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ϵ-SASAKIAN METRIC AS CONFORMAL RICCI SOLITON

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Abstract The main objective of the present paper is to study conformal Ricci soliton in the ϵ-Sasakian manifold and to study its geometric properties associated with the manifold. We have analyzed the associations with the conformal Ricci soliton equation, the curvature conditions resulting in a conformal vector field, and the torse-forming vector field on the ϵ-Sasakian manifold.

1 Introduction

Geometry of manifolds endowed with indefinite metrics has significant use in physics and relativity in the credential of Einstein's theory of general relativity, as the nature of metric depend on the geometric properties of the manifold.

Many scientists and mathematicians have studied manifolds with positive definite and indefinite structures based on Riemannian geometry. The term pseudo- Riemannian (also called semi-Riemannian) geometry has attracted many generations of mathematical physicists for its effectiveness in providing space-models in general relativity. One of the importance of indefinite metric is, that it allows tangent vectors to be classified into timelike, null, and spacelike. These circumstances lead many authors to investigate and explore the importance and applications of the manifolds with indefinite metrics.

The notion of indefinite metrics of almost contact manifolds was firstly initiated by Takahashi[13] in 1969. There were many studies based on Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite Sasakian manifolds proclaimed as ϵ-Sasakian manifolds. Bejancu and Duggal[3] introduced the concept of the ϵ-Sasakian manifold. Further Xufeng and Xioli[15] established these manifolds as real hypersurfaces of ϵ-Kaehlerian manifolds. Further, the studies on this manifold have been done in [5].

In recent years there has been extensive research done in contact geometry. One such research is based on Ricci solitons which emerge as the solutions of the Ricci flow $\frac{\partial}{\partial t}g(t) = -2S(g(t))$, where g is a metric on a smooth manifold M which was introduced by Hamilton[9] in 1982. Ricci solitons were initially studied by Sharma[12] in contact Riemannian geometry. Further study has been made by Bejan and Crasmarenu[2]. Several other authors namely Tripathi[14], Nagaraja and Premalatha[10] and Ghosh[7] extensively studied Ricci solitons.

Diligently ,Ricci soliton on Riemannian manifold (M, g) is a triple (g, V, λ) which satisfies the condition

$$L_v g + 2S = 2\lambda g, \tag{1.1}$$

where, S is the Ricci tensor, L_v denotes the lie-derivative along the vector field V on M and $\lambda \in R$. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is positive, zero and negative respectively.

Fisher[6] developed the concept of the conformal Ricci flow equation which was the modified version of Hamilton's Ricci flow equation. The conformal Ricci flow on a smooth closed connected oriented manifold M of dimension $2n+1$ is defined by the following equation

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{2n+1}\right) = -pg \qquad r(g) = -1, \tag{1.2}$$

where, the scalar field p is a non-dynamical(time-dependent) conformal pressure, $r(g)$ is the scalar curvature of the manifold M.
Basu and Bhattacharyya[1] introduced the notion of conformal Ricci soliton as the solution of (1.2) and the equation is as follows ,

$$L_v g + 2S = \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g, \tag{1.3}$$

where S is the Ricci tensor, λ is constant and p is the time- dependent conformal pressure. The equation (1.3) is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.
In motivation from the above studies, we study ϵ-Sasakian metric as a conformal Ricci soliton. The organization of the paper is as follows: after the introduction, section 2 gives a brief review of the basic notions and formulas of ϵ-Sasakian manifold. In Section 3 conformal Ricci soliton on ϵ-Sasakian manifold has been studied. Section 4 deals with the study of conformal vector fields using some curvature conditions. Section 5 is devoted to the study of torse forming vector field on ϵ-Sasakian manifold. Finally, in section 6 an example of 3-dimensional ϵ-Sasakian metric as conformal Ricci soliton has been illustrated.

2 Preliminaries

A $(2n+1)$ dimensional differentiable manifold (M, g) is said to be an ϵ-almost contact metric manifold if it admits a (1,1) tensor field ϕ, a characteristic vector field ξ, a global 1-form η and an indefinite metric g on M satisfying the following relations

$$\phi^2 = -I + \eta \otimes I, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \eta(X) = \epsilon g(X, \xi), \quad g(\xi, \xi) = \epsilon, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields $X, Y \in \chi(M)$, where $\chi(M)$ is a lie algebra of smooth vector fields and $\epsilon = \pm 1$.
An ϵ-almost contact manifold is said to be an ϵ- Sasakian manifold if the following relations hold.

$$\nabla_X \xi = -\epsilon \phi X, \tag{2.4}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi - \epsilon \eta(X), \tag{2.5}$$

$$(\nabla_X \eta)Y = \epsilon g(Y, \nabla_X \xi), \tag{2.6}$$

where ∇ symbolises the levi civita connection.
In an ϵ-Sasakian manifold we have,

$$\eta(R(X, Y)Z) = \epsilon(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) \tag{2.7}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.8}$$

$$R(\xi, X)Y = \epsilon g(X, Y)\xi - \eta(Y)X \tag{2.9}$$

$$R(X, \xi)Y = -\epsilon g(X, Y)\xi + \eta(Y)X \tag{2.10}$$

$$R(\xi, X)\xi = \epsilon(\eta(X)\xi - X) \tag{2.11}$$

$$S(X, \xi) = \epsilon(n-1)\eta(X), \tag{2.12}$$

for all vector fields $X, Y \in \chi(M)$, where R and S denote the curvature tensor and Ricci tensor respectively.
For a semi-Riemannian manifold the conharmonic curvature tensor is defined as follows

$$\bar{C}(X,Y)Z = \frac{r}{2}[g(X,Z)Y - g(Y,Z)X], \tag{2.13}$$

for all $X, Y, Z \in \chi(M)$, where r is the scalar curvature.

Definition 2.1. [8] A smooth vector field V on a $(2n+1)$- dimensional Riemannian manifold (M, g) is said to be a conformal vector field if

$$(L_V g)(X,Y) = 2\rho g(X,Y), \tag{2.14}$$

where ρ is a smooth function on M.

Definition 2.2. A pseudo Riemannian manifold is called almost quasi-Einstein if its Ricci tensor satisfies

$$S(X,Y) = ag(X,Y) + b(\beta(X)\gamma(Y) + \gamma(X)\beta(Y)), \tag{2.15}$$

where a, b are smooth functions and β, γ are 1-forms.

3 ϵ-Sasakian metric as a conformal Ricci soliton

Theorem 3.1. *Let $2n+1$ dimensional ϵ-Sasakian manifold $(M.g)$ endowed with conformal Ricci soliton (g, V, λ) such that V is pointwise collinear with ξ. Then V becomes constant multiple of ξ and the manifold (M, g) is an Einstein manifold with scalar curvature $r = (2n+1)(n-1)$.*

Proof. Considering conformal Ricci soliton with soliton constant λ and a potential vector field V, let us take $V = \alpha\xi$ in (1.3), where α is a smooth function. Then we get

$$\begin{aligned}(\nabla_X\alpha)g(\xi,Y) + \alpha g(\nabla_X\xi,Y) + (\nabla_Y\alpha)g(X,\xi) + \alpha g(X,\nabla_Y\xi) + 2S(X,Y)\\ = \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y).\end{aligned} \tag{3.1}$$

Using (2.2) and (2.4), the above equation reduces to

$$\epsilon[X(\alpha)\eta(Y) + Y(\alpha)\eta(X)] + 2S(X,Y) = \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y). \tag{3.2}$$

Replacing Y by ξ and making use of (2.2) and (2.12), the previous equation reduces to

$$X(\alpha) = \left[2\lambda - \left(p + \frac{2}{2n+1}\right) - 2(n-1) - \xi(\alpha)\right] \eta(X). \tag{3.3}$$

Again replacing X by ξ and using (2.2), we arrive at

$$\xi(\alpha) = \lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right) - (n-1). \tag{3.4}$$

On substitution of (3.4) in (3.3), we get

$$X(\alpha) = \left[\lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right) - (n-1)\right] \eta(X) \tag{3.5}$$

or,

$$d\alpha = \left[\lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right) - (n-1)\right] \eta. \tag{3.6}$$

By operating $'d'$ on both sides in the above equation and using $d^2 = 0$, we get,

$$\left[\lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right) - (n-1)\right] d\eta = 0. \tag{3.7}$$

Since $d\eta \neq 0$, the previous equation reduces to

$$\lambda = \left(\frac{p}{2} + \frac{1}{2n+1}\right) + (n-1). \tag{3.8}$$

On substitution of (3.8) in (3.6) we obtain $d\alpha = 0$ which results in

$$\alpha = constant. \tag{3.9}$$

Thus this proves that V is a constant multiple of ξ.
Now, at each point of the manifold we take orthonormal basis $e_i : 1 \leq i \leq 2n+1$, by substituting $X = Y = e_i$ in (3.2) and summing over $1 \leq i \leq 2n+1$, we obtain

$$\epsilon\xi(\alpha) + r = (2n+1)\left[\lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right]. \tag{3.10}$$

Since α is constant, the above equation reduces to

$$r = (2n+1)\left[\lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right]. \tag{3.11}$$

By substituting (3.8) in (3.11) we get,

$$r = (n-1)(2n+1). \tag{3.12}$$

In view of equations (3.2) and (3.9) we get,

$$S(X,Y) = \left[\lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right)\right] g(X,Y). \tag{3.13}$$

Thus the result follows from (3.12) and (3.13). □

Considering the integrability formula[12] for Ricci soliton which is as follows:

$$L_v r = -\Delta r + 2R_{ij}R^{ij} - 2\lambda r, \tag{3.14}$$

where $\Delta r = div(\nabla r)$ and ∇ denotes the gradient of a function. Since r is a constant and from (3.12), the above equation yields

$$R_{ij}R^{ij} = \lambda r = \lambda(2n+1)(n-1). \tag{3.15}$$

Now Substituting (3.8) in (3.13), we obtain,

$$S(X,Y) = (n-1)g(X,Y). \tag{3.16}$$

From the above equation, we can obtain

$$R_{ij}R^{ij} = (2n+1)(n-1)^2. \tag{3.17}$$

Comparing (3.15) and (3.17), we get

$$\lambda = n-1. \tag{3.18}$$

Corollary 3.2. *A conformal Ricci soliton in an ϵ-Sasakian manifold M of dimension 2n+1 is shrinking.*

4 Conformal vector field on ϵ-Sasakian manifold

Definition 4.1. An ϵ- Sasakian manifold M of dimension $(2n+1)$ is said to be generalized ϕ recurrent ϵ-Sasakian manifold if its curvature tensor R satisfies the relation

$$\phi^2((\nabla_w R)X,Y)Z) = A(W)(R(X,Y)Z), \tag{4.1}$$

where $X, Y, Z, W \in \chi(M)$ and A is a non vanishing 1- form.

Definition 4.2. [16] A Riemannian manifold M is said to have harmonic conformal curvature tensor if

$$(divC)(X,Y,Z) = 0. \tag{4.2}$$

We Know that [4]

$$\begin{aligned}(divC)(X,Y,Z) = \frac{n-3}{n-1}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] - \\ \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)],\end{aligned} \tag{4.3}$$

where $X, Y, Z \in \chi(M)$ and r is scalar curvature on M.

Theorem 4.3. *An odd dimensional ϵ-Sasakian manifold M admitting a conformal Ricci soliton (g, V, λ) , is an Einstein manifold and the potential vector field is a conformal vector field in the following cases:*

1. *$\bar{C}.S$=0, where $\bar{C}$, S are conharmonic curvature tensor and Ricci tensor respectively in M.*
2. *M is generalized ϕ-reccurent.*
3. *M has harmonic conformal curvature.*

Proof. 1. Consider

$$\bar{C}(X,Y).S(U,V) = 0. \tag{4.4}$$

$$S(\bar{C}(X,Y)U,V) + S(U,\bar{C}(X,Y)V) = 0. \tag{4.5}$$

Put $X = U = \xi$ in the above equation, we get

$$S(\bar{C}(\xi,Y)\xi,V) + S(\xi,\bar{C}(\xi,Y)V) = 0. \tag{4.6}$$

Making use of (2.13) in (4.6), we get

$$\frac{\epsilon r}{2}[S(Y,V) - \eta(Y)S(\xi,V)] + \frac{r}{2}[(n-1)\eta(V)\eta(Y) - g(Y,V)S(\xi,\xi)] = 0. \tag{4.7}$$

Using (2.12), we arrive at

$$S(Y,V) = (n-1)g(Y,V). \tag{4.8}$$

The above equation shows that the manifold is Einstein.
Recalling the conformal Ricci soliton equation and using (4.6) in (1.3), we obtain

$$(L_V g)(X,Y) = \left[2(\lambda - (n-1)) - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y). \tag{4.9}$$

From the definition of conformal vector field, (4.9) can be written in the form
$L_V g = 2\rho g$, where $\rho = \lambda - (n-1) - (\frac{p}{2} + \frac{1}{2n+1})$.
Thus we can conclude that V is a conformal vector field. This completes the proof of Case 1.

2. Suppose M is generalized ϕ-reccurent, i.e.,

$$\phi^2((\nabla_w R)X,Y)Z) = A(W)(R(X,Y)Z), \tag{4.10}$$

where $A(W)$ is a non zero 1-form on M.
Now using (2.1) in the above equation, we get

$$-((\nabla_W R)X,Y)Z) + \eta(\nabla_W R(X,Y)Z) = A(W)(R(X,Y)Z. \tag{4.11}$$

On contraction with respect to X, the above equation yields,

$$-(\nabla_W S)(Y,Z) + g((\nabla_W R)(\xi,Y)Z,\xi) = A(W)S(Y,Z). \tag{4.12}$$

As we know that $g((\nabla_W R)(\xi,Y)Z,\xi) = -g((\nabla_W R)(\xi,Y)\xi,Z)$ and using (2.4) and (2.11), we get $g((\nabla_W R)(\xi,Y)Z,\xi) = 0$.
In view of the above statement, (4.12) reduces to

$$(\nabla_W S)(Y,Z) = -A(W)S(Y,Z). \tag{4.13}$$

Replacing Z by ξ and solving by making use of (2.4) and (2.6), we arrive at

$$-\epsilon(n-1)g(Y,\phi W)+\epsilon S(Y,\phi W)=-A(W)\epsilon(n-1)\eta(Y). \tag{4.14}$$

Take $Y=\phi Y$ and using (2.2) in the above equation, we get

$$\epsilon S(\phi Y,\phi W)=(n-1)[\epsilon g(Y,W)-\eta(Y)\eta(X)]. \tag{4.15}$$

Again taking $X=\phi X$, $Y=\phi Y$ and using (2.1), we obtain

$$S(Y,W)=(n-1)g(Y,W). \tag{4.16}$$

From the conformal Ricci soliton equation and using (4.16) in (1.3), we get

$$(L_V g)(X,Y)=\left[2(\lambda-(n-1))-\left(p+\frac{2}{2n+1}\right)\right]g(X,Y). \tag{4.17}$$

Originating from the definition of conformal vector field, (4.17) can be written in the form $L_V g=2\rho g$, where $\rho=\lambda-(n-1)-(\frac{p}{2}+\frac{1}{2n+1})$.
Thus we can conclude that V is a conformal vector field. This completes the proof of Case 2.
3. Suppose M has the harmonic conformal curvature tensor C. In view of definition (4.2), i.e.,

$$(divC)(X,Y,Z)=0, \tag{4.18}$$

we have,

$$\frac{n-3}{n-1}[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]=\frac{1}{2(n-1)}[g(Y,Z)dr(X)-g(X,Z)dr(Y)]. \tag{4.19}$$

Since ξ is a killing vector, we have,
$L_\xi S=0$ and similarly $L_\xi r=0$ and $dr(\xi)=0$.
We have,

$$(\nabla_\xi S)(Y,Z)=-S(\nabla_Y\xi,Z)-S(Y,\nabla_Z\xi) \tag{4.20}$$

Replacing X by ξ in (4.19), we obtain

$$[(\nabla_\xi S)(Y,Z)-(\nabla_Y S)(\xi,Z)]=\frac{(n-1)}{2(n-1)(n-3)}[g(Y,Z)dr(\xi)-g(\xi,Z)dr(Y)]. \tag{4.21}$$

Using (4.20) in (4.21), we get

$$-S(\nabla_Y\xi,Z)-S(Y,\nabla_Z\xi)-(\nabla_Y S)(\xi,Z)=-\frac{1}{2(n-3)}\eta(Z)dr(Y). \tag{4.22}$$

By making use of (2.4) and (2.12) in the previous equation gives

$$\epsilon(S(Y,\phi Z)+(n-1)g(Z,\phi Y))=-\frac{1}{2(n-3)}\eta(Z)dr. \tag{4.23}$$

If we put $Z=\phi Z$ in the above equation, we obtain

$$(S(Y,\phi^2 Z)+(n-1)g(\phi Z,\phi Y))=0. \tag{4.24}$$

By making use of (2.1), (2.2) and (2.12), we get

$$S(Y,Z)=(n-1)g(Y,Z). \tag{4.25}$$

With the use of (4.25) in (1.3) gives

$$(L_V g)(X,Y)=2\rho g, \tag{4.26}$$

where $\rho=\lambda-(n-1)-(\frac{p}{2}+\frac{1}{2n+1})$.
i.e., V is a conformal vector field. Hence the reult follows.

□

Remark 4.4. In prospect with the above result, we note that the value of $\rho=\lambda-(n-1)-(\frac{p}{2}+\frac{1}{2n+1})$ is found to be same in all the three cases.

5 Conformal Ricci soliton on ϵ-Sasakian manifold with torse-forming vector field

In this section, we make use of the notion torse-forming vector field.

Definition 5.1. [11] A vector field V on $2n+1$-dimensional ϵ-Sasakian manifold M is said to be torse forming vector field if

$$\nabla_X V = fX + \gamma(X)V, \tag{5.1}$$

where f is a smooth function and γ is a 1-form on M.

Theorem 5.2. *If an ϵ- Sasakian manifold of dimension $(2n+1)$ with a torse forming vector field ξ admits conformal Ricci soliton, then the manifold M reduces to almost quasi Einstein manifold and soliton constant $\lambda = \frac{r}{2n+1} + (\frac{p}{2} + \frac{1}{2n+1}) + f(\epsilon - 1)$.*

Proof. Consider conformal Ricci soliton (g, ξ, λ) on an ϵ-Sasakian manifold (M, g). Let us assume that the Reeb vector field ξ of the manifold M is torse forming vector field. Then from the definition (5.1), we have

$$\nabla_X \xi = fX + \gamma(X)\xi, \forall X \in \chi(M). \tag{5.2}$$

From (2.4), we have

$$g(\nabla_X \xi, \xi) = 0. \tag{5.3}$$

Taking inner product of (5.2) with ξ in view of (2.2) , we get

$$g(\nabla_X \xi, \xi) = \epsilon(f\eta(X) + \gamma(X)). \tag{5.4}$$

Combining (5.3) and (5.4), we get

$$\gamma(X) = -f\eta(X). \tag{5.5}$$

Again from the definition (5.1),we have,

$$g(\nabla_X \xi, Y) = fg(X, Y) + \epsilon\gamma(X)\eta(Y). \tag{5.6}$$

We know that,

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi). \tag{5.7}$$

Making use of (5.6) in the preceeding equation we get,

$$(L_\xi g)(X, Y) = 2fg(X, Y) + \xi[\gamma(X)\eta(Y) + \gamma(Y)\eta(X)]. \tag{5.8}$$

Plugging the above equation in (1.3), we obtain,

$$S(X, Y) = \left[\lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right) - f\right] g(X, Y) - \frac{\epsilon}{2}[\gamma(X)\eta(Y) + \gamma(Y)\eta(X)]. \tag{5.9}$$

This proves the first part of the theorem.
Now, we take orthonormal basis $e_i : 1 \leq i \leq 2n+1$. By setting $X = Y = e_i$ in the preceeding equation, summing over $1 \leq i \leq 2n+1$, and by making use of (5.5), we obtain,

$$r = \left[\lambda - \left(\frac{p}{2} + \frac{1}{2n+1}\right) - f\right](2n+1) + f\epsilon. \tag{5.10}$$

Solving for λ, we get,

$$\lambda = \frac{r}{2n+1} + \left(\frac{p}{2} + \frac{1}{2n+1}\right) + f(\epsilon - 1). \tag{5.11}$$

Hence the proof. □

6 Example of 3-dimensional ϵ-Sasakian metric as Conformal Ricci Soliton

We construct 3-dimensional space $\{M = (x_1, x_2, z) \in R^3\}$ where (x_1, x_2, z) are the standard co-ordinates in R^3. Let e_1, e_2, e_3 be the vector fields on M given by
$e_1 = \epsilon z \frac{\partial}{\partial x_1}, \quad e_2 = \epsilon z \frac{\partial}{\partial x_2}, \quad e_3 = -\epsilon z \frac{\partial}{\partial z}$
which are linearly independent forming the basis of T_pM.
Let g be a semi- Riemannian metric on M defined as

$$g(e_i, e_j) = \begin{cases} 0, & \text{if } i \neq j, \\ \epsilon, & \text{if } i = j, \quad where \quad i, j = 1, 2, 3. \end{cases} \tag{6.1}$$

Set $e_3 = \xi$. Let η be a 1-form on M defined by $\eta(X) = \epsilon g(X, e_3) = \epsilon g(X, \xi)$ for all $X \in \chi(M)$. Also, we define (1,1)-tensor ϕ as

$$\phi(e_1) = \epsilon e_2, \quad \phi(e_2) = -\epsilon e_1, \quad \phi(e_3) = 0.$$

In consequence of the above equations, the linearity property of ϕ and g yields

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) + \epsilon\eta(X)\eta(Y), \forall X, Y \in \chi(M).$$

The above relations imply that the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact structure on the manifold M.
Now by direct computations, we obtain

$$\begin{gathered} [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_1, e_3] = \epsilon e_1, \\ [e_2, e_1] = 0, \quad [e_2, e_2] = 0, \quad [e_2, e_3] = \epsilon e_2, \\ [e_3, e_1] = -\epsilon e_1, \quad [e_3, e_2] = -\epsilon e_2, \quad [e_3, e_3] = 0. \end{gathered}$$

By making use of Koszul's formula, we can easily calculate

$$\begin{gathered} \nabla_{e_1} e_1 = -\epsilon e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = \epsilon e_1, \\ \nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -\epsilon e_3, \nabla_{e_2} e_3 = \epsilon e_2, \\ \nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0. \end{gathered}$$

Also, from the above relations we can verify that

$$\nabla_X \xi = -\epsilon\phi X \text{ and } (\nabla_X \phi)Y = g(X, Y)\xi - \epsilon\eta(X).$$

Therefore the manifold $M(\phi, \xi, \eta, g, \epsilon)$ represents a 3-dimensional ϵ Sasakian manifold.
Now we calculate the Riemannian curvature tensor from the well known formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ which can be obtained as follows,

$$\begin{gathered} R(e_1, e_2)e_2 = -e_1, R(e_1, e_3)e_3 = -e_1, R(e_2, e_1)e_1 = -e_2 \\ R(e_2, e_3)e_3 = -e_2, R(e_3, e_1)e_3 = -e_3, R(e_3, e_2)e_2 = -e_3 \\ R(e_1, e_2)e_3 = 0, R(e_2, e_3)e_1 = 0, R(e_3, e_1)e_2 = 0 \end{gathered}$$

From the above values the Ricci tensor can be calculated as follows

$$S(e_1, e_1) = -2, S(e_2, e_2) = -2, S(e_3, e_3) = -2$$

The value of the scalar curvature becomes

$$r = \sum_{i=1}^{3} S(e_i, e_i) = -6.$$

Using the above values in the equation (1.3) we have,

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2S(X, Y) = [2\lambda - (p + \tfrac{2}{3}]g(X, Y).$$

Replacing $X = Y = e_2$ and $\xi = e_3$ in above equation, we get

$$\lambda = \tfrac{1}{2}(p + \tfrac{2}{3}) - 2\epsilon.$$

Therefore, we can say that for $\lambda = \frac{1}{2}(p + \frac{2}{3}) - 2\epsilon$, the data (g,$\xi$,$\lambda$,$\epsilon$) defines a conformal Ricci soliton on 3-dimensional ϵ-Sasakian manifold (M, g, ϕ, ξ, η). Hence by making use of above values in equation (5.11) we obtain $\lambda = -2 + (\frac{p}{2} + \frac{1}{3}) + f(\epsilon - 1)$ which verifies theorem (5.2).

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