Palestine Journal of Mathematics

Vol 13(4)(2024) , 1266–1276

ON POSITIVE INTEGER SOLUTIONS OF A SPECIAL DIOPHANTINE EQUATION

Ahmet Emin

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11B37; Secondary 11B39, 11D09.

Keywords and phrases: Diophantine equation, recurrence relation, generalized Fibonacci and Lucas sequences, Pell equation.

The author would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of the paper.

Abstract In this study, the positive solutions of the Diophantine equation $D: x^2-\left(\sigma^2+4\right)y^2-\left(2\sigma-2\right)x-\left(2\sigma^4+8\sigma^2\right)y-\sigma^6-4\sigma^4+\sigma^2-2\sigma-3=0$ on the set $\mathbb{Z}$ are investigated, along with some recurrence relations that provide the relationships among these solutions. In addition, solutions of the Diophantine equation D in terms of generalized Fibonacci and Lucas sequences are examined. Furthermore, we search for the solutions of this equation over finite fields F_p where p is prime and $p>5$. Finally, an example is given that satisfies our results.

1 Introduction

An equation with all coefficients and all its solutions being integers is called a Diophantine equation. The most common usage, quadratic Diophantine equations, takes the form

$$ax^2+bxy+cy^2+dx+ey+f=0. \tag{1.1}$$

There are many variants of Diophantine equations. Among these, Pell equation is common and is in the form of

$$x^2-dy^2=N, \tag{1.2}$$

where d is not a square number, and N is a constant. Here Eq. (1.2) corresponds to a particular instance of Eq. (1.1); see in [1]. In February 1657, French mathematician Pierre de Fermat (1607-1665) challenged English mathematicians John Wallis (1616-1703) and Lord William V. Brouncker (1620-1684) to solve the nonlinear Diophantine equation $x^2-dy^2=1$. The French mathematician Bernard de Bessey (1605-1675) succeeded in finding the solutions for this equation for $d\leq 150$. The fact that, this equation is referred to as the "Pell equation" is entirely due to an error. This error stems from the Swiss mathematician Leonhard Euler's (1707-1783) mistakenly ascribing Brouncker's work to the English mathematician John Pell (1611-1685), see in [2]. It was proved by the French mathematician Joseph Louis Lagrange (1736-1813) in 1766 that if N equals 1, this equation has infinite integer solutions; see in [3]. It has also been proved by Tekcan in [4] that if N is equal to 2^t, $t\in\mathbb{N}$, there are infinitely many integer solutions. The solutions of Eq. (1.2) are denoted by (x_n,y_n) for $n\geq 1$. The smallest positive integer solution (x_1,y_1) is called the fundamental solution. Other solutions of Eq. (1.2) can be generated from the fundamental solution (x_1,y_1). Readers can consult [5, 6, 7, 8, 9, 10, 11, 12] to learn more about Diophantine and Pell equations.

Let σ and τ be two nonzero integers, and let $\sigma^2-4\tau>0$. Let $\left(U_n\left(\sigma,\tau\right)\right)_{n\geq 1}$ be the generalized Fibonacci sequence defined by $U_0\left(\sigma,\tau\right)=0$, $U_1\left(\sigma,\tau\right)=1$, and

$$U_{n+1}\left(\sigma,\tau\right)=\sigma U_n\left(\sigma,\tau\right)-\tau U_{n-1}\left(\sigma,\tau\right)$$

for $n \geq 1$. Also, let $(V_n(\sigma,\tau))_{n\geq 1}$ be the generalized Lucas sequence defined by $V_0(\sigma,\tau)=2$, $V_1(\sigma,\tau)=\sigma$, and

$$V_{n+1}(\sigma,\tau)=\sigma V_n(\sigma,\tau)-\tau V_{n-1}(\sigma,\tau)$$

for $n \geq 1$. The Binet's formula for generalized Fibonacci and Lucas sequences are given by

$$U_n(\sigma,\tau)=\frac{\gamma^n-\delta^n}{\gamma-\delta} \quad \text{and} \quad V_n(\sigma,\tau)=\gamma^n+\delta^n$$

for $n \geq 0$, respectively, where γ and δ are the roots of the characteristic equation $x^2-\sigma x+\tau=0$, which are

$$\gamma=\frac{\sigma+\sqrt{\sigma^2-4\tau}}{2} \quad \text{and} \quad \delta=\frac{\sigma-\sqrt{\sigma^2-4\tau}}{2}.$$

Also, it is clear that $\gamma+\delta=\sigma$, $\gamma-\delta=\sqrt{\sigma^2-4\tau}$, and $\gamma\delta=\tau$. If $\sigma=1$ and $\tau=-1$, then $U_n(\sigma,\tau)=F_n$ and $V_n(\sigma,\tau)=L_n$, where F_n and L_n are the classical Fibonacci and Lucas sequences. For further information on generalized Fibonacci and Lucas sequences, see [2, 13, 14, 15, 16, 17].

In this paper, the positive integer solutions of the Diophantine equation

$$D: x^2-\left(\sigma^2+4\right)y^2-(2\sigma-2)x-\left(2\sigma^4+8\sigma^2\right)y-\sigma^6-4\sigma^4+\sigma^2-2\sigma-3=0 \tag{1.3}$$

are examined.

The primary objective of examining the Diophantine equation D provided in (1.3) in this paper is to demonstrate that a seemingly complicated Diophantine equation can be simplified through various transformations. Another reason is the presence of hidden Fibonacci and Lucas sequences among the solutions of the Diophantine equation D in Eq. (1.3).

2 Integer Solutions of a Diophantine Equation and Some Obtained Recursive Relations

In this section, the positive integer solutions of the Diophantine equation

$$D: x^2-\left(\sigma^2+4\right)y^2-(2\sigma-2)x-\left(2\sigma^4+8\sigma^2\right)y-\sigma^6-4\sigma^4+\sigma^2-2\sigma-3=0$$

and some properties of these solutions will be researched. It is a difficult task to determine whether D has a solution or not. Therefore, applying an appropriate linear transformation to D will transform it into another Diophantine equation, which will more easily determine the solution set of D. By applying these linear transformations,

$$T:=\begin{cases} x=u+h \\ y=v+k \end{cases} \tag{2.1}$$

will be used to D, for some $h,k\in\mathbb{Z}$. As a result of this applied linear transformation, the equation

$$\begin{aligned} T(D)=\widetilde{D}\ :\ &(u+h)^2-\left(\sigma^2+4\right)(v+k)^2 \\ &-(2\sigma-2)(u+h)-\left(2\sigma^4+8\sigma^2\right)(v+k)-\sigma^6-4\sigma^4+\sigma^2-2\sigma-3=0 \end{aligned} \tag{2.2}$$

is obtained. After making the necessary adjustments in Eq. (2.2), the coefficients u and v must be zero. Therefore $2h-2\sigma+2=0$ and $-2\left(\sigma^2+4\right)k-2\sigma^2\left(\sigma^2+4\right)=0$. From here, $h=\sigma-1$ and $k=-\sigma^2$. As a result, with $x=u+\sigma-1$ and $y=v-\sigma^2$ in Eq. (2.2), it becomes the Pell equation

$$\widetilde{D}\ :\ u^2-\left(\sigma^2+4\right)v^2=4. \tag{2.3}$$

It is much easier to search the solution set of Eq. (2.3) than Eq. (1.3); see [7].

Theorem 2.1. *Let $\widetilde{D}$ be the Diophantine equation given in (2.3). Then, the continued fraction expansion of $\sqrt{\sigma^2+4}$ is*

$$\sqrt{\sigma^2+4}=\begin{cases}\left[2;\overline{4}\right] & ,\sigma=1\\ \left[\sigma;\overline{\frac{\sigma}{2},2\sigma}\right] & ,\sigma>1 \ \textit{is even}\\ \left[\sigma;\overline{\frac{\sigma-1}{2},1,1,\frac{\sigma-1}{2},2\sigma}\right] & ,\sigma>1 \ \textit{is odd}\end{cases}$$

Proof. If $\sigma=1$, then $\sqrt{\sigma^2+4}=\sqrt{5}$. So,

$$\sqrt{5}=2+\left(\sqrt{5}-2\right)=2+\frac{1}{\frac{1}{\sqrt{5}-2}}=2+\frac{1}{\sqrt{5}+2}=2+\frac{1}{4+\left(\sqrt{5}-2\right)}$$
$$=2+\frac{1}{4+\frac{1}{\frac{1}{\sqrt{5}-2}}}=2+\frac{1}{4+\frac{1}{\sqrt{5}+2}}=2+\frac{1}{4+\frac{1}{4+\left(\sqrt{5}-2\right)}}.$$

Hence, $\sqrt{5}=[2;4,4,4,\ \ldots]=\left[2;\overline{4}\right]$.

If $\sigma>1$ is even, then

$$\sqrt{\sigma^2+4}=\sigma+\left(\sqrt{\sigma^2+4}-\sigma\right)=\sigma+\frac{1}{\frac{1}{\sqrt{\sigma^2+4}-\sigma}}=\sigma+\frac{1}{\frac{\sqrt{\sigma^2+4}+\sigma}{4}}$$
$$=\sigma+\frac{1}{\frac{\sigma}{2}+\frac{\sqrt{\sigma^2+4}-\sigma}{4}}=\sigma+\frac{1}{\frac{\sigma}{2}+\frac{1}{\frac{4}{\sqrt{\sigma^2+4}-\sigma}}}=\sigma+\frac{1}{\frac{\sigma}{2}+\frac{1}{\sqrt{\sigma^2+4}+\sigma}}$$
$$=\sigma+\frac{1}{\frac{\sigma}{2}+\frac{1}{2\sigma+\left(\sqrt{\sigma^2+4}-\sigma\right)}}.$$

Hence, $\sqrt{\sigma^2+4}=\left[\sigma;\overline{\frac{\sigma}{2},2\sigma}\right]$. Also, if $\sigma>1$ is odd, a similar path is followed and it is easily proved that $\sqrt{\sigma^2+4}=\left[\sigma;\overline{\frac{\sigma-1}{2},1,1,\frac{\sigma-1}{2},2\sigma}\right]$. □

Theorem 2.2. *The fundamental solution of $\widetilde{D}$ is given by*

$$(u_1,v_1)=\left(\sigma^2+2,\sigma\right).$$

Proof. The Fundamental solution of $\widetilde{D}$ is $(u_1,v_1)=\left(\sigma^2+2,\sigma\right)$. Indeed, If σ is written instead of v in the equation $u^2-\left(\sigma^2+4\right)v^2=4$, then $u^2=\left(\sigma^2+4\right)v^2+4=\left(\sigma^2+2\right)^2$ and $u=\sigma^2+2$ is found. From that point, $(u_1,v_1)=\left(\sigma^2+2,\sigma\right)$ is the fundamental solution of $\widetilde{D}$. □

Theorem 2.3. *Let define the set $\{(u_n,v_n)\}_{n\geq1}$, where*

$$\begin{pmatrix}u_n\\ v_n\end{pmatrix}=\frac{1}{2^{n-1}}\begin{pmatrix}\sigma^2+2 & \sigma\left(\sigma^2+4\right)\\ \sigma & \sigma^2+2\end{pmatrix}^{n-1}\begin{pmatrix}\sigma^2+2\\ \sigma\end{pmatrix} \tag{2.4}$$

for $n\geq1$. Then, (u_n,v_n) is the n^{th} solution of $\widetilde{D}$.

Proof. The proof of the theorem will be done using mathematical induction on n. From Eq. (2.4) and for $n=1$, we have

$$\begin{pmatrix}u_1\\ v_1\end{pmatrix}=\frac{1}{2^0}\begin{pmatrix}\sigma^2+2 & \sigma\left(\sigma^2+4\right)\\ \sigma & \sigma^2+2\end{pmatrix}^{0}\begin{pmatrix}\sigma^2+2\\ \sigma\end{pmatrix}=\begin{pmatrix}\sigma^2+2\\ \sigma\end{pmatrix}$$

and can be seen as $(u_1, v_1) = (\sigma^2+2, \sigma)$, which is the fundamental solution of $\widetilde{D}$. Let us assume that Eq. (2.4) is true for n. That is, $u_n^2 - (\sigma^2+4)\, v_n^2 = 4$.

$$\begin{aligned}\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} &= \frac{1}{2^n}\begin{pmatrix} \sigma^2+2 & \sigma(\sigma^2+4) \\ \sigma & \sigma^2+2 \end{pmatrix}^n \begin{pmatrix} \sigma^2+2 \\ \sigma \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \sigma^2+2 & \sigma(\sigma^2+4) \\ \sigma & \sigma^2+2 \end{pmatrix} \frac{1}{2^{n-1}}\begin{pmatrix} \sigma^2+2 & \sigma(\sigma^2+4) \\ \sigma & \sigma^2+2 \end{pmatrix}^{n-1} \begin{pmatrix} \sigma^2+2 \\ \sigma \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \sigma^2+2 & \sigma(\sigma^2+4) \\ \sigma & \sigma^2+2 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(\sigma^2+2)\, u_n + \frac{1}{2}\sigma(\sigma^2+4)\, v_n \\ \frac{1}{2}\sigma u_n + \frac{1}{2}(\sigma^2+2)\, v_n \end{pmatrix}.\end{aligned}$$

Thus, we obtain

$$u_{n+1} = \frac{1}{2}(\sigma^2+2)\, u_n + \frac{1}{2}\sigma(\sigma^2+4)\, v_n \quad \text{and} \quad v_{n+1} = \frac{1}{2}\sigma u_n + \frac{1}{2}(\sigma^2+2)\, v_n.$$

If these equations are written in $u^2 - (\sigma^2+4)\, v^2$, then we have

$$\begin{aligned} u_{n+1}^2 - (\sigma^2+4)\, v_{n+1}^2 &= \left[\frac{1}{2}(\sigma^2+2)\, u_n + \frac{1}{2}\sigma(\sigma^2+4)\, v_n\right]^2 \\ &- (\sigma^2+4)\left[\frac{1}{2}\sigma u_n + \frac{1}{2}(\sigma^2+2)\, v_n\right]^2 \\ &= \frac{1}{4}(\sigma^2+2)^2 u_n^2 + 2.\frac{1}{4}(\sigma^2+2)\,\sigma(\sigma^2+4)\, u_n v_n \\ &+ \frac{1}{4}\sigma^2(\sigma^2+4)^2 v_n^2 - \frac{1}{4}(\sigma^2+4)\,\sigma^2 u_n^2 \\ &- 2.\frac{1}{4}(\sigma^2+2)\,\sigma(\sigma^2+4)\, u_n v_n \\ &- \frac{1}{4}(\sigma^2+4)(\sigma^2+2)^2 v_n^2 \\ &= u_n^2 - (\sigma^2+4)\, v_n^2 \\ &= 4. \end{aligned}$$

Therefore, (u_{n+1}, v_{n+1}) becomes a solution of $\widetilde{D}$ too. □

Corollary 2.4. *Let (u_n, v_n) and (u_{n+1}, v_{n+1}) be two consecutive solutions of $\widetilde{D}$. Then,*

$$u_{n+1} = \frac{1}{2}(\sigma^2+2)\, u_n + \frac{1}{2}\sigma(\sigma^2+4)\, v_n \quad \textit{and} \quad v_{n+1} = \frac{1}{2}\sigma u_n + \frac{1}{2}(\sigma^2+2)\, v_n \tag{2.5}$$

for $n \geq 1$.

Theorem 2.5. *Let (u_{n-2}, v_{n-2}), (u_{n-1}, v_{n-1}), and (u_n, v_n) be three consecutive solutions of $\widetilde{D}$. Then,*

$$u_n = (\sigma^2+2)\, u_{n-1} + u_{n-2} \quad \textit{and} \quad v_n = (\sigma^2+2)\, v_{n-1} + v_{n-2} \tag{2.6}$$

Proof. From Eq. (2.4) we have

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \frac{1}{2^{n-1}}\begin{pmatrix} \sigma^2+2 & \sigma(\sigma^2+4) \\ \sigma & \sigma^2+2 \end{pmatrix}^{n-1} \begin{pmatrix} \sigma^2+2 \\ \sigma \end{pmatrix}$$

for $n \geq 1$. If $\frac{1}{2}\begin{pmatrix} \sigma^2+2 & \sigma\left(\sigma^2+4\right) \\ \sigma & \sigma^2+2 \end{pmatrix}$ is taken M here, that is,

$$M = \frac{1}{2}\begin{pmatrix} \sigma^2+2 & \sigma\left(\sigma^2+4\right) \\ \sigma & \sigma^2+2 \end{pmatrix},$$

then Eq. (2.4) becomes

$$\begin{aligned} \begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \frac{1}{2^{n-1}}\begin{pmatrix} \sigma^2+2 & \sigma\left(\sigma^2+4\right) \\ \sigma & \sigma^2+2 \end{pmatrix}^{n-1}\begin{pmatrix} \sigma^2+2 \\ \sigma \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \sigma^2+2 & \sigma\left(\sigma^2+4\right) \\ \sigma & \sigma^2+2 \end{pmatrix}\frac{1}{2^{n-2}}\begin{pmatrix} \sigma^2+2 & \sigma\left(\sigma^2+4\right) \\ \sigma & \sigma^2+2 \end{pmatrix}^{n-2}\begin{pmatrix} \sigma^2+2 \\ \sigma \end{pmatrix} \\ &= M\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix}. \end{aligned} \tag{2.7}$$

From the characteristic polynomial of M,

$$\begin{aligned} |\lambda I_2 - M| = 0 \Rightarrow &\begin{vmatrix} \lambda - \frac{1}{2}\left(\sigma^2+2\right) & -\frac{1}{2}\sigma\left(\sigma^2+4\right) \\ -\frac{1}{2}\sigma & \lambda - \frac{1}{2}\left(\sigma^2+2\right) \end{vmatrix} = 0 \\ &\Rightarrow \lambda^2 - \lambda\left(\sigma^2+2\right) + I_2 = 0 \end{aligned}$$

for $\lambda \in \mathbb{R}$. If M is written instead of λ here, we get $M^2 - M\left(\sigma^2+2\right) + I_2 = 0$ and $M^2 = M\left(\sigma^2+2\right) - I_2$. From Eq. (2.7),

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = M\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} = M.M\begin{pmatrix} u_{n-2} \\ v_{n-2} \end{pmatrix} = M^2\begin{pmatrix} u_{n-2} \\ v_{n-2} \end{pmatrix} = \left[M\left(\sigma^2+2\right) - I_2\right]\begin{pmatrix} u_{n-2} \\ v_{n-2} \end{pmatrix}$$

and so,

$$\left(\sigma^2+2\right)M\begin{pmatrix} u_{n-2} \\ v_{n-2} \end{pmatrix} - I_2\begin{pmatrix} u_{n-2} \\ v_{n-2} \end{pmatrix} = \left(\sigma^2+2\right)\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} - \begin{pmatrix} u_{n-2} \\ v_{n-2} \end{pmatrix} = \begin{pmatrix} \left(\sigma^2+2\right)u_{n-1} - u_{n-2} \\ \left(\sigma^2+2\right)v_{n-1} - v_{n-2} \end{pmatrix}.$$

Thus, $u_n = \left(\sigma^2+2\right)u_{n-1} - u_{n-2}$ and $v_n = \left(\sigma^2+2\right)v_{n-1} - v_{n-2}$. □

Theorem 2.6. *Let* (u_{n-3}, v_{n-3}), (u_{n-2}, v_{n-2}), (u_{n-1}, v_{n-1}), *and* (u_n, v_n) *be four consecutive solutions of* $\widetilde{D}$. *Then,*

$$u_n = \left(\sigma^2+1\right)\left(u_{n-1}+u_{n-2}\right) - u_{n-3} \quad \textit{and} \quad v_n = \left(\sigma^2+1\right)\left(v_{n-1}+v_{n-2}\right) - v_{n-3} \tag{2.8}$$

for $n \geq 4$.

Proof. The proof of the theorem will be done using mathematical induction on n. From Eq. (2.5) we know that $u_{n+1} = \frac{1}{2}\left(\sigma^2+2\right)u_n + \frac{1}{2}\sigma\left(\sigma^2+4\right)v_n$ and $v_{n+1} = \frac{1}{2}\sigma u_n + \frac{1}{2}\left(\sigma^2+2\right)v_n$ for $n \geq 1$. Also, if we consider that the fundamental solution of $\widetilde{D}$ is $(u_1, v_1) = \left(\sigma^2+2, \sigma\right)$

from Theorem 2.2, we have

$$\begin{aligned}
u_2 &= \frac{1}{2}\left[\left(\sigma^2+2\right)u_1+\sigma\left(\sigma^2+4\right)v_1\right] \\
&= \frac{1}{2}\left[\left(\sigma^2+2\right)\left(\sigma^2+2\right)+\sigma\left(\sigma^2+4\right)\sigma\right] \\
&= \sigma^4+4\sigma^2+2, \\
v_2 &= \frac{1}{2}\left[\sigma u_1+\left(\sigma^2+2\right)v_1\right] \\
&= \frac{1}{2}\left[\sigma\left(\sigma^2+2\right)+\left(\sigma^2+2\right)\sigma\right] \\
&= \sigma^3+2\sigma, \\
u_3 &= \frac{1}{2}\left[\left(\sigma^2+2\right)u_2+\sigma\left(\sigma^2+4\right)v_2\right] \\
&= \frac{1}{2}\left[\left(\sigma^2+2\right)\left(\sigma^4+4\sigma^2+2\right)+\sigma\left(\sigma^2+4\right)\left(\sigma^3+2\sigma\right)\right] \\
&= \sigma^6+6\sigma^4+9\sigma^2+2, \\
v_3 &= \frac{1}{2}\left[\sigma u_2+\left(\sigma^2+2\right)v_2\right] \\
&= \frac{1}{2}\left[\sigma\left(\sigma^4+4\sigma^2+2\right)+\left(\sigma^2+2\right)\left(\sigma^3+2\sigma\right)\right] \\
&= \sigma^5+4\sigma^3+3\sigma, \\
u_4 &= \frac{1}{2}\left[\left(\sigma^2+2\right)u_3+\sigma\left(\sigma^2+4\right)v_3\right] \\
&= \frac{1}{2}\left[\left(\sigma^2+2\right)\left(\sigma^6+6\sigma^4+9\sigma^2+2\right)+\sigma\left(\sigma^2+4\right)\left(\sigma^5+4\sigma^3+3\sigma\right)\right] \\
&= \sigma^8+8\sigma^6+20\sigma^4+16\sigma^2+2, \\
v_4 &= \frac{1}{2}\left[\sigma u_3+\left(\sigma^2+2\right)v_3\right] \\
&= \frac{1}{2}\left[\sigma\left(\sigma^6+6\sigma^4+9\sigma^2+2\right)+\left(\sigma^2+2\right)\left(\sigma^5+4\sigma^3+3\sigma\right)\right] \\
&= \sigma^7+6\sigma^5+10\sigma^3+4\sigma.
\end{aligned}$$

If these values are substituted in Eqs. (2.8), we get

$$\begin{aligned}
\left(\sigma^2+1\right)\left(u_3+u_2\right)-u_1 &= \left(\sigma^2+1\right)\left(\sigma^6+6\sigma^4+9\sigma^2+2+\sigma^4+4\sigma^2+2\right) \\
&\quad -\left(\sigma^2+2\right) \\
&= \left(\sigma^2+1\right)\left(\sigma^6+7\sigma^4+13\sigma^2+4\right)-\left(\sigma^2+2\right) \\
&= \sigma^8+8\sigma^6+20\sigma^4+16\sigma^2+2 \\
&= u_4
\end{aligned}$$

and

$$\begin{aligned}
\left(\sigma^2+1\right)\left(v_3+v_2\right)-v_1 &= \left(\sigma^2+1\right)\left(\sigma^5+4\sigma^3+3\sigma+\sigma^3+2\sigma\right)-\sigma \\
&= \left(\sigma^2+1\right)\left(\sigma^5+5\sigma^3+5\sigma\right)-\sigma \\
&= \sigma^7+6\sigma^5+10\sigma^3+4\sigma \\
&= v_4.
\end{aligned}$$

Hence, it is seen that the equations are true for $n=4$. Let us assume that the Eqs. (2.8) are true for $n-1$. That is,

$$u_{n-1}=\left(\sigma^2+1\right)\left(u_{n-2}+u_{n-3}\right)-u_{n-4} \text{ and } v_{n-1}=\left(\sigma^2+1\right)\left(v_{n-2}+v_{n-3}\right)-v_{n-4}$$

for $n \geq 4$. We will again use Eqs. (2.5) to show that these equations are also true for n.

$$\begin{aligned}
u_n &= \frac{1}{2}\left[\left(\sigma^2+2\right)u_{n-1}+\sigma\left(\sigma^2+4\right)v_{n-1}\right] \\
&= \frac{1}{2}\left(\sigma^2+2\right)\left(\left(\sigma^2+1\right)\left(u_{n-2}+u_{n-3}\right)-u_{n-4}\right) \\
&+ \frac{1}{2}\sigma\left(\sigma^2+4\right)\left(\left(\sigma^2+1\right)\left(v_{n-2}+v_{n-3}\right)-v_{n-4}\right) \\
&= \frac{1}{2}\left(\sigma^2+1\right)\left(\sigma^2+2\right)\left(u_{n-2}+u_{n-3}\right)-\frac{1}{2}\left(\sigma^2+2\right)u_{n-4} \\
&+ \frac{1}{2}\left(\sigma^2+1\right)\sigma\left(\sigma^2+4\right)\left(v_{n-2}+v_{n-3}\right) \\
&- \frac{1}{2}\sigma\left(\sigma^2+4\right)v_{n-4} \\
&= \left(\sigma^2+1\right)\underbrace{\frac{1}{2}\left[\left(\sigma^2+2\right)u_{n-2}+\sigma\left(\sigma^2+4\right)v_{n-2}\right]}_{u_{n-1}} \\
&+ \left(\sigma^2+1\right)\underbrace{\frac{1}{2}\left[\left(\sigma^2+2\right)u_{n-3}+\sigma\left(\sigma^2+4\right)v_{n-3}\right]}_{u_{n-2}} \\
&- \underbrace{\frac{1}{2}\left[\left(\sigma^2+2\right)u_{n-4}+\sigma\left(\sigma^2+4\right)v_{n-4}\right]}_{u_{n-3}} \\
&= \left(\sigma^2+1\right)\left(u_{n-1}+u_{n-2}\right)-u_{n-3}.
\end{aligned}$$

And similarly, we get

$$\begin{aligned}
v_n &= \frac{1}{2}\left[\sigma u_{n-1}+\left(\sigma^2+2\right)v_{n-1}\right] \\
&= \frac{1}{2}\sigma\left(\left(\sigma^2+1\right)\left(u_{n-2}+u_{n-3}\right)-u_{n-4}\right) \\
&+ \frac{1}{2}\left(\sigma^2+2\right)\left(\left(\sigma^2+1\right)\left(v_{n-2}+v_{n-3}\right)-v_{n-4}\right) \\
&= \left(\sigma^2+1\right)\underbrace{\frac{1}{2}\left[\sigma u_{n-2}+\left(\sigma^2+2\right)v_{n-2}\right]}_{v_{n-1}}+\left(\sigma^2+1\right)\underbrace{\frac{1}{2}\left[\sigma u_{n-3}+\left(\sigma^2+2\right)v_{n-3}\right]}_{v_{n-2}} \\
&- \underbrace{\frac{1}{2}\left[\sigma u_{n-4}+\left(\sigma^2+2\right)v_{n-4}\right]}_{v_{n-3}} \\
&= \left(\sigma^2+1\right)\left(v_{n-1}+v_{n-2}\right)-v_{n-3}.
\end{aligned}$$

So, the desired has been achieved.

□

Remark 2.7. The problem of finding integer solutions of the Pell equation over finite fields was investigated in [9] and [10] and concluded by Özkoç and Tekcan. Since this problem has an important place in mathematics, whether the same problem solution was valid for the problem investigated in this paper was also investigated. The problem in the new section below is solved using the same method in [9] and [10].

3 Solving the Pell Equation $\widetilde{D}$: $u^2-\left(\sigma^2+4\right)v^2=4$ Over Finite Fields F_p

Now, we will investigate integer solutions of the Pell equation $\widetilde{D}$ over finite fields F_p.

Let $\left(\sigma^2+4\right) \equiv d\,(\bmod p)$, for $\sigma \geq 1$, primes $p>5$ and $p \nmid d$. Let us define the Pell equation

$$\widetilde{D}\,\xi_p\,(d): \; u^2-dv^2 \equiv 4 \;(\bmod p)\,. \tag{3.1}$$

Let $\overline{\widetilde{D}\,\xi_p\,(d)}$ be the set of integer solutions of Eq. (3.1) over the finite fields F_p. Then,

$$\overline{\widetilde{D}\,\xi_p\,(d)} = \left\{(u,v): \; u,v \in F_p, \; u^2-dv^2 \equiv 4 \;(\bmod p)\right\}.$$

Also, let $\overline{\overline{\widetilde{D}\,\xi_p\,(d)}}$ represent the number of integer solutions over the finite fields F_p and let Q_p be the set of quadratic residues modulo p. Then, we can state the following theorem.

Theorem 3.1. *Let $\widetilde{D}\,\xi_p\,(d)$ be the Pell Eq. in (3.1). So*

$$\overline{\overline{\widetilde{D}\,\xi_p\,(d)}} = \begin{cases} p-1 & for\ d \in Q_p \\ p+1 & for\ d \notin Q_p\,. \end{cases}$$

where $\sigma \geq 1$, primes $p>5$ and $p \nmid d$.

Proof. A similar proof applied in [9] and [10] is used; thus, the proof is completed. □

Example 3.2. Let $\sigma=2$. In this case, Eq. (2.3) turns into the Pell equation $u^2-8v^2=4$. The fundamental solution of this equation is $(u_1,v_1)=(6,2)$. Also, Eqs. (2.5), $u_n=3u_{n-1}+8v_{n-1}$ and $v_n=u_{n-1}+3v_{n-1}$, for $n \geq 2$. Eqs. (2.6), $u_n=6u_{n-1}+u_{n-2}$ and $v_n=6v_{n-1}+v_{n-2}$, for $n \geq 3$. And finally, Eqs. (2.8), $u_n=5\left(u_{n-1}+u_{n-2}\right)-u_{n-3}$ and $v_n=5\left(v_{n-1}+v_{n-2}\right)-v_{n-3}$, for $n \geq 4$. Using these equations we get $(u_2,v_2)=(34,12)\,,\,(u_3,v_3)=(198,70)$, $(u_4,v_4)=(1154,408)\,,\,(u_5,v_5)=(6726,2378)$, ... which are solutions to the Pell equation $u^2-8v^2=4$. In addition, the continued fraction expansion of $\sqrt{8}$ is $\left[2;\overline{1,4}\right]$.

Moreover, Let us take $p=17$. So $Q_{17}=\{1,2,4,8,9,13,15,16,\}$ and $\widetilde{D}\,\xi_{17}\,(8): \; u^2-8v^2 \equiv 4 \;(\bmod 17)$. Since $8 \in Q_{17}$, $\overline{\overline{\widetilde{D}\,\xi_{17}\,(8)}}=17-1=16$. Indeed,

$$\overline{\widetilde{D}\,\xi_{17}\,(8)} = \left\{(u,v): \; u,v \in F_{17}, \; u^2-8v^2 \equiv 4 \;(\bmod 17)\right\},$$

that is, $\overline{\widetilde{D}\,\xi_{17}\,(8)} = \{(0,5)\,,\ (0,12)\,,(2,0)\,,(5,3)\,,(5,14)\,,\underline{(6,2)\,,(6},15)\,,(8,4)\,,(8,13)\,,(9,4)\,,(9,13)\,,(11,2),(11,15)\,,(12,3)\,,(12,14)\,,(15,0)\}$. Thus, $\overline{\overline{\widetilde{D}\,\xi_{17}\,(8)}}=16$.

4 Solutions of the Diophantine Equation $\widetilde{D}: u^2-\left(\sigma^2+4\right)v^2=4$ in Terms of Generalized Fibonacci and Lucas Sequences

In this section, we investigate the solutions of the Diophantine equation $\widetilde{D}: u^2-\left(\sigma^2+4\right)v^2=4$ in the form of generalized Fibonacci and Lucas sequences.

Theorem 4.1. *Positive integer solutions of $\widetilde{D}$ are given by*

$$(u_n,v_n) = \left(V_{2n}\,(\sigma,-1)\,,U_{2n}\,(\sigma,-1)\right)$$

for $n \geq 1$.

Proof. If the generalized Fibonacci and Lucas sequences are written as Binet's formula, we get

$$U_{2n}\,(\sigma,-1) = \frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta} \;\text{ and }\; V_{2n}\,(\sigma,-1) = \gamma^{2n}+\delta^{2n}$$

for $n \geq 1$, respectively, where

$$\gamma = \frac{\sigma+\sqrt{\sigma^2+4}}{2} \;\text{ and }\; \delta = \frac{\sigma-\sqrt{\sigma^2+4}}{2}.$$

So,

$$\begin{aligned} u^2 - \left(\sigma^2+4\right) v^2 &= V_{2n}^2\left(\sigma,-1\right) - \left(\sigma^2+4\right) U_{2n}^2\left(\sigma,-1\right) \\ &= \left(\gamma^{2n}+\delta^{2n}\right)^2 - \left(\sigma^2+4\right)\left(\frac{\gamma^{2n}-\delta^{2n}}{\gamma-\delta}\right)^2 \\ &= 4\gamma^{4n}\delta^{4n} \\ &= 4(\gamma\delta)^{4n} \\ &= 4 \end{aligned}$$

for $n \geq 1$. Thus, the proof is completed. □

Corollary 4.2. *If σ is taken as 1, then the Diophantine equation $\widetilde{D}$ becomes $u^2 - 5v^2 = 4$, and the positive integer solutions of $\widetilde{D}$ are given by*

$$(u_n, v_n) = (L_{2n}, F_{2n})$$

for $n \geq 1$.

5 Main Results

Until this chapter, certain properties between these solutions have been proven by transforming the Diophantine equation D into a Diophantine equation $\widetilde{D}$ through a linear transformation T and finding the solution set of $\widetilde{D}$. The relationship of the solution set of $\widetilde{D}$ with the generalized Fibonacci and Lucas sequences is given. It has also been explained that $x = u + \sigma - 1$ and $y = v - \sigma^2$. In this section, all the results obtained through the inverse of T will be transformed back from $\widetilde{D}$ to D. The following theorems will be given, expressing the main results of this paper.

Theorem 5.1. *Let D be the Diophantine equation given in (1.3). So,*
1. The fundamental solution of D is $(x_1, y_1) = \left(\sigma^2+\sigma+1, \sigma-\sigma^2\right)$.
2. (u_n, v_n) as defined in Eq. (2.4) and let the sequence

$$\{(x_n, y_n)\}_{n\geq 1} = \left\{u_n + \sigma - 1, v_n - \sigma^2\right\}$$

be given. Then (x_n, y_n) is the n^{th} solution of D. Hence, D has infinitely many $(x_n, y_n) \in \mathbb{Z}\times\mathbb{Z}$ solutions.
3. Let (x_{n-1}, y_{n-1}) and (x_n, y_n) be two consecutive solutions of D. Then,

$$x_n = \frac{1}{2}\left[\left(\sigma^2+2\right)x_{n-1} + \sigma\left(\sigma^2+4\right)y_{n-1} + \sigma^5 + 3\sigma^3 + \sigma^2 - 1\right]$$

and

$$y_n = \frac{1}{2}\left[\sigma x_{n-1} + \left(\sigma^2+2\right)y_{n-1} + \sigma^4 - \sigma^2 + \sigma\right]$$

for $n \geq 1$.
4. Let (x_{n-3}, y_{n-3}), (x_{n-2}, y_{n-2}), (x_{n-1}, y_{n-1}), and (x_n, y_n) be four consecutive solutions of D. Then,

$$x_n = \left(\sigma^2+1\right)\left(x_{n-1}+x_{n-2}\right) - x_{n-3} - 2\sigma^3 + 2\sigma^2 \text{ and } y_n = \left(\sigma^2+1\right)\left(y_{n-1}+y_{n-2}\right) - y_{n-3} + 2\sigma^4$$

for $n \geq 4$.

Proof. A similar procedure is followed to the proofs of the theorems presented in Chapter 2, and then the proof is completed. □

Theorem 5.2. *Let D be the Diophantine equation given in (1.3). So, positive integer solutions of D are given by*

$$(x_n, y_n) = \left(V_{2n}\left(\sigma,-1\right) + \sigma - 1, U_{2n}\left(\sigma,-1\right) - \sigma^2\right)$$

for $n \geq 1$.

Proof. A similar procedure is followed to the proofs of theorem 4.1, and then the proof is completed. □

Corollary 5.3. *If σ is taken as 1, then the Diophantine equation D becomes $x^2 - 5(y+1)^2 = 4$, and the positive integer solutions of D are given by*

$$(x_n, y_n) = (L_{2n}, F_{2n} - 1)$$

for $n > 1$.

Let D be the Diophantine equation given in (1.3), $\left(\sigma^2+4\right) \equiv d\,(\mathrm{mod}p)$, for $\sigma \geq 1$, primes $p > 5$ and $p \nmid d$. Also, let us define the Pell equation

$$D\xi_p\left(d\right) : \, x^2 - \left(\sigma^2+4\right)y^2 - \left(2\sigma-2\right)x - \left(2\sigma^4+8\sigma^2\right)y - \sigma^6 - 4\sigma^4 + \sigma^2 - 2\sigma - 3 \equiv 0 \;(\mathrm{mod}p) \quad (5.1)$$

and set

$$\overline{D\xi_p\left(d\right)} = \{(x,y) : \, x, y \in F_p, \; x^2 - \left(\sigma^2+4\right)y^2 - \left(2\sigma-2\right)x - \left(2\sigma^4+8\sigma^2\right)y - \sigma^6 - 4\sigma^4 + \sigma^2 - 2\sigma - 3 \equiv 0 \;(\mathrm{mod}p)\,\}.$$

Let $\overline{\overline{D\,\xi_p\left(d\right)}}$ represent the number of integer solutions over the finite fields F_p and let Q_p be the set of quadratic residues modulo p. Then we can state the following theorem, which is one of the main results of this paper.

Theorem 5.4. *Let $D\,\xi_p\left(d\right)$ be the Pell equation in (5.1). So*

$$\overline{\overline{D\,\xi_p\left(d\right)}} = \begin{cases} p-1 & for \quad \sigma^2+4 \in Q_p \\ p+1 & for \quad \sigma^2+4 \notin Q_p \,, \end{cases}$$

where $\sigma \geq 1$, primes $p > 5$ and, $p \nmid (\sigma^2+4)$.

Proof. A similar procedure is followed to the proofs of theorem 3.1, and then the proof is completed. □

6 Discussion and Conclusions

In this study, positive integer solutions of the diophantine equation

$$D : x^2 - \left(\sigma^2+4\right)y^2 - \left(2\sigma-2\right)x - \left(2\sigma^4+8\sigma^2\right)y - \sigma^6 - 4\sigma^4 + \sigma^2 - 2\sigma - 3 = 0$$

have been investigated. Since finding the solution set of the equation in this state has been challenging, the linear transformation T defined by $x = u + h$ and $y = v + k$ has been applied to D, and the Diophantine equation D has been converted to the Pell equation $\widetilde{D} : u^2 - \left(\sigma^2+4\right)v^2 = 4$. Finding positive solutions to this Pell equation is relatively easier now. Firstly, the fundamental solutions of the Pell equation have been found, and other solutions have been found using the fundamental solution. Various recursive formulas have been given with the help of the solutions found, and their accuracy has been proven. An example is given that confirms the results found for the equation $\widetilde{D} : u^2 - \left(\sigma^2+4\right)v^2 = 4$. Finally, all the results for the Pell equation $\widetilde{D} : u^2 - \left(\sigma^2+4\right)v^2 = 4$ are transferred to D with the help of the inverse of the T linear transformation. The importance of this study also emerges here. It is possible to transform seemingly complex Diophantine equations into different Diophantine equations using certainly linear transformations; in this way, it can be a source for researching more easily whether there is another number sequence among the solutions of the equation. In addition, solutions of the Diophantine equation D in terms of generalized Fibonacci and Lucas sequences have been examined. Furthermore, the solutions of this equation have been searched for over finite fields F_p where p is prime and $p > 5$. In future studies, the search for other number sequences (such as Pell, Pell – Lucas, and Jacobsthal sequences) among the solutions of D appears to be an open problem.

References

[1] E. J. Barbeau, *Pell's equation*, Springer, New York, (2003).

[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, New York, (2019).

[3] T. Andreescu, D. Andrica and I. Cucurezeanu, *An Introduction to Diophantine Equations: A Problem-Based Approch*, Birkhauser, New York, (2010).

[4] A. Tekcan, *The Pell Equation* $x^2 - (k^2 - k)y^2 = 2^t$, Int. J. Comput. Math. Sci., **2(1)**, 5-9, (2008).

[5] A. Chandoul, *The Pell equation* $x^2 - Dy^2 = \mp k^2$, Advances in Pure Mathematics, **1(02)**, 16, (2011).

[6] C. Baltus, *Continued fractions and the Pell equations: the work of Euler and Lagrange*, Comm. Anal Theory Contin. Fractions, **3**, 4-31, (1994).

[7] J. Kannan, M. Somanath and K. Raja, *On a class of solutions for the hyperbolic diophantine equation*, Int. J. Appl. Math., **32(3)**, 443, (2019).

[8] K. Matthews, *The Diophantine Equation* $x^2 - Dy^2 = N, D > 0$, Expositiones Mathematicae, **18(4)**, 323-332, (2000).

[9] Ö. Özkoç and A. Tekcan, *Integer Solutions of a Special Diophantine Equation*, Int. Aip. Conference Proceedings, American Institute of Physics, **1389(1)**, 371-374 (2011).

[10] Ö. Özkoç, A. Tekcan and I. N. Cangul, *Solving some parametric quadratic Diophantine equation over* $\mathbb{Z}$ *and* F_p , Appl. Math. and Comput., **218(3)**, 703-706, (2011).

[11] R. Keskin and M. G. Duman, *Positive integer solutions of some Pell equations* , Palestine Journal of Mathematics, **8(2)**, 213-226, (2019).

[12] A. Emin, *Some multi figurate numbers in terms of generalized fibonacci and lucas numbers*, The Aligarh Bulletin of Mathematics, **42(1)**, 107-123, (2023).

[13] B. Demirtürk and R. Keskin, *Integer solutions of some Diophantine equations via Fibonacci and Lucas numbers*, J. Integer Seq., **12(8)**, 8, (2009).

[14] T. Koshy, *Elementary number theory with applications*, Academic press, New York, (2002).

[15] Y. K. Panwar, B. Singh and V. K. Gupta, *Generalized fibonacci sequences and its properties*, Palestine Journal of Mathematics, **3(1)**, 141-147, (2014).

[16] A. Taane, I. E. Djellas and M. Mekkaoui, *On some nested sums involving q-Fibonacci numbers*, Palestine Journal of Mathematics, **12(2)**, 183-193, (2023).

[17] K. Adegoke, R. Frontczak and T. Goy, *New Binomial Fibonacci Sums*, Palestine Journal of Mathematics, **13(1)**, 323-339, (2024).

Author information

Ahmet Emin, Department of Mathematics, Faculty of Science, Karabuk University, Turkey.
E-mail: ahmetemin@karabuk.edu.tr

Received: 2023-12-26
Accepted: 2024-05-07