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On Certain Inequalities for Isotonic Linear Functionals

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Abstract In this paper, Giaccardi inequality is derived for isotonic linear functions using h-convex functions. Petrović inequality for h-convex functions is recaptured for isotonic linear functionals. Several subsequenct results are obtained by using specific kind of utilized functions. Moreover, some examples of isotonic linear functionals are presented to get integral and discrete variants for measurable spaces. Additionally, mean value theorems are derived for funtionals linked with newly derived inequalities.

1 Introduction

In 2007, Varošanec [17] introduced the class of h-convex functions which unifies a number of classes of convex, P-, s-convex, Godunova-Levin and s-Godunova-Levin functions. An h-convex function is defined as a non-negative function $\alpha : I \to \mathbb{R}$ which satisfies the inequality

$$\alpha(\beta l + (1-\beta)m) \leq h(\beta)\alpha(l) + h(1-\beta)\alpha(m), \tag{1.1}$$

where $h : J \supseteq (0,1) \to \mathbf{R}$ be non-negative function, $\beta \in (0,1)$, $l, m \in I$, and I, J are intervals in $\mathbb{R}$. A function α is said to be h-concave, if the inequality (1.1) is reversed. It is usual to write $\alpha \in SX(h, I)$ if α is h-convex function on I. In the same paper, it is given that a function $l^{\frac{1}{2}}$ is $\beta^{\frac{1}{3}}-$convex on interval $(0,1)$. Analytically,

$$\alpha(l) = l^{\frac{1}{2}} \text{ and } h(\beta) = \beta^{\frac{1}{3}} \text{ for } l, \beta \in (0,1), \tag{1.2}$$

satisfy the inequality (1.1). Note that $\alpha(l) = l^{\frac{1}{2}}$ is not convex on $(0,1)$.

Alomari in [2], introduced the idea of h-chord and stated its geometrical interpretation with h-convex function as follows: For any h-convex function $\alpha : I \to \mathbb{R}$, the graph of α must be on or below the h-chord joining the endpoints $(l, \alpha(l))$ and $(m, \alpha(m))$ for all $l, m \in I$, that is,

$$\alpha(t) \leq [\alpha(m) - \alpha(l)]h\left(\frac{\beta - l}{m - l}\right) + \alpha(l) = L(\beta; h),$$

for any $l \leq t \leq m$ and $l, m \in I$, where $L(\beta; h)$ is representing h-chord. For example, if we draw the graph of the function given in (1.2) and some h-chords (see Figure 1), then one can note, for any two values from $(0,1)$, the corresponding graph of the function between these two values is below the h-chord joining the corresponding points on the graph. This shows that h-convex functions are quite interesting and should be further researched and explored. Before discussing the motivation, let's look at the definition of isotonic linear functionals, (see [15, Page 47]).

Definition 1.1. Let $\mathfrak{L}(\mathbb{E})$ be the class of functions defined on a non-empty set $\mathbb{E}$ with the following properties:

L1: If α and φ are in $\mathfrak{L}(\mathbb{E})$, so is $a\alpha + b\varphi$ for any $a, b \in \mathbb{R}$.

L2: $1(\kappa) = 1$ for $\kappa \in E$ is in $\mathfrak{L}(\mathbb{E})$.

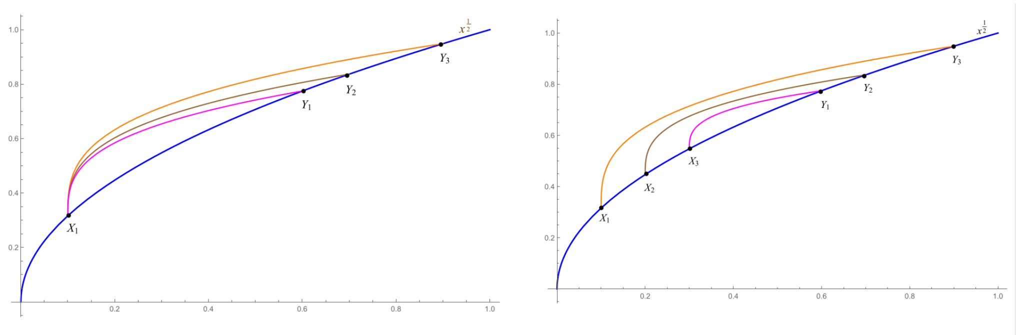

Figure 1. Note that graph of the function is below the h-chord joining any two points on the graph, that is, graph of the function is below the h-chord joining points X_1 and Y_1 or X_1 and Y_2 or X_1 and Y_3 on left and similarly function is below h-chord joining points X_1 and Y_1 or X_2 and Y_2 or X_3 and Y_3 on right.

A functional $\mathcal{A}$ defined on $\mathfrak{L}(\mathbb{E})$ is an isotonic linear functional if it satisfies:

A1: If $\alpha, \varphi \in \mathfrak{L}(\mathbb{E})$, $a, b \in \mathbb{R}$, then $\mathcal{A}(a\alpha + b\varphi) = a\mathcal{A}(\alpha) + b\mathcal{A}(\varphi)$.

A2: If $\alpha \geq 0$ for $\alpha \in \mathfrak{L}(\mathbb{E})$, then $\mathcal{A}(\alpha) \geq 0$.

One can see various examples of isotonic linear functionals discovered on timescale in [3]. Examples which are very common and simple included:

$$\mathcal{A}(\varphi) = \sum_{n \in \mathbb{E}} q_n \varphi_n$$

when $\mathbb{E}$ is a subset of {1,2,3, ...} with all $q_n > 0$ and

$$\mathcal{A}(\varphi) = \int_{\mathbb{E}} \varphi d\mu$$

when μ defined as a positive measure on $\mathbb{E}$.

In literature, scientists and mathematicians considered convex functions and isotonic linear functionals to produce compelling and interesting results. See for similar literature in [5, 8, 9, 10, 11, 16, 20] and the reference therein. The older examples include generalization of a famous Jensen inequality for isotonic linear functional given by Jessen [12] in 1931. Many researchers and mathematicians later presented functional generalizations of famous inequalities, for example, Jensen's inequality [1], Hadamard inequality [6], Grüss inequality [7], Hölder inequality [18], Ostrowski inequality [21]. Applications of such generalizations in various branches of mathematics and related sciences provide motivation for their further investigations.

The aim of this paper is to get a generalized version of Giaccardi inequality for isotonic linear functionals via h-convexity. By applying this inequality with specific conditions, subsequent results provide Petrović inequality and several other versions in discrete, integral and time scale forms. Next, we give the classical Giaccardi inequality and its varients for isotonic linear functionals.

Theorem 1.2. *[19] Let* $\mathbf{l} := (l_1, l_2, ..., l_n) \in I^n$ *be non-negative n-tuples* $\mathbf{q} := (q_1, q_2, ..., q_n)$ *be positive numbers and* $l_0, \tilde{l}_n := \sum_{i=1}^n q_i l_i \in I$ *such that*

$$(l_i - l_0)(\tilde{l}_n - l_i) \geq 0, \text{ where } \tilde{l}_n \neq l_0. \tag{1.3}$$

If $\alpha : I \to \mathbb{R}$ *is a convex function, then*

$$\sum_{i=1}^n q_i \alpha(l_i) \leq \frac{\sum_{i=1}^n q_i (l_i - l_0)}{\tilde{l}_n - l_0} \alpha(\tilde{l}_n) + \frac{\sum_{i=1}^n q_i \left(\tilde{l}_n - l_i\right)}{\tilde{l}_n - l_0} \alpha(l_0), \tag{1.4}$$

where I *is an interval in* $\mathbb{R}$.

Theorem 1.3. *[4] Consider an isotonic linear functional $\mathcal{A}$ defined on $\mathfrak{L}(\mathbb{E})$, $l_0 \in \mathbb{R}$ and $\varphi \in \mathfrak{L}(\mathbb{E})$ such that $l_0 \neq \mathcal{A}(\varphi)$ and*

$$(\mathcal{A}(\varphi) - \varphi(\kappa)) (\varphi(\kappa) - l_0) \geq 0 \, \text{for all } \kappa \in \mathbb{E}. \tag{1.5}$$

If α is convex on interval $[l_0, \mathcal{A}(\varphi)]$ or on interval $[\mathcal{A}(\varphi), l_0]$ and $\alpha(\varphi) \in \mathfrak{L}(\mathbb{E})$, then

$$\mathcal{A}(\alpha(\varphi)) \leq \{[\mathcal{A}(\varphi) - \mathcal{A}(1)l_0]\alpha(\mathcal{A}(\varphi)) + [\mathcal{A}(1) - 1]\mathcal{A}(\varphi)\alpha(l_0)\}/[\mathcal{A}(\varphi) - l_0] \tag{1.6}$$

is valid.

Further, we organize the work as follows: Functional version of Giaccardi inequality for $h-$convex functions is proved and Petrović inequality is stated as its special case in Section 2. In the same section in terms of applications, integral versions of the aforesaid inequalities are given in particular cases. Section 3 consists of Lagrange mean value theorems constructed for functionals associated with the generalized Giaccardi and Petrović inequalities.

2 Main results

To present the key findings of the paper, following definition and result are very important.

Definition 2.1. A function $h : J \to \mathbb{R}$ be a supermultiplicative function if

$$h(m_1 m_2) \geq h(m_1)h(m_2) \tag{2.1}$$

for all $m_1, m_2 \in J$ and $m_1, m_2 \geq 1$. If inequality (2.1) is reversed, then h is said to be submultiplicative function. If the equality holds in (2.1), then h is said to be a multiplicative function.

Lemma 2.2. *[17] Let $h : J \to \mathbb{R}$ be a non-negative supermultiplicative function and let $f : I \to \mathbb{R}$ be a function such that $\alpha \in SX(h, I)$. Then for $u, v, w \in I, u < v < w$ such that $w - u, w - v, v - u \in J$, the following inequality holds:*

$$h(w - v)\alpha(u) - h(w - u)\alpha(v) + h(v - u)\alpha(w) \geq 0. \tag{2.2}$$

The following theorem establishes the Giaccardi inequality for isotonic linear functionals with respect to h-convex functions.

Theorem 2.3. *Let $\mathcal{A}$ be an isotonic linear functional defined on $\mathfrak{L}(\mathbb{E})$, $l_0 \in \mathbb{R}$ and $\varphi \in \mathfrak{L}(\mathbb{E})$ such that $l_0 \neq \mathcal{A}(\varphi)$ and condition (1.5) is satisfied. If $h : J \to \mathbb{R}$ be a positive supermultiplicative function and $\alpha \in SX\,(h, [l_0, \mathcal{A}(\varphi)])$ or $\alpha \in SX\,(h, [\mathcal{A}(\varphi), l_0])$, then*

$$\mathcal{A}(\alpha(\varphi)) \leq \frac{\mathcal{A}(h(\varphi(\kappa) - l_0))}{h(\mathcal{A}(\varphi) - l_0)}\alpha(\mathcal{A}(\varphi)) + \frac{\mathcal{A}(h(\mathcal{A}(\varphi) - \varphi(\kappa)))}{h(\mathcal{A}(\varphi) - l_0)}\alpha(l_0). \tag{2.3}$$

Proof. From condition (1.5), we have that

$$l_0 \leq \varphi(\kappa) \leq \mathcal{A}(\varphi) \text{ for } \kappa \in \mathbb{E} \tag{2.4}$$

or

$$\mathcal{A}(\varphi) \leq \varphi(\kappa) \leq l_0 \text{ for } \kappa \in \mathbb{E}. \tag{2.5}$$

Keeping in mind the condition (2.4), h is supermultiplicative and $\alpha \in SX\,(h, [l_0, \mathcal{A}(\varphi)])$, we set $u = l_0$, $v = \varphi(\kappa)$ and $w = \mathcal{A}(\varphi)$ in Lemma 2.2 to get the following inequality

$$h\left(\mathcal{A}(\varphi) - \varphi(\kappa)\right)\alpha(l_0) - h\left(\mathcal{A}(\varphi) - l_0\right)\alpha\left(\varphi(\kappa)\right) + h\left(\varphi(\kappa) - l_0\right)\alpha\left(\mathcal{A}(\varphi)\right) \geq 0,$$

A function h is given to be positive, so we can write

$$\frac{h(\varphi(\kappa) - l_0)}{h(\mathcal{A}(\varphi) - l_0)}\alpha(\mathcal{A}(\varphi)) + \frac{h(\mathcal{A}(\varphi) - \varphi(\kappa))}{h(\mathcal{A}(\varphi) - l_0)}\alpha(l_0) - \alpha(\varphi) \geq 0.$$

As the right hand side of above inequality is in $\mathfrak{L}(\mathbb{E})$, by using the property stated in clause (A2) of the isotonic linear functionals, one has

$$\mathcal{A}\left(\frac{h(\varphi(\kappa)-l_0)}{h(\mathcal{A}(\varphi)-l_0)}\alpha(\mathcal{A}(\varphi))+\frac{h(\mathcal{A}(\varphi)-\varphi(\kappa))}{h(\mathcal{A}(\varphi)-l_0)}\alpha(l_0)-\alpha(\varphi)\right)\geq 0.$$

Since $h(\mathcal{A}(\varphi)-l_0)$, $\alpha(l_0)$ and $\alpha\left(\mathcal{A}(\varphi)\right)$ are reals, by using clause (A1) of the isotonic linear functionals, one get

$$\frac{\mathcal{A}\left(h(\varphi(\kappa)-l_0)\right)}{h(\mathcal{A}(\varphi)-l_0)}\alpha(\mathcal{A}(\varphi))+\frac{\mathcal{A}\left(h(\mathcal{A}(\varphi)-\varphi(\kappa))\right)}{h(\mathcal{A}(\varphi)-l_0)}\alpha(l_0)-\mathcal{A}\left(\alpha(\varphi)\right)\geq 0.$$

This is equivalent to inequality (2.3).

In a similar way, for condition (2.5), h is supermultiplicative and $\alpha\in SX\left(h,[\mathcal{A}(\varphi),l_0]\right)$, we set $u=\mathcal{A}(\varphi)$, $v=\varphi(\kappa)$ and $w=l_0$ in Lemma 2.2 to get required result. □

In [13], using the h-convexity of the function, the Ginaccardi inequality has been proved with the stronger condition on h that $h(\beta)+h(1-\beta)\leq 1$ for $\beta\in(0,1)$. Such condition restrict the particular values of h for which h-convex functions becomes P-functions, s-convex, Godunova-Levin functions and s-Godunova-Levin functions. It is critical to provide the theorem in the absence of such stronger conditions. It is stated as follows:

Theorem 2.4. *Let $l_1,l_2,...,l_n\in I$, where $I\subseteq\mathbb{R}$ be an interval, $q_1,q_2,...,q_n\in(0,\infty)$ and $l_0,\tilde{l}_n:=\sum_{i=1}^{n}q_il_i\in I$ such that*

$$(l_i-l_0)(\tilde{l}_n-l_i)\geq 0, \tag{2.6}$$

where $\tilde{l}_n\neq l_0$. If $\alpha\in SX\left(h,I\right)$, then

$$\sum_{i=1}^{n}q_i\alpha(l_i)\leq\mathcal{C}\alpha(\tilde{l}_n)+\mathcal{D}\alpha(l_0), \tag{2.7}$$

where

$$\mathcal{C}=\frac{\sum_{i=1}^{n}q_ih(l_i-l_0)}{h(\tilde{l}_n-l_0)}\quad\text{and }\mathcal{D}=\frac{\sum_{i=1}^{n}q_ih(\tilde{l}_n-l_i)}{h(\tilde{l}_n-l_0)}. \tag{2.8}$$

Proof. Consider a set $\mathbb{E}$ to be $\{1,2,...,n\}$ and $\mathfrak{L}(\mathbb{E})=\{\varphi:\mathbb{E}\to I\mid\varphi(i)=l_i,i\in\mathbb{E}\}$. Then it is easy to see that $\mathfrak{L}(\mathbb{E})$ satisfies L1 and L2 of Definition 1.1. Also consider

$$\mathcal{A}(\varphi)=\sum_{i\in\mathbb{E}}q_i\varphi(i)=\sum_{i=1}^{n}q_il_i.$$

It is also easy to see that $\mathcal{A}$ satisfies A1, A2 of Definition 1.1. Substituting the values of $\mathcal{A}$ and φ in Theorem 2.3 to obtain

$$(\varphi(\kappa)-l_0)(\mathcal{A}(\varphi)-\varphi(\kappa))=(l_i-l_0)(\tilde{l}_n-l_i)\geq 0$$

and

$$\mathcal{A}(\varphi)=\sum_{i=1}^{n}q_il_i\neq l_0.$$

At last, expression (2.3) becomes the outcome that we need. □

The integral analogues of the above results is given in the following theorem.

Theorem 2.5. *Consider a measurable space (Ω,Λ,ζ), where $\zeta(\Omega)$ is positive finite measure. Also consider a measureable function $\varphi:\Omega\to I$ and $l_0,\int_\Omega\varphi(\kappa)d\zeta\in I$ such that $\int_\Omega\varphi(\kappa)d\zeta\neq l_0$ and*

$$(\varphi(\kappa)-l_0)\left(\int_\Omega\varphi(\kappa)d\zeta-\varphi(\kappa)\right)\geq 0. \tag{2.9}$$

If $\alpha \in SX(h, I)$*, then*

$$\int_{\Omega} \alpha(\varphi) d\zeta \leq \mathcal{C}\alpha\left(\int_{\Omega} \varphi(\kappa) d\zeta\right) + \mathcal{D}\alpha(l_0), \tag{2.10}$$

is valid, where

$$\mathcal{C} = \frac{\int_{\Omega} h\left(\varphi(\kappa) - l_0\right) d\zeta}{h\left(\int_{\Omega} \varphi(\kappa) d\zeta - l_0\right)} \quad \textit{and} \quad \mathcal{D} = \frac{\int_{\Omega} h\left(\int_{\Omega} \varphi(\kappa) d\zeta - \varphi(\kappa)\right) d\zeta}{h\left(\int_{\Omega} \varphi(\kappa) d\zeta - l_0\right)}.$$

Proof. Assume that $\mathbb{E} = \Omega$ and

$$\mathfrak{L}(\mathbb{E}) = \left\{\varphi : \Omega \to I \mid \int_{\Omega} \varphi(\kappa) d\zeta \text{ exists}\right\},$$

then $\mathfrak{L}(\mathbb{E})$ satisfies conditions L1 and L2 of Definition 1.1. If we take

$$\mathcal{A}(\varphi) = \int_{\Omega} \varphi(\kappa) d\zeta,$$

then $\mathcal{A}$ satisfies conditions A1 and A2 of Definition 1.1. Using the above values of an istonic linear functional $\mathcal{A}$ in Theorem 2.3 to get

$$(\varphi(\kappa) - l_0)(\mathcal{A}(\varphi) - \varphi(\kappa)) = (\varphi(\kappa) - l_0)\left(\int_{\Omega} \varphi(\kappa) d\zeta - \varphi(\kappa)\right) \geq 0$$

and

$$\mathcal{A}(\varphi) = \int_{\Omega} \varphi(\kappa) d\zeta \neq l_0.$$

Finally, an inequality (2.3) becomes (2.10) as our required result.

□

Remark 2.6. In Theorems 2.3, 2.4 and 2.5, if we take particular values of h, that are, $h(\beta) = \beta$, $h(\beta) = 1$, $h(\beta) = \beta^s$, $h(\beta) = \frac{1}{\beta}$ and $h(\beta) = \frac{1}{\beta^s}$ for $\beta, s \in (0, 1)$, we get respective variant of Giacardi inequality for convex functions, $P-$functions, $s-$convex functions in second sense, Godunova-Levin functions and $s-$Godunova-Levin functions of second sense respectively.

The Petrović inequality is particular case of the Giaccardi inequality. Becasue of its importance in the literature, we are stating its variant for h-convex functions in the following corollaries.

Corollary 2.7. *Let* $\mathcal{A}$ *be an isotonic linear functional defined on* $\mathfrak{L}(\mathbb{E})$ *and* $\varphi \in \mathfrak{L}(\mathbb{E})$ *such that* $\mathcal{A}(\varphi) \geq \varphi(\kappa) \geq 0$ *or* $\mathcal{A}(\varphi) \leq \varphi(\kappa) \leq 0$*. If* $h : J \to \mathbb{R}$ *be a positive supermultiplicative function and* $\alpha \in SX(h, [0, \mathcal{A}(\varphi)])$ *or* $\alpha \in SX(h, [\mathcal{A}(\varphi), 0])$*, then*

$$\mathcal{A}(\alpha(\varphi)) \leq \frac{\mathcal{A}(h(\varphi(\kappa)))}{h(\mathcal{A}(\varphi))}\alpha(\mathcal{A}(\varphi)) + \frac{\mathcal{A}(h(\mathcal{A}(\varphi) - \varphi(\kappa)))}{h(\mathcal{A}(\varphi))}\alpha(0). \tag{2.11}$$

Proof. It is easy to see that if we take $l_0 = 0$ in Theorem 2.3, then the condition (1.5) becomes the condition of this corollary and inequality (2.3) becomes the result we required. □

The following result presents Petrović inequality for h-convex functions. It is pertinent to mention that a similar result was published in [13] but with stronger conditions on function h.

Corollary 2.8. *Let* $l_1, l_2, ..., l_n \in [0, a]^n$ *or* $[a, 0]^n$*,* $q_1, q_2, ..., q_n$ *be positive numbers such that*

$$\tilde{l}_n \geq l_j \text{ or } l_j \geq \tilde{l}_n \text{ for } j = 1, 2, ..., n, \tag{2.12}$$

where $\tilde{l}_n := \sum_{i=1}^{n} q_i x_i \neq 0$*. If* $h : J \to \mathbb{R}$ *be a positive supermultiplicative function and* $\alpha \in SX(h, [0, a])$ *or* $\alpha \in SX(h, [a, 0])$*, then*

$$\sum_{i=1}^{n} q_i \alpha(l_i) \leq \sum_{i=1}^{n} q_i h\left(\frac{l_i}{\tilde{l}_n}\right) \alpha(\tilde{l}_n) + \sum_{i=1}^{n} q_i \left(1 - h\left(\frac{l_i}{\tilde{l}_n}\right)\right) \alpha(0). \tag{2.13}$$

Proof. Put $l_0 = 0$ in Theorem 2.4 to get the required results. □

The following corollary provides integral analogues of the well-known Petrović inequality for h-convex functions.

Corollary 2.9. *Consider a measurable space* (Ω, Λ, ζ) *with a positive finite measure* $\zeta(\Omega)$. *Also consider a measurable function* $\varphi : \Omega \to I$ *and* $\int_\Omega \varphi(\kappa) d\zeta \in I$ *such that*

$$\int_\Omega \varphi(\kappa) d\zeta \geq \varphi(\kappa) \text{ or } \int_\Omega \varphi(\kappa) d\zeta \leq \varphi(\kappa) \text{ for } \kappa \in \Omega. \tag{2.14}$$

If $\alpha \in SX(h, I)$, *then*

$$\int_\Omega \alpha(\varphi) d\zeta \leq \frac{\int_\Omega h(\varphi(\kappa)) d\zeta}{h\left(\int_\Omega \varphi(\kappa) d\zeta\right)} \alpha\left(\int_\Omega \varphi(\kappa) d\zeta\right) + \frac{\int_\Omega h\left(\int_\Omega \varphi(\kappa) d\zeta - \varphi(\kappa)\right) d\zeta}{h\left(\int_\Omega \varphi(\kappa) d\zeta\right)} \alpha(0). \tag{2.15}$$

is valid.

Proof. Take $l_0 = 0$ in Theorem 2.5, then condition (2.9) becomes (2.14) and inequality (2.10) becomes (2.15). □

In the following corollary integral variant of the Petrović inequality is stated. A resulting inequality is provided at [15, Page 158] but with different conditions.

Corollary 2.10. *Consider a measurable space* (Ω, Λ, ζ) *with a positive finite measure* $\zeta(\Omega)$. *Also consider a measurable function* $\varphi : \Omega \to I$ *and* $\int_\Omega \varphi(\kappa) d\zeta \in I$ *such that*

$$\int_\Omega \varphi(\kappa) d\zeta \geq \varphi(\kappa) \text{ or } \int_\Omega \varphi(\kappa) d\zeta \leq \varphi(\kappa) \text{ for } \kappa \in \Omega. \tag{2.16}$$

If α *is convex on* I, *then*

$$\int_\Omega \alpha(\varphi) d\zeta \leq \alpha\left(\int_\Omega \varphi(\kappa) d\zeta\right) + \left(\int_\Omega d\zeta - 1\right) \alpha(0) \tag{2.17}$$

is valid.

3 Mean value theorems

In this section we give Mean Values Theorems (MVTs) for the non-negative difference of the inequality (2.3). We defined the linear functional in the following way:

Consider the closed interval $\tilde{I}$, $l_0 \in \tilde{I}$ and $(l_1, ..., l_n) \in \tilde{I}^n$. Then for $\alpha : \tilde{I} \to \mathbb{R}$, we define a functional

$$\mathfrak{G}(\alpha, h; l_0) = \frac{\mathcal{A}(h(\varphi(\kappa) - l_0))}{h(\mathcal{A}(\varphi) - l_0)} \alpha(\mathcal{A}(\varphi)) + \frac{\mathcal{A}(h(\mathcal{A}(\varphi) - \varphi(\kappa)))}{h(\mathcal{A}(\varphi) - l_0)} \alpha(l_0) - \mathcal{A}(\alpha(\varphi)). \tag{3.1}$$

where $\mathcal{A}$ is an isotonic linear functional.

By taking $l_0 = 0$ in (3.1), one can get the linear functional for Petrović's inequality as follows:

$$\mathfrak{T}(\alpha, h) = \mathfrak{G}(\alpha, h; 0). \tag{3.2}$$

In the proof of [13, Theorem 2.1], it has been proved that if f is h-convex function with the conditions that h is super-multiplicative and $h(\alpha) + h(1-\alpha) \leq 1$ for $\alpha \in (0, 1)$, then $\frac{f(x) - f(c)}{h(x-c)}$ is increasing for $x > c$.

This fact has been utilized to prove the subsequent lemma, which introduces two h-convex functions under specific circumstances to prove MVT of Lagrange type.

Lemma 3.1. *Let* $\alpha : \tilde{I} \to \mathbb{R}$ *and* $h : (0, \infty) \to \mathbb{R}^+$ *be differentiable functions such that*

$$n \leqslant \frac{h(l - l_0)\alpha'(l) - (\alpha(l) - \alpha(l_0))h'(l - l_0)}{2lh(l - l_0) - (l^2 - l_0^2)h'(l - l_0)} \leqslant N \ \forall l, l_0 \in \tilde{I}. \tag{3.3}$$

The functions $\psi_1, \psi_2 : \tilde{I} \to \mathbb{R}$ *are* $h-$*convex function on* $\tilde{I}$, *if*

$$\psi_1(l) = Nl^2 - \alpha(l) \text{ and } \psi_2(l) = \alpha(l) - nl^2.$$

Proof. First consider that

$$\begin{aligned}\mathcal{G}(\psi_1, h; l) =& \frac{\psi_1(l) - \psi_1(l_0)}{h(l - l_0)} \\ =& \frac{N(l^2 - l_0^2)}{h(l - l_0)} - \frac{\alpha(l) - \alpha(l_0)}{h(l - l_0)}.\end{aligned}$$

After differentiating, one has

$$\begin{aligned}\mathcal{G}'(\psi_1, h; l) = N&\frac{h(l - l_0)2l - (l^2 - l_0^2)h'(l - l_0)}{h^2(l - l_0)} \\ &- \frac{h(l - l_0)\alpha'(l) - (\alpha(l) - \alpha(l_0))h'(l - l_0)}{h^2(l - l_0)}.\end{aligned}$$

From (3.3), one has

$$\begin{aligned}&h(l - l_0)\alpha'(l) - (\alpha(l) - \alpha(l_0))h'(l - l_0) \\ &\leq N(2lh(l - l_0) - (l^2 - l_0^2)h'(l - l_0)).\end{aligned}$$

This leads to

$$\begin{aligned}&\frac{h(l - l_0)\alpha'(l) - (\alpha(l) - \alpha(l_0))h'(l - l_0)}{h^2(l - l_0)} \\ &\leq N\frac{2lh(l - l_0) - (l^2 - l_0^2)h'(l - l_0)}{h^2(l - l_0)}.\end{aligned}$$

This implies

$$\begin{aligned}&N\frac{2lh(l - l_0) - (l^2 - l_0^2)h'(l - l_0)}{h^2(l - l_0)} \\ &- \frac{h(l - l_0)\alpha'(l) - (\alpha(l) - \alpha(l_0))h'(l - l_0)}{h^2(l - l_0)} \geq 0.\end{aligned}$$

Hence $\mathcal{G}'(\psi_1, h; l) \geq 0$.

Similar to the above, one may show $\mathcal{G}'(\psi_2, h; l) \geq 0$.

It means $\mathcal{G}(\psi_1, h; l)$ and $\mathcal{G}(\psi_2, h; l)$ are increasing for $l > l_0$. Hence, ψ_1 and ψ_2 are $h-$convex functions. □

Lagrange type MVT for the functional defined in (3.1) is given in the following theorem.

Theorem 3.2. *Assuming the functional $\mathfrak{G}$, which is given in* (3.1)*. For bounded functions h, h' and $\alpha \in C^1(\tilde{I})$, there exists ς in the interior of $\tilde{I}$ such that*

$$\mathfrak{G}(\alpha, h; l_0) = \frac{h(\varsigma - l_0)\alpha'(\varsigma) - (\alpha(\varsigma) - \alpha(l_0))h'(\varsigma - l_0)}{2\varsigma h(\varsigma - l_0) - (\varsigma^2 - l_0^2)h'(\varsigma - l_0)}\mathfrak{G}(\varphi, h; l_0), \tag{3.4}$$

where $\varphi(l) = l^2$, provided that $\mathfrak{G}(\varphi, h; l_0)$ is non-zero.

Proof. As it is given that h, h' are bounded$and \alpha \in C^1(\tilde{I})$, so there are real numbers n and N such that

$$n \leqslant \frac{h(l - l_0)\alpha'(l) - (\alpha(l) - \alpha(l_0))h'(l - l_0)}{2lh(l - l_0) - (l^2 - l_0^2)h'(l - l_0)} \leqslant N, \forall l, l_0 \in \tilde{I}.$$

Considering an $h-$convex function ψ_1 defined in Lemma 3.1 to get

$$\mathfrak{G}(\psi_1, h; l_0) \geq 0,$$

that is

$$\mathfrak{G}(Nl^2 - \alpha(l), h; l_0) \geq 0.$$

This implies

$$N\mathfrak{G}(\varphi,h;l_0)\geq\mathfrak{G}(\alpha,h;l_0). \tag{3.5}$$

In a similar manner, one can consider an $h-$convex function ψ_2 defined in Lemma 3.1 to get

$$n\mathfrak{G}(\varphi,h;l_0)\leq\mathfrak{G}(\alpha,h;l_0). \tag{3.6}$$

Combining inequalities (3.5) and (3.6), one has

$$n\leq\frac{\mathfrak{G}(\alpha,h;l_0)}{\mathfrak{G}(\varphi,h;l_0)}\leq N.$$

So there exist ς in the interior of $\tilde{I}$ such that

$$\frac{\mathfrak{G}(\alpha,h;l_0)}{\mathfrak{G}(\varphi,h;l_0)}=\frac{h(\varsigma-l_0)\alpha'(\varsigma)-(\alpha(\varsigma)-\alpha(l_0))h'(\varsigma-l_0)}{2\varsigma h(\varsigma-l_0)-(\varsigma^2-l_0^2)h'(\varsigma-l_0)}.$$

This is equivalent to (3.7). □

Remark 3.3. We now discuss Theorem 3.2 and the non-negative linear functional given in (3.1) when h and m have different particular values.

(i) Taking $h(\upsilon)=\upsilon^s$, one gets the Lagrange-type MVT for $s-$convex functions.

(ii) By taking $h(\upsilon)=\upsilon$, one gets the generalization of a result given by A. U. Rehman et al. in [14, Corollary 2.2].

The following theorem states the Lagrange type MVT for functional attributed by the Petrović's inequality for $h-$convex function.

Theorem 3.4. *Consider a functional $\mathfrak{T}$ defined in (3.2). For bounded functions h, h' and $\alpha\in C^1(\tilde{I})$ there exists ς in the interior of $\tilde{I}$ such that*

$$\mathfrak{T}(\alpha,h)=\frac{h(\varsigma)\alpha'(\varsigma)-(\alpha(\varsigma)-\alpha(0))h'(\varsigma)}{2\varsigma h(\varsigma)-\varsigma^2h'(\varsigma)}\mathfrak{T}(\varphi,h), \tag{3.7}$$

where $\varphi(l)=l^2$, provided that $\mathfrak{T}(\varphi,h;0)$ is non-zero.

Proof. By taking $\upsilon_0=0$ in (3.4), one gets the required result. □

Remark 3.5. We analyse the instances for different values of h and m in Theorem 3.4 and the non-negative linear functional given in (3.2).

(i) To get the result for $s-$convex function, take $h(\upsilon)=\upsilon^s$.

(ii) By setting $h(\upsilon)=\upsilon$ gives the result for convex functions.

The following theorem consists of a Cauchy type MVT for the functional defined in (3.1).

Theorem 3.6. *Consider a functional $\mathfrak{G}$ defined in (3.1). If $\alpha_1,\alpha_2\in C^1(\tilde{I})$, then there exist ς in the interior of $\tilde{I}$ such that*

$$\frac{\mathfrak{G}(\alpha_1;h,l_0)}{\mathfrak{G}(\alpha_2;h,l_0)}=\frac{h(\varsigma-l_0)\alpha_1'(\varsigma)-\alpha_1(\varsigma)h'(\varsigma-l_0)+\alpha_1(l_0)h'(\varsigma-l_0)}{h(\varsigma-l_0)\alpha_2'(\varsigma)-\alpha_2(\varsigma)h'(\varsigma-l_0)+\alpha_2(l_0)h'(\varsigma-l_0)}, \tag{3.8}$$

provided that the denominators are non-zero.

Proof. Consider a function $\mathcal{K}$ from $C^1(\tilde{I})$ defined as follows:

$$\mathcal{K}=t_1\alpha_1-t_2\alpha_2,\ \text{where } t_1=\mathfrak{G}(\alpha_2;h,l_0)\text{ and } t_2=\mathfrak{G}(\alpha_1;h,l_0).$$

Replace α with $\mathcal{K}$ in Theorem 3.2, then one has

$$\begin{aligned}0 =&h(\varsigma - l_0)((t_1\alpha_1 - t_2\alpha_2)(\varsigma))' - (t_1\alpha_1 - t_2\alpha_2)(\varsigma)h'(\varsigma - l_0)\\ &+ (t_1\alpha_1 - t_2\alpha_2)(l_0)h'(\varsigma - l_0)\\ =&h(\varsigma - l_0)(t_1\alpha_1'(\varsigma) - t_2\alpha_2'(\varsigma)) - t_1\alpha_1(\varsigma)h'(\varsigma - l_0) + t_2\alpha_2(\varsigma)h'(\varsigma - l_0)\\ &+ t_1\alpha_1(l_0)h'(\varsigma - l_0) - t_2\alpha_2(l_0)h'(\varsigma - l_0)\\ =&t_1\left\{h(\varsigma - l_0)\alpha_1'(\varsigma) - \alpha_1(\varsigma)h'(\varsigma - l_0) + \alpha_1(l_0)h'(\varsigma - l_0)\right\}\\ &- t_2\left\{h(\varsigma - l_0)\alpha_2'(\varsigma) - \alpha_2(\varsigma)h'(\varsigma - l_0) + \alpha_2(l_0)h'(\varsigma - l_0)\right\}.\end{aligned}$$

This gives

$$\frac{t_2}{t_1} = \frac{h(\varsigma - l_0)\alpha_1'(\varsigma) - \alpha_1(\varsigma)h'(\varsigma - l_0) + \alpha_1(l_0)h'(\varsigma - l_0)}{h(\varsigma - l_0)\alpha_2'(\varsigma) - \alpha_2(\varsigma)h'(\varsigma - l_0) + \alpha_2(l_0)h'(\varsigma - l_0)}.$$

Now put the values of t_1 and t_2 in above expression to get the required result. □

Remark 3.7. Now we discuss the cases when h and m have different particular values in Theorem 3.6 and the isotonic linear functional defined in (3.1).

(i) In a case where $h(\upsilon) = \upsilon^s$, the result for $s-$convex functions can be obtained.

(ii) Cauchy-type MVT for convex function can be obtained by setting $h(\upsilon) = \upsilon$.

In the following theorem, Cauchy type MVT related to functional due to Petrović's inequality for $h-$convex functions is given.

Theorem 3.8. *Let the conditions of Theorem 3.2 are valid. If $\alpha_1, \alpha_2 \in C^1(\tilde{I})$, then there exist ς in the interior of $\tilde{I}$ such that*

$$\frac{\mathfrak{T}(\alpha_1, h)}{\mathfrak{T}(\alpha_2, h)} = \frac{h(\varsigma)\alpha_1'(\varsigma) - \alpha_1(\varsigma)h'(\varsigma) + \alpha_1(0)h'(\varsigma)}{h(\varsigma)\alpha_2'(\varsigma) - \alpha_2(\varsigma)h'(\varsigma) + \alpha_2(0)h'(\varsigma)}, \tag{3.9}$$

provided that the denominators are non-zero.

Proof. By setting $\upsilon_0 = 0$ in (3.8), one gets the required result. □

Remark 3.9. In Theorem 3.8, and isotonic linear functional defined in (3.2), we discuss the cases for different values of h and m.

(i) To get the result for $s-$convex functions, take $h(\upsilon) = \upsilon^s$.

(ii) A case when $h(\upsilon) = \upsilon$ can be considered as Cauchy-type MVT for the Petrović's inequality for convex functions.

Conclusion

In this paper, famous inequality known as a Giaccardi inequality is derived for isotonic linear functional via h-convexity. Another well-known Petrović inequality is established as a particular case of the Giaccardi inequality. We derive important variants of Giaccardi and Petrović inequalities by using popular cases of an isotonic linear functional.

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