp} \vee J^{\perp}$ and hence $(I \vee J)^{\perp} \subseteq (I^{\perp} \vee J^{\perp})^{\perp}$. Let $a \in (I^{\perp} \vee J^{\perp})^{\perp}$. Then there exists $s \in I^{\perp} \vee J^{\perp}$ such that $(a, D) \subseteq (s, D)$. Since $s \in I^{\perp} \vee J^{\perp}$, there exist $b \in I^{\perp}$ and $c \in J^{\perp}$ such that $s = b \vee c$. Since $b \in I^{\perp}$, there exists $b' \in I$ such that $(b, D) \subseteq (b', D)$. Since $c \in J^{\perp}$, there exists $c' \in J$ such that $(c, D) \subseteq (c', D)$. Since $(b, D) \subseteq (b', D)$ and $(c, D) \subseteq (c', D)$, we get that $((b', D), D) \subseteq ((b, D), D)$ and $((c', D), D) \subseteq ((c, D), D)$. Then $((b' \vee c', D), D) = ((b', D), D) \cap ((c', D), D) \subseteq ((b, D), D) \cap ((c, D), D) = ((b \vee c, D), D)$. Therefore $(a, D) \subseteq (s, D) = (b \vee c, D) \subseteq (b' \vee c', D)$. Since $b' \vee c' \in I \vee J$, we get that $a \in (I \vee J)^{\perp}$. Hence $(I^{\perp} \vee J^{\perp})^{\perp} \subseteq (I \vee J)^{\perp}$. □

Proposition 3.6. *Let R be an ADL with maximal element m and I be any ideal of R. Then the following conditions are equivalent:*

(1) $I^{\perp} = R$,
(2) $I^{\perp} \cap D \neq \emptyset$,
(3) $I \cap D \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2): Assume that $I^{\perp} = R$. Then $m \in I^{\perp}$. Since $m \in D$, we get $I^{\perp} \cap D \neq \emptyset$.
(2) $\Rightarrow$ (3): Assume that $I^{\perp} \cap D \neq \emptyset$. Then choose $a \in I^{\perp} \cap D$. Since $a \in D$, by Proposition 2.8(4), we get $(a, D) = R$. Since $a \in I^{\perp}$, there exists $s \in I$ such that $R = (a, D) \subseteq (s, D)$. Then $(s, D) = R$ and hence $s \in D$. Therefore $s \in I \cap D$. Thus $I \cap D \neq \emptyset$.
(3) $\Rightarrow$ (1): Assume that $I \cap D \neq \emptyset$. Then choose $s \in I \cap D$. Since $s \in D$, we have $(s, D) = R$. That implies $(a, D) \subseteq (s, D) = R$, for all $a \in R$. Therefore $a \in I^{\perp}$, for all $a \in R$. Hence $I^{\perp} = R$. □

Now we introduce the definition of intrinsic ideal in an ADL.

Definition 3.7. An ideal I of an ADL R is called *intrinsic* if $I = I^{\perp}$.

Example 3.8. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and define $\vee$, $\wedge$ on R as follows:

$\wedge$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	1	2	3	4	5	6	7
3	0	3	3	3	0	0	3	0
4	0	4	5	0	4	5	7	7
5	0	4	5	0	4	5	7	7
6	0	6	6	3	7	7	6	7
7	0	7	7	0	7	7	7	7

$\vee$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	1	2	3	1	2	6	6
4	4	1	1	1	4	4	1	4
5	5	2	2	2	5	5	2	5
6	6	1	2	6	1	2	6	6
7	7	1	2	6	4	5	6	7

Then $(R, \vee, \wedge)$ is an ADL. Clearly, we have that $D = \{1, 2, 6\}$. We have that $(0, D) = D, (3, D) = \{1, 2, 4, 5, 6, 7\}, (4, D) = (5, D) = (7, D) = \{1, 2, 3, 6\}$ and $(1, D) = (2, D) = (6, D) = R$. Then clearly $(0)^{\perp} = \{0\}, (3)^{\perp} = \{0, 3\}, (4)^{\perp} = (5)^{\perp} = (7)^{\perp} = \{0, 4, 5, 7\},$ and $(1)^{\perp} = (2)^{\perp} = (6)^{\perp} = R$. Consider the ideals $I_1 = \{0, 3\}$ and $I_2 = \{0, 3, 6, 7\}$. Clearly, I_1 is an intrinsic ideal. But I_2 is not intrinsic, because $I_2^{\perp} = L \neq I_2$.

Proposition 3.9. *Let R be an ADL. If N is maximal in the class of all ideals which are not meeting D, then M is an intrinsic ideal.*

Proof. Let N be an ideal which is maximal with respect to the property of $M \cap D = \emptyset$. By Proposition 3.6, we get $N^{\perp} \cap D = \emptyset$ and $N^{\perp} \neq R$. Then $N^{\perp}$ is a proper ideal of R such that $N \subseteq N^{\perp}$. By the maximality of N, we get $N = N^{\perp}$. Therefore N is an intrinsic ideal of R. □

Let us denote the class of all intrinsic ideals of the ADL R by $\mathcal{N}(R)$. Then it is clear that $\mathcal{N}(R)$ need not be a sublattice of the distributive lattice $\mathfrak{I}(R)$ of all ideals of R. However, in the following, we derive that $\mathcal{N}(R)$ forms a distributive lattice on its own.

Theorem 3.10. *For any ADL R, the set $\mathcal{N}(R)$ of all intrinsic ideals of R forms a distributive lattice with greatest element R.*

Proof. For any $I, J \in \mathcal{N}(R)$, define the operations $\cap$ and $\sqcup$ on $\mathcal{N}(R)$ as follows:

$$I \cap J = (I \cap J)^{\perp} \text{ and } I \sqcup J = (I^{\perp} \vee J^{\perp})^{\perp} = (I \vee J)^{\perp}$$

Clearly, $(I \cap J)^{\perp}$ is the infimum of I and J in $\mathcal{N}(R)$. Also $(I \vee J)^{\perp}$ is an upper bound of I and J. By Lemma 3.5(2), we have $(I^{\perp} \vee J^{\perp})^{\perp} = (I \vee J)^{\perp}$. Suppose $K \in \mathcal{N}(R)$ such that $I \subseteq K$ and $J \subseteq K$. Let $a \in (I \vee J)^{\perp}$. Then there exist $i \in I \subseteq K$ and $j \in J \subseteq K$ such that $(a, D) \subseteq (i \vee j, D)$. Since $i \vee j \in K$, we get $a \in K^{\perp} = K$. Therefore $(I \vee J)^{\perp}$ is the supremum of both I and J in $\mathcal{N}(R)$. Then it can be easily verified that $(\mathcal{N}(R), \cap, \sqcup)$ is a distributive lattice. Clearly, R is the greatest element of the lattice $(\mathcal{N}(R), \cap, \sqcup)$. □

In the following, we characterize the intrinsic ideal of an ADL in element wise.

Theorem 3.11. *For any ideal I of an ADL R, the following are equivalent:*

(1) *I is intrinsic,*
(2) *for any $a \in R$, $a \in I$ if and only if $(a)^{\perp} \subseteq I$,*
(3) *for any $a, b \in R$, $(a, D) = (b, D)$ and $a \in I$ imply $b \in I$,*
(4) *for any $a, b \in R$, $(a)^{\perp} = (b)^{\perp}$ and $a \in I$ imply $b \in I$,*
(5) *$I = \bigcup_{a \in I} (a)^{\perp}$.*

Proof. (1) $\Rightarrow$ (2): Assume (1). Suppose $a \in I$. Let $s \in (a)^{\perp}$. Then $(s, D) \subseteq (a, D)$. Since $a \in I$,we get that $s \in I^{\perp} = I$. Hence $(a)^{\perp} \subseteq I$. Converse is clear.

(2) $\Rightarrow$ (3): Assume (2). Let $a, b \in R$ with $(a, D) = (b, D)$ and $a \in I$. By our assumption, we get that $(a)^{\perp} \subseteq I$. Since $(a, D) = (b, D)$, we have that $(a)^{\perp} = (b)^{\perp}$ and hence $b \in (b)^{\perp} \subseteq I$. Therefore $b \in I$.

(3) $\Rightarrow$ (4): By Lemma 3.3(8), it is clear.

(4) $\Rightarrow$ (5): Assume (4). Clearly, we have that $(a] \subseteq (a)^{\perp}$, for all $a \in I$ and hence $I = \bigcup_{a \in I} (a] \subseteq$

$\bigcup_{a\in I}(a)^{\perp}$. Let $b \in \bigcup_{a\in I}(a)^{\perp}$. Then there exists $x \in I$ such that $b \in (x)^{\perp}$. Then we get $(b)^{\perp} \subseteq (x)^{\perp}$. That implies $(b)^{\perp} = (b)^{\perp} \cap (x)^{\perp} = (x \wedge b)^{\perp}$. Since $x \wedge b \in I$, by condition (4), we get $b \in I$. Therefore $\bigcup_{a\in I}(a)^{\perp} \subseteq I$. Hence $\bigcup_{a\in I}(a)^{\perp} \subseteq I$. Thus $I = \bigcup_{a\in I}(a)^{\perp}$.

(5) $\Rightarrow$ (1): Assume (5). Clearly, we have that $I \subseteq I^{\perp}$. Let $a \in I^{\perp}$. Then there exists $s \in I$ such that $(a, D) \subseteq (s, D)$. That implies $a \in (s)^{\perp}$. Since $s \in I$, we get that $a \in \bigcup_{s\in I}(s)^{\perp} = I$. Therefore $I^{\perp} \subseteq I$ and hence I is intrinsic. □

Proposition 3.12. *Let I be an intrinsic ideal and P a prime ideal of an ADL R such that $P \cap D = \emptyset$ and $I \subseteq P$. If P is minimal, then P is an intrinsic ideal.*

Proof. Suppose P is not an intrinsic ideal of R. Then there exists elements $a, b \in R$ such that $(a)^{\perp} = (b)^{\perp}$, $a \in P$ and $b \notin P$. Consider $K = (R \setminus P) \vee [a \wedge b)$. Suppose $I \cap K \neq \emptyset$. Then choose an element $x \in I \cap K$. Then $x \in I$ and $x \in K$. Since $x \in K$, there exist $r \in R \setminus P$ and $s \in [a \wedge b)$ such that $x = r \wedge s$. Since $s \in [a \wedge b)$, we get that $s \vee (a \wedge b) = s$. Now, $x = r \wedge s = r \wedge (s \vee (a \wedge b)) = (r \wedge s) \vee (r \wedge a \wedge b)$. That implies $r \wedge a \wedge b = (r \wedge s) \wedge (r \wedge a \wedge b)$. Since $x = r \wedge s \in I$, we get that $r \wedge a \wedge b \in I$. Since $(a)^{\perp} = (b)^{\perp}$, by Lemma 3.3(9), we get $(r \wedge b)^{\perp} = (r \wedge a \wedge b)^{\perp}$. Since I is a intrinsic ideal and $r \wedge a \wedge b \in I$, we get $r \wedge b \in I \subseteq P$. That implies $r \in P$ or $b \in P$, which is a contradiction. Therefore $I \cap K = \emptyset$. Hence there exists a prime ideal M such that $K \cap M = \emptyset$ and $I \subseteq M$. Since $K \cap M = \emptyset$, we get $M \subseteq R \setminus K \subseteq P$ because of $R \setminus P \subseteq K$. Since $a \wedge b \in K$, we get $a \wedge b \notin M$. That implies $a \wedge b \in P$ and $a \wedge b \notin M$. Therefore $I \subseteq M \subset P$. Since $P \cap D = \emptyset$, we have $M \cap D = \emptyset$. Hence P is not a minimal in the class of all prime ideals with $P \cap D = \emptyset$ and containing I, which is a contradiction. Therefore P is an intrinsic ideal. □

Corollary 3.13. *If $\{0\}$ is an intrinsic ideal of an ADL R, then every minimal prime ideal of R is an intrinsic ideal.*

Proof. Let P be a minimal prime ideal of R. Suppose there exists $a \in R$ such that $a \in P \cap D$. Since P is minimal, there exists a non-zero element $b \notin P$ such that $a \wedge b = 0$. That implies $a \notin D$, which is a contradiction. Therefore $P \cap D = \emptyset$. Since $\{0\} \subseteq P$, by Proposition 3.12, P is intrinsic. □

Proposition 3.14. *Let R be an ADL. Then the following are equivalent:*

(1) *for $a, b \in R$, $(a, D) = (b, D)$ implies $(a] = (b]$,*
(2) *for $a, b \in R$, $(a)^{\perp} = (b)^{\perp}$ implies $(a] = (b]$,*
(3) *every ideal I with $I \cap D = \emptyset$ is intrinsic,*
(4) *every prime ideal P with $P \cap D = \emptyset$ is intrinsic.*

Proof. (1) $\Rightarrow$ (2): By Lemma 3.3(6), it is clear.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1): Assume that condition (4) holds. Let $a, b \in R$ with $(a, D) = (b, D)$. By Lemma 3.3(8), we have that $(a)^{\perp} = (b)^{\perp}$. We prove that $(a] = (b]$. Suppose $(a] \neq (b]$. Without loss of generality, assume that $(a] \nsubseteq (b]$. Then $a \notin (b]$. Then there exists a prime ideal P of R such that $b \in (b] \subseteq P$ and $a \notin P$. Suppose $P \cap D \neq \emptyset$. Then $P^{\perp} = R$, which gives $a \in P^{\perp}$. That implies $P \nsubseteq P^{\perp}$, we get a contradiction to $P \subseteq P^{\perp}$. Therefore $P \cap D = \emptyset$. By our assumption, we get that P is intrinsic. Since $b \in P$ and $(a)^{\perp} = (b)^{\perp}$, we get that $a \in P$, which is a contradiction to $a \notin P$. Hence $(a] = (b]$. □

The ideal of the form $(a)^{\perp}$ is called an *inlet* of R. Since $(0, D) = D$, we can observe that $(0)^{\perp} = \{x \in R \mid (x, D) = D\}$.

Proposition 3.15. *Let R be an ADL and $a \in R$. Then the following are equivalent:*

(1) $(a)^{\perp} = R$,
(2) $(a)^{\perp} \cap D \neq \emptyset$,
(3) $a \in D$.

Lemma 3.16. *Let R be an ADL with maximal elements. For any $a, b \in R$, the following properties hold:*

(1) $a \vee b$ *is a maximal element implies* $(a)^{\perp} \vee (b)^{\perp} = R$,
(2) *For any* $a \notin D$, $(a, D) \cap (a)^{\perp} = \emptyset$,
(3) $(a \vee b)^{\perp} = ((a)^{\perp} \vee (b)^{\perp})^{\perp}$.

Proof. (1) Let $a, b \in R$ be such that $a \vee b$ is a maximal element. Then $R = (a \vee b] = (a] \vee (b] \subseteq (a)^{\perp} \vee (b)^{\perp}$. That implies $(a)^{\perp} \vee (b)^{\perp} = R$.
(2) Let $a \in R$ be such that $a \notin D$. Suppose $s \in (a, D) \cap (a)^{\perp}$. Then $((s, D), D) \subseteq (a, D)$ and $(s, D) \subseteq (a, D)$. That implies $a \in ((a, D), D) \subseteq ((s, D), D) \subseteq (a, D)$. Therefore $a = a \vee a \in D$, which is a contradiction. Hence $(a, D) \cap (a)^{\perp} = \emptyset$.
(3) It is clear by Lemma 3.5(2). □

Obviously each inlet is an intrinsic ideal and hence for any two inlets $(a)^{\perp}$ and $(b)^{\perp}$ their supremum in $\mathcal{N}(R)$ is given by

$$(a)^{\perp} \sqcup (b)^{\perp} = ((a] \vee (b])^{\perp} = ((a \vee b])^{\perp} = (a \vee b)^{\perp}$$

Also their infimum in $\mathcal{N}(R)$ is $(a)^{\perp} \cap (b)^{\perp} = (a \wedge b)^{\perp}$.

Theorem 3.17. *For any ADL R, the class $\mathcal{N}_+(R)$ of all inlets is a lattice $(\mathcal{N}_+(R), \cap, \sqcup)$ and sublattice to the distributive lattice $(\mathcal{N}(R), \cap, \sqcup, R)$ of all intrinsic ideals of R. Moreover, $\mathcal{N}_+(R)$ has the same greatest element $R = (d)^{\perp}$, $d \in D$ as $\mathcal{N}(R)$ while $\mathcal{N}_+(R)$ has the smallest element $(s)^{\perp}$ if and only if R has an element s of the form $(s, D) = D$.*

Proof. Clearly $(\mathcal{N}_+(R), \cap, \sqcup)$ is a sublattice to the distributive lattice $(\mathcal{N}(R), \cap, \sqcup)$. It is remaining to prove the statement concerning the smallest element of $\mathcal{N}_+(R)$. Suppose $(s)^{\perp}$ is the smallest element of $\mathcal{N}_+(R)$. Let $a \in (s, D)$. Then $a \vee s \in D$. Now, for any $a \in R$ $(a)^{\perp} = (a)^{\perp} \sqcup (s)^{\perp} = (a \vee s)^{\perp} = R$ which gives that $s \in D$. Hence $(s, D) \subseteq D$. Therefore $(s, D) = D$. Conversely, assume that R has an element s such that $(s, D) = D$. Let $a \in (s)^{\perp}$. Then $(a, D) \subseteq (s, D) = D \subseteq (t, D)$, for all $t \in R$. Hence $a \in (t)^{\perp}$ for all $t \in R$. Thus $(s)^{\perp} \subseteq (t)^{\perp}$ for all $t \in T$. Hence $(s)^{\perp}$ is the smallest element of $\mathcal{N}_+(R)$. □

In any ADL R, it is a well known fact that the quotient algebra $R/\phi = \{[a]_{\phi} \mid a \in L\}$, where $[a]_{\phi}$ is the congruence class of a with respect to ϕ, is a quotient lattice with respect to the operations given by $[a]_{\phi} \cap [b]_{\phi} = [a \wedge b]_{\phi}$ and $[a]_{\phi} \vee [b]_{\phi} = [a \vee b]_{\phi}$, for all $a, b \in R$.

Proposition 3.18. *Define a binary relation ϕ on an ADL R by*

$$(a, b) \in \phi \text{ if and only if } (a)^{\perp} = (b)^{\perp}$$

for all $a, b \in R$. Then ϕ is a congruence on R where $(0)^{\perp}$ is the smallest congruence class and D is the unit congruence class of R/ϕ. Furthermore, $ker\ \phi$ is an intrinsic ideal of R.

Proof. From (9) of Lemma 3.3, ϕ is a congruence on R. Clearly, $(0)^{\perp}$ is the smallest congruence class of R/ϕ. Let $a, b \in D$. By Proposition 3.15, we get $(a)^{\perp} = (b)^{\perp} = R$. Thus $(a, b) \in \phi$. Therefore D is a congruence class of R/ϕ. Now, let $s \in D$ and $a \in R$. Since D is a filter, we get $s \vee a \in D$. Since D is a congruence class with respect to ϕ, we get $[a]_{\phi} \vee [s]_{\phi} = [a \vee s]_{\phi} = D$. Therefore D is the unit congruence class of R/ϕ. Clearly, we have that $ker\ \phi$ is an ideal of R. Let $a \in ker\ \phi$. Then $(a)^{\perp} = (0)^{\perp}$. Let $s \in (a)^{\perp}$. Then $(s)^{\perp} \subseteq (a)^{\perp} = (0)^{\perp}$. Since $0 \leq s$, we get $(0)^{\perp} \subseteq (s)^{\perp}$. Hence $(s)^{\perp} = (0)^{\perp}$, which means that $s \in ker\ \phi$. Hence $(a)^{\perp} \subseteq ker\ \phi$. Therefore $ker\ \phi$ is an intrinsic ideal of R. □

Definition 3.19. An ADL R is said to be *weakly quasi-complemented* if to each $a \in R$, there exists $b \in R$ such that $(a \wedge b)^{\perp} = (0)^{\perp}$ and $a \vee b \in D$.

From the Example 3.8, it is clearly observed that R is weakly quasi-complemented. We now characterize weakly quasi-complemented ADLs with help of inlets and the congruence ϕ.

Theorem 3.20. *The following conditions are equivalent in an ADL R:*

(1) *R is weakly quasi-complemented,*
(2) *$\mathcal{N}_+(R)$ is a Boolean algebra,*
(3) *R/ϕ is a Boolean algebra.*

Proof. (1) $\Rightarrow$ (2): Assume that R is weakly quasi-complemented. Let $(a)^{\perp} \in \mathcal{N}_{+}(R)$. Then there exists $a' \in R$ such that $a \wedge a' = 0$ and $a \vee a' \in D$. Hence $(a)^{\perp} \cap (a')^{\perp} = (a \wedge a')^{\perp} = (0)^{\perp}$ and $(a)^{\perp} \sqcup (a')^{\perp} = (a \vee a')^{\perp} = R$. Therefore $\mathcal{N}_{+}(R)$ is a Boolean algebra.
(2) $\Rightarrow$ (3): Assume that $\mathcal{N}_{+}(R)$ is a Boolean algebra. Let $[a]_{\phi} \in L/\phi$. Then $(a)^{\perp} \in \mathcal{N}_{+}(R)$. Hence there exists $(b)^{\perp} \in \mathcal{N}_{+}(R)$ such that $(a \wedge b)^{\perp} = (a)^{\perp} \cap (b)^{\perp} = (0)^{\perp}$ and $(a \vee b)^{\perp} = (a)^{\perp} \sqcup (b)^{\perp} = R$. Hence $a \wedge b \in [0]_{\phi}$ and $a \vee b \in D$. Thus $[a]_{\phi} \cap [b]_{\phi} = [a \wedge b]_{\phi} = [0]_{\phi}$ and $[a]_{\phi} \vee [b]_{\phi} = [a \vee b]_{\phi} = D$. Therefore R/ϕ is a Boolean algebra.
(3) $\Rightarrow$ (1): Assume that R/ϕ is a Boolean algebra. Let $a \in R$. Then $[a]_{\phi} \in R/\phi$. Since R/ϕ is a Boolean algebra, there exists $[a']_{\phi} \in R/\phi$ such that $[a \wedge a']_{\phi} = [a]_{\phi} \cap [a']_{\phi} = [0]_{\phi}$ and $[a \vee a']_{\phi} = [a]_{\phi} \vee [a']_{\phi} = D$. Thus $(a \wedge a')^{\perp} = (0)^{\perp}$ and $a \vee a' \in D$. Therefore R is weakly quasi-complemented. □

Theorem 3.21. *Every ADL R is epimorphic to the lattice $(\mathcal{N}_{+}(R), \sqcup, \cap)$ of inlets.*

Proof. Define a mapping $\psi : R \longrightarrow \mathcal{N}_{+}(R)$ by $\psi(x) = (x)^{\perp}$ for all $x \in R$. Clearly, ψ is well-defined. Let $a, b \in R$. Then $\psi(a \wedge b) = (a \wedge b)^{\perp} = (a)^{\perp} \cap (b)^{\perp} = \psi(a) \cap \psi(b)$. By Lemma 3.16(3), we get $\psi(a \vee b) = (a \vee b)^{\perp} = ((a)^{\perp} \vee (b)^{\perp})^{\perp} = (a)^{\perp} \sqcup (b)^{\perp} = \psi(a) \sqcup \psi(b)$. Therefore ψ is a homomorphism. Clearly, ψ is surjective. □

Proposition 3.22. *Every maximal intrinsic ideal of an ADL R is prime.*

Proof. Let N be a maximal intrinsic ideal of an ADL R. Let $a, b \in R$ be such that $a \notin N$ and $b \notin N$. Then $N \sqcup (a)^{\perp} = R$ and $N \sqcup (b)^{\perp} = R$. Now, $R = R \cap R = \{N \sqcup (a)^{\perp}\} \cap \{N \sqcup (b)^{\perp}\} = N \sqcup \{(a)^{\perp} \cap (b)^{\perp}\} = N \sqcup (a \wedge b)^{\perp}$.

Suppose $a \wedge b \in N$. Since N is intrinsic, we get $(a \wedge b)^{\perp} \subseteq N$. Hence $N = R$, which is a contradiction. Therefore N is prime. □

Theorem 3.23. *Let I be an intrinsic ideal and F a filter of an ADL R such that $I \cap F = \emptyset$. Then there exists a prime intrinsic ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Let I be an intrinsic ideal and F a filter of a R such that $I \cap F = \emptyset$. Consider $\mathcal{G} = \{J \mid J \text{ is an intrinsic ideal}, I \subseteq J \text{ and } J \cap F = \emptyset\}$. Clearly $I \in \mathcal{G}$. Clearly, $\mathcal{G}$ satisfies the hypothesis of Zorn's Lemma and hence $\mathcal{G}$ has a maximal element, let it be N. Suppose $a, b \in R$ such that $a \notin N$ and $b \notin N$. Then $N \subset N \vee (a] \subseteq N \vee (a)^{\perp}$ and $N \subset N \vee (b] \subseteq N \vee (b)^{\perp}$. By the maximality of N, we get $\{N \vee (a)^{\perp}\} \cap F \neq \emptyset$ and $\{N \vee (b)^{\perp}\} \cap F \neq \emptyset$. Choose $s \in \{N \vee (a)^{\perp}\} \cap F$ and $t \in \{N \vee (b)^{\perp}\} \cap F$. Then $s \wedge t \in F$. Now, $s \wedge t \in \{N \vee (a)^{\perp}\} \cap \{N \vee (b)^{\perp}\} = N \vee \{(a)^{\perp} \cap (b)^{\perp}\} = N \vee (a \wedge b)^{\perp}$. Suppose $a \wedge b \in N$. Since N is intrinsic, we get $(a \wedge b)^{\perp} \subseteq N$. Hence $s \wedge t \in N$ and thus $s \wedge t \in N \cap F \neq \emptyset$, which is a contradiction. Therefore N is prime. □

Corollary 3.24. *Let I be an intrinsic ideal of an ADL R and $x \notin I$. Then there exists a prime intrinsic ideal P of R such that $I \subseteq P$ and $x \notin P$.*

Corollary 3.25. *For any intrinsic ideal I of an ADL R, we have*

$$I = \bigcap \{P \mid P \text{ is a prime intrinsic ideal of } R \text{ such that } I \subseteq P\}$$

Corollary 3.26. *The intersection of all prime intrinsic ideals is equal to $(0)^{\perp}$.*

Let I be an intrinsic ideal and P be a prime intrinsic ideal of an ADL R such that $I \subseteq P$. Then P is called a minimal prime intrinsic ideal belonging to I if there exists no prime intrinsic ideal Q such that $I \subseteq Q \subset P$. A minimal prime intrinsic ideal belonging to $(0)^{\perp}$ is simply called minimal prime intrinsic.

In the following theorem, a necessary and sufficient condition is proved for a prime intrinsic ideal of an ADL to become minimal.

Theorem 3.27. *Let I be an intrinsic ideal and P a prime intrinsic ideal of an ADL R such that $I \subseteq P$. Then P is a minimal prime intrinsic ideal belonging to I if and only if to each $a \in P$, there exists $b \notin P$ such that $a \wedge b \in I$.*

Proof. Assume that P is a minimal prime intrinsic ideal belonging to I. Since P is a proper intrinsic ideal, by Proposition 3.6, we get $P \cap D = \emptyset$. Then $R \setminus P$ is a maximal filter with respect to the property that $(R \setminus P) \cap I = \emptyset$. Let $x \in P$. Then clearly $R \setminus P \subset (R \setminus P) \vee [a)$. By the maximality of $R \setminus P$, we get $\{(R \setminus P) \vee [a)\} \cap I \neq \emptyset$. Choose $s \in \{(R \setminus P) \vee [a)\} \cap I$. Then we get $s = t \wedge a$ for some $t \in R \setminus P$ and $s \in I$. Therefore $t \wedge a = s \in I$ where $t \notin P$. Conversely, assume that the condition holds. Suppose P is not a minimal prime intrinsic ideal belonging to I. Then there exists a prime intrinsic ideal K of R such that $I \subseteq K \subset P$. Choose $a \in P \setminus K$. Then, by the assumed condition, there exists $b \notin P$ such that $a \wedge b \in I \subseteq K$. Since $a \notin k$, it gives that $b \in K \subset P$, which is a contradiction. Therefore P is a minimal prime intrinsic ideal belonging to I. ◻

By taking $\{0\}$ in place of I in Theorem 3.27, we get the following

Corollary 3.28. *A prime intrinsic ideal P of an ADL is minimal if and only if to each $a \in P$, there exists $b \notin P$ such that $a \wedge b \in (0)^{\perp}$.*

4 Prime spectrum of intrinsic ideals

In this section, we discuss some algebraic properties of prime intrinsic ideals of an ADL. A set of equivalent conditions is given for the space of prime intrinsic ideals of an ADL to become a Hausdorff space. For any ADL R, let us denote the class of all prime intrinsic ideals of R by $Spec^{\perp}(R)$. For any $X \subseteq R$, let $\mathcal{K}(X) = \{P \in Spec^{\perp}(R) \mid X \nsubseteq P\}$ and for any $a \in R, \mathcal{K}(a) = \mathcal{K}(\{a\})$.

We have the following result which can be verified directly.

Lemma 4.1. *Let R be an ADL. For any $a, b \in R$, the following properties hold:*

(1) $\bigcup_{a \in R} \mathcal{K}(a) = Spec^{\perp}(R)$,
(2) $\mathcal{K}(a) \cap \mathcal{K}(b) = \mathcal{K}(a \wedge b)$,
(3) $\mathcal{K}(a) \cup \mathcal{K}(b) = \mathcal{K}(a \vee b)$,
(4) $\mathcal{K}(a) = \emptyset$ *if and only if* $a \in (0)^{\perp}$,
(5) $\mathcal{K}(a) = Spec^{\perp}(R)$ *if and only if* $a \in D$.

From the above lemma, it can be easily observed that the collection $\{\mathcal{K}(x) | x \in R\}$ forms a base for a topology on $Spec^{\perp}(R)$ which is called a hull-kernel topology.

Theorem 4.2. *In any ADL R, the following properties hold:*

(1) *For any $x \in R$, $\mathcal{K}(x)$ is compact in $Spec^{\perp}(R)$,*
(2) *Let C be a compact open subset of $Spec^{\perp}(R)$. Then $C = \mathcal{K}(a)$ for some $a \in R$,*
(3) *$Spec^{\perp}(R)$ is a T_0-space,*
(4) *The map $a \mapsto \mathcal{K}(a)$ is a homomorphism from R onto the lattice of all compact open subsets of $Spec^{\perp}(R)$.*

Proof. (1) Let $a \in R$ and $A \subseteq R$ be such that $\mathcal{K}(a) \subseteq \bigcup_{b \in A} \mathcal{K}(b)$. Let I be the ideal generated by the set A. Suppose $a \notin I^{\perp}$. By Corollary 3.24, there exits a prime intrinsic ideal P such that $I^{\perp} \subseteq P$ and $a \notin P$. Hence $P \in \mathcal{K}(a) \subseteq \bigcup_{b \in A} \mathcal{K}(b)$. Therefore $b \notin P$ for some $b \in A$, which is a contradiction to that $b \in A \subseteq I \subseteq I^{\perp} \subseteq P$. Therefore $a \in I^{\perp}$. Then $a \in (s)^{\perp}$ for some $s \in I$. Since I is the ideal generated by A and $s \in I$, there exist we get $s_1, s_2, \ldots, s_n \in A$ and $t \in R$ such that $s = (\bigvee_{i=1}^{n} s_i) \wedge t$. That implies $(s)^{\perp} = ((\bigvee_{i=1}^{n} s_i) \wedge t)^{\perp} \subseteq (\bigvee_{i=1}^{n} s_i)^{\perp}$. That implies $\mathcal{K}(a) \subseteq \bigcup_{i=1}^{n} \mathcal{K}(s_i)$, which is a finite subcover of $\mathcal{K}(a)$. Hence $\mathcal{K}(a)$ is compact in $Spec^{\perp}(R)$. Thus for each $a \in R$, $\mathcal{K}(a)$ is a compact open subset of $Spec^{\perp}(R)$.

(2) Let C be a compact open subset of $Spec^{\perp}(R)$. Since C is open, we get $C = \bigcup_{a \in A} \mathcal{K}(a)$ for

some $A \subseteq R$. Since C is compact, there exists $a_1, a_2, \ldots, a_n \in A$ such that $C = \bigcup_{i=1}^{n} \mathcal{K}(a_i) = \mathcal{K}(\bigvee_{i=1}^{n} a_i)$. Therefore $C = \mathcal{K}(x)$ for some $x \in R$.

(3) Let M and N be two distinct prime intrinsic ideals of R. Without loss of generality, assume that $M \nsubseteq N$. Choose $a \in R$ such that $a \in M$ and $a \notin N$. Hence $M \notin \mathcal{K}(a)$ and $N \in \mathcal{K}(a)$. Therefore $Spec^{\perp}(R)$ is a T_0-space.

(4) It can be obtained from (2) and (3) of Lemma 4.1. ◻

Lemma 4.3. *The following properties hold in an ADL R:*

(1) *for any $a \in R$, $\mathcal{K}(a) = \mathcal{K}((a)^{\perp})$,*
(2) *for any ideal I of R, $\mathcal{K}(I) = \mathcal{K}(I^{\perp})$,*
(3) *for any intrinsic ideal I of R, $\mathcal{K}(I) = \bigcup_{a \in I} \mathcal{K}((a)^{\perp})$.*

Proof. (1) Let $P \in \mathcal{K}(x) \cap Spec^{\perp}(R)$. Then $a \notin P$. Since P is intrinsic, we get $(a)^{\perp} \nsubseteq P$. Hence $P \in \mathcal{K}((a)^{\perp})$. Therefore $\mathcal{K}(a) \subseteq \mathcal{K}((a)^{\perp})$. Similarly, the other inclusion holds.

(2) Since $I \subseteq I^{\perp}$, we get $\mathcal{K}(I) \subseteq \mathcal{K}(I^{\perp})$. Let $P \in \mathcal{K}(I^{\perp}) \cap Spec^{\perp}(R)$. Then $I^{\perp} \nsubseteq P$. Choose $x \in I^{\perp}$ and $x \notin P$. Then $(a, D) \subseteq (s, D)$, for some $s \in I$. Hence $a \in (a)^{\perp} \subseteq (s)^{\perp}$. If $P \notin \mathcal{K}(I)$, then $s \in I \subseteq P$. Since P is intrinsic, we get $a \in (a)^{\perp} \subseteq (s)^{\perp} \subseteq P$, which is a contradiction. Thus $P \in \mathcal{K}(I)$. Therefore $\mathcal{K}(I^{\perp}) \subseteq \mathcal{K}(I)$.

(3) Let $P \in \mathcal{K}(I) \cap Spec^{\perp}(R)$. Then $I \nsubseteq P$. Choose $a \in I$ such that $a \notin P$. Then $P \in \mathcal{K}(a)$. Since $a \in I$, we get $P \in \bigcup_{a \in I} \mathcal{K}(a)$. Hence $\mathcal{K}(I) \subseteq \bigcup_{a \in I} \mathcal{K}(a)$. Conversely, let $P \in \bigcup_{a \in I} \mathcal{K}(a)$. Then $P \in \mathcal{K}(a)$ for some $a \in I$. Then $a \notin P$ for some $a \in I$. Hence $I \nsubseteq P$. Thus $P \in \mathcal{K}(I)$. Therefore $\bigcup_{a \in I} \mathcal{K}(a) \subseteq \mathcal{K}(I)$. ◻

Theorem 4.4. *For any ADL R, the lattice $(\mathcal{N}(R), \sqcup, \cap)$ of all intrinsic ideals of R is isomorphic to the lattice of all open subsets in $Spec^{\perp}(R)$.*

Proof. Denote the class of all open subsets of the space $Spec^{\perp}(R)$ by $\Im$. Clearly $(\Im, \cap, \cup)$ is a lattice. Define $\varphi : \mathcal{N}(R) \longrightarrow \Im$ by $\varphi(I^{\perp}) = \mathcal{K}(I)$ for all $I \in \mathcal{N}(R)$. By Lemma 4.3(2), every open subset of $Spec^{\perp}(R)$ is of the form $\mathcal{K}(I)$ for some $I \in \mathcal{N}(R)$. Hence the mapping φ is onto. Let $I, J \in \mathcal{N}(R)$ and suppose $\varphi(I) = \varphi(J)$. If $I \neq J$, then there exists $a \in J$ such that $a \notin I$. By Corollary 3.24, there exists $P \in Spec^{\perp}(R)$ such that $I \subseteq P$ and $a \notin P$. Thus $P \in \mathcal{K}(a)$ for $a \in J$. By Lemma 4.3(3), we get $P \in \bigcup_{a \in J} \mathcal{K}(a) = \mathcal{K}(J)$. Since $\varphi(I) = \varphi(J)$, we get $\mathcal{K}(I) = \mathcal{K}(J)$. Hence $P \in \mathcal{K}(J) = \mathcal{K}(I)$. Thus $I \nsubseteq P$ which contradicts the choice of P. Hence $I = J$ and therefore φ is one-one.

For any $I, J \in \mathcal{N}(R)$, we have $\varphi(I \cap J) = \mathcal{K}(I \cap J) = \mathcal{K}(I) \cap \mathcal{K}(J) = \varphi(I) \cap \varphi(J)$. Now,

$$\begin{aligned}
\varphi(I \sqcup J) &= \mathcal{K}(I \sqcup J) \\
&= \mathcal{K}((I \vee J)^{\perp}) && \text{by Theorem 3.10} \\
&= \mathcal{K}(I \vee J) && \text{by Lemma 4.3(2)} \\
&= \mathcal{K}(I) \cup \mathcal{K}(J) && \text{by Lemma 4.1(3)} \\
&= \varphi(I) \cup \varphi(J).
\end{aligned}$$

Hence φ is a homomorphism. Therefore $\mathcal{N}(R)$ is isomorphic to $\Im$. ◻

For any $A \subseteq R$, denote $\mathcal{H}(A) = \{P \in Spec^{\perp}(R) \mid A \subseteq P\}$. Then clearly $\mathcal{H}(A) = Spec^{\perp}(R) \setminus \mathcal{K}(A)$. Therefore $\mathcal{H}(A)$ is a closed set in $Spec^{\perp}(R)$. Also every closed set in $Spec^{\perp}(R)$ is of the form $\mathcal{H}(A)$ for some $A \subseteq R$.

Now, we have the following result.

Theorem 4.5. *For any ADL R and $X \subseteq Spec^{\perp}(R)$, the closure of A is given by $\overline{A} = \mathcal{H}(\bigcap_{P \in A} (P))$.*

Proof. Let $A \subseteq Spec^{\perp}(R)$ and $Q \in A$. Then $\bigcap_{P \in A} P \subseteq Q$. Thus $Q \in \mathcal{H}(\bigcap_{P \in A} P)$. Therefore $\mathcal{H}(\bigcap_{P \in A} P)$ is a closed set containing A. Let C be any closed set in $Spec^{\perp}(R)$. Then $C = \mathcal{H}(B)$ for some $B \subseteq R$. Since $A \subseteq C = H(B)$, we get that $B \subseteq P$ for all $P \in A$. Hence $B \subseteq \bigcap_{P \in A} P$. Therefore $H(\bigcap_{P \in A} P) \subseteq H(B) = C$. Hence $H(\bigcap_{P \in A} P)$ is the smallest closed set containing A. Therefore $\overline{A} = H(\bigcap_{P \in A} P)$. □

Theorem 4.6. *The following conditions are equivalent in an ADL R,*

(1) *every prime intrinsic ideal is maximal,*
(2) *every prime intrinsic ideal is minimal,*
(3) $Spec^{\perp}(R)$ *is a T_1-space,*
(4) $Spec^{\perp}(R)$ *is a Hausdorff space,*
(5) *for any $a, b \in R$, there exists $c \in R$ such that $a \wedge c \in (0)^{\perp}$ and $\mathcal{K}(b) \cap \{Spec^{\perp}(R) \backslash \mathcal{K}(a)\} = \mathcal{K}(b \wedge c)$.*

Proof. (1) ⇔ (2): Since every maximal intrinsic ideal is prime, it is clear.
(2) ⇒ (3): Assume that every prime intrinsic ideal is minimal. Let P and Q be two distinct prime intrinsic ideals of R. By (2), P and Q are minimal. Hence, we get $P \nsubseteq Q$ and $Q \nsubseteq P$. Choose $a \in P \setminus Q$ and $b \in Q \setminus P$. Then $Q \in \mathcal{K}(a) \setminus \mathcal{K}(b)$ and $P \in \mathcal{K}(b) \setminus \mathcal{K}(a)$. Therefore $Spec^{\perp}(R)$ is a T_1-space.
(3) ⇒ (4): Assume that $Spec^{\perp}(R)$ is a T_1-space. Let P be a prime intrinsic ideal of R. By Theorem 4.5, $\{P\} = \overline{\{P\}} = \{Q \in Spec^{\perp}_{F}(R) \mid P \subseteq Q\}$. Therefore P is maximal. Thus every prime intrinsic ideal is a maximal intrinsic ideal. Since every maximal intrinsic ideal is prime, we get that every prime intrinsic ideal is a minimal prime intrinsic ideal. Let $P, Q \in Spec^{\perp}(R)$ be such that $P \neq Q$. Choose $a \in P$ and $a \notin Q$. Since P is minimal, there exists $b \notin P$ such that $a \wedge b \in (0)^{\perp}$. Thus $P \in \mathcal{K}(b), Q \in \mathcal{K}(a)$ and $\mathcal{K}(a) \cap \mathcal{K}(b) = \mathcal{K}(a \wedge b) = \emptyset$. Therefore $Spec^{\perp}(R)$ is a Hausdorff space.
(4) ⇒ (5): Assume that $Spec^{\perp}(R)$ is Hausdorff. Hence $\mathcal{K}(a)$ is a compact subset of $Spec^{\perp}(R)$, for each $a \in R$. Then $\mathcal{K}(a)$ is a clopen subset of $Spec^{\perp}(R)$. Let $a, b \in R$ such that $a \neq b$. Then $\mathcal{K}(b) \cap \{Spec^{\perp}(R) \setminus \mathcal{K}(a)\}$ is a compact subset of the compact space $\mathcal{K}(b)$. Since $\mathcal{K}(b)$ is open in $Spec^{\perp}(R)$, we get $\mathcal{K}(b) \cap \{Spec^{\perp}(R) \setminus \mathcal{K}(a)\}$ is a compact open subset of $Spec^{\perp}(R)$. Hence by Theorem 4.2(2), there exists $c \in R$ such that $\mathcal{K}(c) = \mathcal{K}(b) \cap \{Spec^{\perp}(R) \setminus \mathcal{K}(a)\}$ Therefore $\mathcal{K}(b) \cap \{Spec^{\perp}(R) \setminus \mathcal{K}(a)\} \setminus \mathcal{K}(b) \cap \mathcal{K}(c) = \mathcal{K}(b \wedge c)$. Also $\mathcal{K}(a \wedge c) = \mathcal{K}(a) \cap \mathcal{K}(c) = \emptyset$. Therefore $a \wedge c \in (0)^{\perp}$.
(5) ⇒ (2): Let P be a prime intrinsic ideal of R. Choose $a, b \in R$ such that $a \in P$ and $b \notin P$. Then by condition (5), there exists $c \in R$ such that $a \wedge c \in (0)^{\perp}$ and $\mathcal{K}(b) \cap \{Spec^{\perp}(R) \backslash \mathcal{K}(a)\} = \mathcal{K}(b \wedge c)$. Then clearly $P \in \mathcal{K}(b) \cap \{Spec^{\perp}(R) \setminus \mathcal{K}(a)\} = \mathcal{K}(b \wedge c)$. If $c \in P$, then $b \wedge c \in P$, which is a contradiction to $P \in \mathcal{K}(b \wedge c)$. Hence $c \notin P$. Thus for each $a \in P$, there exists $c \notin P$ such that $a \wedge c \in (0)^{\perp}$. Therefore P is a minimal prime intrinsic ideal. □

For any ADL R, it is clear that $\mathcal{H}(A) = Spec^{\perp}(R) \setminus \mathcal{K}(A)$ and hence $\mathcal{H}(A)$ is a closed set in $Spec^{\perp}(R)$. In the following result, a necessary and sufficient condition is derived for the space $Spec^{\perp}(R)$ to become regular.

Theorem 4.7. *For any ADL R, the space $Spec^{\perp}(R)$ is a regular space if and only if for any $P \in Spec^{\perp}(R)$ and $a \notin P$, there exist an ideal I of L and $b \in R$ such that $P \in \mathcal{K}(b) \subseteq \mathcal{H}(I) \subseteq \mathcal{K}(a)$.*

Proof. Assume that $Spec^{\perp}(R)$ is a regular space. Let $P \in Spec^{\perp}(R)$ and $a \notin P$ for some $a \in R$. Then $P \notin \mathcal{H}(a)$. Since $Spec^{\perp}(R)$ is a regular space, there exist two disjoint open sets G and H in $Spec^{\perp}(R)$ such that $P \in G$ and $\mathcal{H}(a) \subseteq H$. Therefore $Spec^{\perp}(R) \setminus H \subseteq \mathcal{K}(a)$. Since $Spec^{\perp}(R) \setminus H$ is a closed set, we get that $Spec^{\perp}(R) \setminus H = \mathcal{H}(I)$, for some intrinsic ideal I in R. Thus $\mathcal{H}(I) = Spec^{\perp}(R) \setminus H \subseteq \mathcal{K}(a)$. Now $G \cap H = \emptyset$ will imply that $H \subseteq Spec^{\perp}(R) \setminus G$. Since $Spec^{\perp}(R) \setminus G$ is closed, we get $Spec^{\perp}(R) \setminus G = \mathcal{H}(J)$ for some intrinsic ideal J of R. Since $P \in G$, we get $P \notin Spec^{\perp}(R) \setminus G = \mathcal{H}(J)$ and hence $J \notin P$. Choose $b \in J$ such that $b \notin P$.

Then $P \in \mathcal{K}(b)$. Let $T \in H$. Then $J \subseteq T$ because of $H \subseteq \mathcal{H}(J)$. Since $b \in J \subseteq T$, we get $T \in \mathcal{H}(b)$. Thus $H \subseteq \mathcal{H}(b)$. Hence by (1),$\mathcal{K}(b) = Spec^{\perp}(R) \setminus \mathcal{H}(b) \subseteq Spec^{\perp}(R) \setminus H = \mathcal{H}(I)$. which means $\mathcal{K}(b) \subseteq \mathcal{H}(I)$. Thus for any $P \in Spec^{\perp}(R)$ and $a \notin P$, there exist an ideal I of R and $b \in R$ such that $P \in \mathcal{K}(b) \subseteq \mathcal{H}(I) \subseteq \mathcal{K}(a)$. Conversely, assume that for any $P \in Spec^{\perp}(R)$ and $a \notin P$, there exist an ideal I of R and $b \in R$ such that $P \in \mathcal{K}(b) \subseteq \mathcal{H}(I) \subseteq \mathcal{K}(a)$. To show that the space $Spec^{\perp}(R)$ is regular, let $P \in Spec^{\perp}(R)$ and $\mathcal{H}(K)$ be any closed set of $Spec^{\perp}(R)$ such that $P \notin \mathcal{H}(K)$. Then $K \nsubseteq P$. Hence there exist $a \in K$ such that $a \notin P$. Thus $P \in \mathcal{K}(a)$. Since $a \notin P$, by the assumption, there exists an ideal I of L and $b \in R$ such that $P \in \mathcal{K}(b) \subseteq \mathcal{H}(I) \subseteq \mathcal{K}(a)$. Hence $\mathcal{K}(a) \cap \mathcal{H}(K) = \emptyset$, because of $K \in \mathcal{H}(a)$ for $a \in K$. Thus $\mathcal{H}(K) \subseteq Spec^{\perp}(R) \setminus \mathcal{K}(a) \subseteq Spec^{\perp}(R) \setminus \mathcal{H}(I)$. Therefore $\mathcal{H}(K) \subseteq Spec^{\perp}(R) \setminus \mathcal{K}(a) \subseteq Spec^{\perp}(R) \setminus \mathcal{H}(I)$. Also $\mathcal{K}(b) \cap \mathcal{K}(I) = \emptyset$. Thus there exist two disjoint open sets $\mathcal{K}(b)$ and $\mathcal{K}(I)$ such that $P \in \mathcal{K}(b)$ and $\mathcal{H}(K) \subseteq \mathcal{K}(I)$. Therefore $Spec^{\perp}(R)$ is a regular space. □

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