Palestine Journal of Mathematics

Vol 13(4)(2024) , 1286–1298

POST-QUANTUM-BERNSTEIN OPERATORS ON A TRIANGULAR DOMAIN AND ASSOCIATED APPROXIMATIONS

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 41A35, 41A36; Secondary 41A80.

Keywords: (p,q) Bernstein Operators, Product Operators, Boolean Sum Opeartors, Approximation formulae, Triangular domain.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract *The objective of this paper is to construct the (p,q)-analogue of univariate Bernstein operators, denoted by $\left(\mathcal{B}^{y,l}_{p,q}f\right)(y,z)$ and $\left(\mathcal{B}^{z,n}_{p,q}f\right)(y,z)$, their products $\left(\mathcal{P}^{ln}_{p,q}f\right)(y,z)$ and $\left(\mathcal{Q}^{nl}_{p,q}f\right)(y,z)$ and their Boolean sums $\left(\mathcal{S}^{ln}_{p,q}f\right)(y,z)$ and $\left(\mathcal{T}^{nl}_{p,q}f\right)(y,z)$ on a triangular domain $\mathcal{T}_h$, using post quantum calculus and study their interpolating and approximating properties on this domain. Further the remainders for approximation formulae for the corresponding operators are evaluated. The theoretical findings are then supported by graphical representations and analysis of the operators and the functions being interpolated and approximated by them. This lead us to establish the fact that the parameters p,q provide flexibility for approximation by letting us maintain error within the desired limits.*

1 Introduction

Weierstrass, in 1885 asserted that every continuous function on a compact interval of the real line can be approximated by an algebraic polynomial. In 1912, S.N. Bernstein constructed these polynomials, named after his name as Bernstein Polynomials, in order to give a constructive proof of Weierstrass approximation theorem [25]. Since then various generalisations have been given for Bernstein polynomials which find applications in Numerical analysis, Computer-aided geometric design(CAGD), finite element analysis and finding solutions of differential equations. Approximation operators on polygonal domains are needed in the finite element method for solving differential equations with known boundary conditions. As a result, several researchers created some more operators for better approximation and generalised Bernstein type operators on many domains. After the papers [26, 27, 28] of R.E. Barnhill et al., Lagrange, Birkhoff and Hermite type operators have been studied, which interpolate a given function and some of its derivative on the boundary of triangle (as in Dirichlet, Neumann or Robin boundary conditions for differential equation problems).They investigated interpolation operations on triangles with curved sides (one, two, or all curved sides), many of which were related to the finite element method and computer-aided geometric design. D. D. Stancu investigated polynomial interpolation on triangular boundary data and the error bound for smooth interpolation [31, 32]. P. Blaga and G. Coman extended Bernstein operator on triangle and defined boolean sum operators [29]. Catinas extended some interpolation operators to triangle with one curved side [30]. T. Acar et al. studied approximation properties of Bivariate Bernstein-Stancu-Chlodowsky, Bernstein-Kantorovich type operators etc. in [1, 2, 3]. Better unifrom approximation by a new set of Bivariate Bernstein Operators is studied in [43].Also, Inverse result in simultaneous approximation by Baskakov-Durmeyer-Stancu operators are given by [34]. Q. B. Cai constructed approximating operator λ-Bernstein operators based on parameter λ in [6, 7]. N. Braha et al.

studied λ-Bernstein operators via power series summability methods in [5]. Mursaleen et al. investigated approximation behaviors of q-Bernstein shifted operators and q-Bernstein Schurer operators in [17, 18]. Khalid et al. recently generalised Bernstein type operators and investigated their applications in Computer Aided Geometric Design (CAGD) [12, 13]. Other applications of Bernstein type operators for the creation of Bezier curves and surfaces can be found in [4, 8, 9, 10, 22, 23, 24]. To provide flexibility about a particular control point, weighted lupas bernstein bezier curves using (p,q)-integers are constructed in [11]. Using q-calculus (quantum analogue), Lupaş [33] and Phillips [20] generalised the Bernstein polynomials to q-Bernstein polynomials. More on q-Bernstein polynomials can be found in [21]. In 2015 Mursaleen et al. [19] defined (p,q)-analogue of Bernstein polynomials with the help of post quantum calculus. Convergence of sequences $\frac{1}{[n]_{p,q}}$ and $\frac{[n-1]_{p,q}}{[n]_{p,q}}$ is of importance for convergence of sequence of operators based on (p,q)-calculus. All six different possibilities of convergence with proof are discussed in [15]. One can also see the convergence behavior of different type of operators based on (p,q)-calculus in [16, 35, 36, 37, 38, 39, 40, 41]. Motivated by the papers of Khan et al. [14] and P. Blaga at al. [29], we construct (p,q)-Bernstein operators on a triangular domain. The products and Boolean sums of these operators are used to construct new set of operators. We shall then inspect the interpolating and approximating properties of these operators. Using the concept of modulus of continuity and Peano's theorem, the remainders of the corresponding approximation formulae are also evaluated. Towards the end of the paper the accuracy of the approximation and the role played by the parameters p and q is illustrated by graphical representation of given functions with suitable (p,q)-Bernstein operators.

Now we recall some definitions of (p,q)-calculus. For any $p,q>0$ and positive integer l, the (p,q)-integer written as $[l]_{p,q}$ is defined by

$$[l]_{p,q}=\frac{p^l-q^l}{p-q}.$$

The (p,q) Binomial expansion and the (p,q) Binomial coefficients are respectively given by

$$(ay+bz)^l_{p,q}=\sum_{i=0}^{l}\begin{bmatrix} l \\ i \end{bmatrix}_{p,q} a^{l-i}b^iy^{l-i}z^i,$$

and

$$\begin{bmatrix} l \\ i \end{bmatrix}_{p,q}:=\frac{[l]_{p,q}!}{[i]_{p,q}![l-i]_{p,q}!}.$$

Further, the recurrence relations of (p,q) Binomial coefficients are given by

$$\begin{bmatrix} l+1 \\ i \end{bmatrix}_{p,q}=q^{l-i+1}\begin{bmatrix} l \\ i-1 \end{bmatrix}_{p,q}+p^i\begin{bmatrix} l \\ i \end{bmatrix}_{p,q} \tag{1.1}$$

and

$$\begin{bmatrix} l+1 \\ i \end{bmatrix}_{p,q}=p^{l-i+1}\begin{bmatrix} l \\ i-1 \end{bmatrix}_{p,q}+q^i\begin{bmatrix} l \\ i \end{bmatrix}_{p,q}. \tag{1.2}$$

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2 Construction of (p,q)-Bernstein Operators on triangular domain

Consider the standard triangular domain given as follows:

$$\mathcal{T}_h=\{(y,z)\in\mathbb{R}^2 \mid\ y\geq 0, z\geq 0, y+z\leq h\},\qquad for\quad h>0.$$

Let us consider a real-valued function f defined on $\mathcal{T}_h$.Through the point $(y,z)\in\mathcal{T}_h$, let us consider the parallel lines also parallel to the coordinate axes intersecting the edges $\Gamma_j, j=1,2,3,$ of the triangle respectively at the points $(0,z)$, $(h-z,z)$, $(y,0)$ and $(y,h-y)$.

Let us consider $\triangle_l^y = \{[i]_{p,q}\frac{h-z}{[l]_{p,q}}, i = \overline{0,l}\}$ *and* $\triangle_n^z = \{[j]_{p,q}\frac{h-y}{[n]_{p,q}}, j = \overline{0,n}\}$ *which act as partitions of the intervals* $[0, h-z]$ *and* $[0, h-y]$*, respectively.*

Now, the (p,q)*-Bernstein Operators* $\mathcal{B}_{p,q}^{y,l}$ *and* $\mathcal{B}_{p,q}^{z,n}$ *on triangle* $\mathcal{T}_h$ *are given by*

$$(\mathcal{B}_{p,q}^{y,l}f)(y,z) = \begin{cases} \sum\limits_{i=0}^{l} \hat{p}_{l,i}(y,z) f\left(\frac{[i]_{p,q}}{p^{i-l}[l]_{p,q}}(h-z), z\right), & (y,z) \in \mathcal{T}_h \setminus (0,h), \\ \\ f(0,h), & (0,h) \in \mathcal{T}_h \end{cases}$$

and

$$(\mathcal{B}_{p,q}^{z,n}f)(y,z) = \begin{cases} \sum\limits_{j=0}^{n} \hat{q}_{n,j}(y,z) f\left(y, \frac{[j]_{p,q}}{p^{j-n}[n]_{p,q}}(h-y)\right), & (y,z) \in \mathcal{T}_h \setminus (h,0), \\ \\ f(h,0), & (h,0) \in \mathcal{T}_h \end{cases}$$

where

$$\hat{p}_{l,i}(y,z) = \frac{\begin{bmatrix} l \\ i \end{bmatrix}_{p,q} y^i \prod_{s=0}^{l-i-1}((h-z)p^s - q^s y) p^{\frac{i(i-1)-l(l-1)}{2}}}{(h-z)^l}, \quad 0 \le y+z \le h \tag{2.1}$$

and

$$\hat{q}_{n,j}(y,z) = \frac{\begin{bmatrix} n \\ j \end{bmatrix}_{p,q} z^j \prod_{t=0}^{n-j-1}((h-y)p^t - q^t z) p^{\frac{j(j-1)-n(n-1)}{2}}}{(h-y)^n}, \quad 0 \le y+z \le h. \tag{2.2}$$

It can be clearly seen that for $p = 1$*, the above operators turns out to Phillips-type* q*-Bernstein operators on triangles as in* [14]*. Using the Principle of Mathematical Induction and the recurrence relations (1.1) and (1.2), it can be easily proved that*

$$(h-z)^l := \sum_{i=0}^{l} \begin{bmatrix} l \\ i \end{bmatrix}_{p,q} y^i \prod_{s=0}^{l-i-1} ((h-z)p^s - q^s y) p^{\frac{i(i-1)-l(l-1)}{2}}, \tag{2.3}$$

and

$$(h-y)^n := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} z^j \prod_{t=0}^{n-j-1} ((h-y)p^t - q^t z) p^{\frac{j(j-1)-n(n-1)}{2}}. \tag{2.4}$$

Definition 2.1. If the operator $\mathcal{B}_{p,q}^{y,l}$ preserve the monomial of highest degree say k, i.e., $\mathcal{B}_{p,q}^{y,l} y^k = y^k$ then we say that operator $\mathcal{B}_{p,q}^{y,l}$ has degree of exactness k. Then we write $dex(\mathcal{B}_{p,q}^{y,l}) = k$.

Theorem 2.2. *For a real-valued function* f *defined on* $\mathcal{T}_h$*, we have*

$$(i)\mathcal{B}_{p,q}^{y,l} f = f \text{ on } \Gamma_2 \cup \Gamma_3;$$

$$(ii)(\mathcal{B}_{p,q}^{y,l} e_{i0})\,(y,z) = y^i, \;\; i = 0,1 \; (dex(\mathcal{B}_{p,q}^{y,l}) = 1),$$

$$(iii)(\mathcal{B}_{p,q}^{y,l} e_{20})\,(y,z) = y^2 + \frac{p^{l-1} y(h-y-z)}{[l]_{p,q}},$$

$$(\mathcal{B}_{p,q}^{y,l} e_{ij})\,(y,z) = \begin{cases} z^j y^i, & i = 0,1, \;\; j \in \mathbb{N}; \\ \\ z^j\left(y^2 + \frac{p^{l-1} y(h-y-z)}{[l]_{p,q}}\right), & i = 2, \;\; j \in \mathbb{N}; \end{cases}$$

where $e_{ij}(y,z) = y^i z^j$ *and dex* $(\mathcal{B}_{p,q}^{y,l})$ *denotes degree of exactness of the operator* $\mathcal{B}_{p,q}^{y,l}$.

Proof. *By the definition of the operator,* $(B^{y,l}_{p,q}f)(0,h) = f(0,h)$. *So we caculate the moments only on* $\mathcal{T}_h \setminus (0,h)$. *The interpolation property represented by* (i) *follows from the relations*

$$\hat{p}_{l,i}\,(0,z) = \begin{cases} 1, & \text{if } i = 0, \\ \\ 0, & i \neq 0 \end{cases}$$

and

$$\hat{p}_{l,i}(h-z,z) = \begin{cases} 1, & \text{if } i = l, \\ \\ 0, & i \neq l. \end{cases}$$

To prove properties (ii), *we proceed in the following way*

$$\begin{aligned}
(\mathcal{B}^{y,l}_{p,q}e_{00})\,(y,z) &= \sum_{i=0}^{l} \frac{\begin{bmatrix} l \\ i \end{bmatrix}_{p,q} y^i \prod_{s=0}^{l-i-1}((h-z)p^s - q^s y)p^{\frac{i(i-1)-l(l-1)}{2}}}{(h-z)^l} \\
&= \frac{(h-z)^l}{(h-z)^l} = 1; \\
(\mathcal{B}^{y,l}_{p,q}e_{10})(y,z) &= \sum_{i=0}^{l} \frac{\begin{bmatrix} l \\ i \end{bmatrix}_{p,q} p^{\frac{i(i-1)-l(l-1)}{2}} u^i \prod_{s=0}^{l-i-1}((h-z)p^s - q^s y)}{(h-z)^l} \frac{[i]_{p,q}}{p^{i-l}[l]_{p,q}}(h-z) \\
&= \sum_{i=0}^{l} \frac{\frac{[i]_{p,q}}{p^{i-l}[l]_{p,q}} \begin{bmatrix} l \\ i \end{bmatrix}_{p,q} p^{\frac{i(i-1)-l(l-1)}{2}} y^i \prod_{s=0}^{l-i-1}((h-z)p^s - q^s y)}{(h-z)^{l-1}} \\
&= \sum_{i=0}^{l-1} \frac{\begin{bmatrix} l-1 \\ i \end{bmatrix}_{p,q} p^{\frac{i(i-1)-(l-1)(l-2)}{2}} y^{i+1} \prod_{s=0}^{l-i-2}((h-z)p^s - q^s y)}{(h-z)^{l-1}} \\
&= y\sum_{i=0}^{l-1} \frac{\begin{bmatrix} l-1 \\ i \end{bmatrix}_{p,q} p^{\frac{i(i-1)-(l-1)(l-2)}{2}} y^{i} \prod_{s=0}^{(l-1)-i-1}((h-z)p^s - q^s y)}{(h-z)^{l-1}} \\
&= y;
\end{aligned}$$

$$\begin{aligned}
(\mathcal{B}^{y,l}_{p,q}e_{20})\,(y,z) &= \sum_{i=0}^{l} \frac{\begin{bmatrix} l \\ i \end{bmatrix}_{p,q} p^{\frac{i(i-1)-l(l-1)}{2}} y^i \prod_{s=0}^{l-i-1}((h-z)p^s - q^s y)}{(h-z)^l} \frac{[i]^2_{p,q}(h-z)^2}{p^{2(i-1)}[l]^2_{p,q}} \\
&= \frac{1}{p^{l(l-1)/2}}(h-z)^2 y \sum_{i=0}^{l-1} \frac{\begin{bmatrix} l-1 \\ i \end{bmatrix}_{p,q} p^{\frac{i(i+1)}{2}} y^i \frac{[i+1]_{p,q}}{[l]_{p,q}} \prod_{s=0}^{l-i-2}((h-z)p^s - q^s y)}{(h-z)^l} \frac{1}{p^{2(i-l+1)}}(h-z)^2 \\
&= \frac{1}{p^{l(l-1)/2}}(h-z)^2 y \sum_{i=0}^{l-1} \frac{\begin{bmatrix} l-1 \\ i \end{bmatrix}_{p,q} p^{\frac{i(i+1)}{2}} y^i \frac{p^i + q[i]_{p,q}}{[l]_{p,q}} \prod_{s=0}^{l-i-2}((h-z)p^s - q^s y)}{(h-z)^l} \frac{1}{p^{2(i-l+1)}} \\
&= \frac{(h-z)^2 y p^{l-1}}{[l]_{p,q}} + \frac{(h-z)^2 y}{p^{l(l-1)/2}} \sum_{i=0}^{l-1} \frac{\frac{q[i]_{p,q}}{[l]_{p,q}} \begin{bmatrix} l-1 \\ i \end{bmatrix}_{p,q} p^{\frac{i(i+1)}{2}} y^i \prod_{s=0}^{l-i-2}((h-y)p^s - q^s y)}{(h-z)^l} \frac{1}{p^{2(i-l+1)}} \\
&= \frac{(h-z)^2 y p^{l-1}}{[l]_{p,q}} + \frac{q[l-1]_{p,q} y^2}{[l-1]_{p,q}} \sum_{i=0}^{l-2} \frac{\begin{bmatrix} l-1 \\ i \end{bmatrix}_{p,q} p^{\frac{i(i-1)-(l-2)(l-3)}{2}} y^i \prod_{s=0}^{l-2-i-1}((h-z)p^s - q^s y)}{(h-z)^{l-2}} \\
&= \frac{(h-z)^2 y p^{l-1}}{[l]_{p,q}} + \frac{q[l-1]_{p,q} y^2}{[l-1]_{p,q}} \\
&= y^2 + \frac{(h-z-y)yp^{l-1}}{[l]_{p,q}}.
\end{aligned}$$

This completes the proof of theorem 2.2.

Remark 2.3. For a real-valued function f defined on $\mathcal{T}_h$, we can prove the following in the similar way as we have done earlier.

$$(i)\mathcal{B}_{p,q}^{z,n} f = f \; on \; \Gamma_1 \cup \Gamma_3;$$

$$(ii)(\mathcal{B}_{p,q}^{z,n} e_{0j})\,(y,z) = z^j,\;\; j = 0,1\; (dex(\mathcal{B}_{p,q}^{z,n}) = 1),$$

$$(iii)(\mathcal{B}_{p,q}^{z,n} e_{02})\,(y,z) = z^2 + \frac{z(h-y-z)}{[n]_{p,q}},$$

$$(\mathcal{B}_{p,q}^{z,n} e_{ij})\,(y,z) = \begin{cases} y^i z^j, & j = 0,1,\;\; i \in \mathbb{N}; \\ y^i \left(z^2 + \frac{p^{n-1} z(h-y-z)}{[n]_{p,q}} \right), & j = 2,\;\; i \in \mathbb{N}. \end{cases}$$

Consider the approximation formula

$$f = \mathcal{B}_{p,q}^{y,l} f + \mathcal{R}_{p,q}^{y,l} f,$$

we prove the following result.

Theorem 2.4. *If f is a continuous function on the interval $[0,\ h-z]$ i.e. $f(.,z) \in C[0,\ h-z]$, then*

$$\left|(\mathcal{R}_{p,q}^{y,l} f)(y,z)\right| \leq \left(1 + \frac{h\sqrt{p^{l-1}}}{2\delta\sqrt{[l]_{p,q}}}\right) M\big(f(.,z);\delta\big), \quad z \in [0,h].$$

where $M(f(.,z);\delta)$. modulus of continuity of the function f with respect to the first variable y. Moreover, if $\delta = \frac{1}{\sqrt{[l]_{p,q}}}$ and $0 < q < p \leq 1$, then

$$\left|(\mathcal{R}_{p,q}^{y,l} f)(y,z)\right| \leq \left(1 + \frac{h}{2}\right) M\left(f(.,z); \frac{1}{\sqrt{[l]_{p,q}}}\right), \quad z \in [0,h].$$

Proof. *Since by definition, $(B_{p,q}^{y,l} f)(0,h) = f(0,h)$ so because of the interpolation property remainder will be zero at $(0,h)$. Further we have*

$$\left|(\mathcal{R}_{p,q}^{y,l} f)(y,z)\right| \leq \sum_{i=0}^{l} \hat{p}_{l,i}(y,z) \left| f(y,z) - f\left(\frac{[i]_{p,q}(h-z)}{p^{i-l}[l]_{p,q}}, z\right)\right|.$$

Since

$$\left| f(y,z) - f\left(\frac{[i]_q(h-z)}{p^{i-l}[l]_{p,q}}, z\right)\right| \leq \left(\frac{1}{\delta}\left| y - \frac{[i]_{p,q}(h-z)}{p^{i-l}[l]_{p,q}}\right| + 1\right) M(f(.,z);\delta),$$

one obtains

$$\begin{aligned} \left|(\mathcal{R}_{p,q}^{y,l} f)(y,z)\right| &\leq \sum_{i=0}^{l} \hat{p}_{l,i}(y,z) \left(\frac{1}{\delta}\left| y - \frac{[i]_{p,q}(h-z)}{p^{i-l}[l]_{p,q}}\right| + 1\right) M(f(.,z);\delta) \\ &\leq \left[1 + \frac{1}{\delta}\left(\sum_{i=0}^{l} \hat{p}_{l,i}(y,z) \left(y - \frac{[i]_{p,q}(h-z)}{p^{i-l}[l]_{p,q}}\right)^2\right)^{1/2}\right] M(f(.,z);\delta) \\ &= \left[1 + \frac{1}{\delta}\sqrt{\frac{y(h-y-z)p^{l-1}}{[l]_{p,q}}}\;\right] M(f(.,z);\delta). \end{aligned}$$

As

$$\max_{\mathcal{T}_h}[y(h-y-z)] = \frac{h^2}{4},$$

it follows that

$$\left|(\mathcal{R}^{y,l}_{p,q}f)(y,z)\right| \leq \left(1+\frac{\sqrt{p^{l-1}}h}{2\delta\sqrt{[l]_{p,q}}}\right)M(f(.,z);\delta).$$

Clearly for $\delta = \frac{1}{\sqrt{[l]_{p,q}}}$ *and* $0<q<p\leq 1$, *we obtain*

$$\left|(\mathcal{R}^{y,l}_{p,q}f)(y,z)\right| \leq \left(1+\frac{h}{2}\right) M\left(f(.,z);\frac{1}{\sqrt{[l]_{p,q}}}\right). \qquad \square$$

Theorem 2.5. *If* $f(.,z)\in C^2[0,h]$ *and 0*$<q<p\leq 1$, *then*

$$(\mathcal{R}^{y,l}_{p,q}f)(y,z) = -\frac{y(h-y-z)}{2[l]_{p,q}}f^{(2,0)}(\xi,z), \qquad \xi\in[0,h-z] \tag{2.5}$$

and

$$\left|(\mathcal{R}^{y,l}_{p,q}f)(y,z)\right| \leq \frac{h^2}{8[l]_{p,q}}\mathcal{M}_{20}f, \quad (y,z)\in\mathcal{T}_h,$$

where

$$\mathcal{M}_{ij}f = \max_{\mathcal{T}_h}\left|f^{(i,j)}(y,z)\right|.$$

Proof. *As* $dex(\mathcal{B}^{y,l}_{p,q}) = 1$, *using Peano's theorem, we obtain*

$$(\mathcal{R}^{y,l}_{p,q}f)(y,z) = \int_0^{h-z}\mathcal{K}_{20}(y,z;t)f^{(2,0)}(t,z)dt,$$

here the kernel given by

$$\mathcal{K}_{20}(u,v;t) := \mathcal{R}^{y,l}_{p,q}\left[(y-t)_+\right] = (y-t)_+ - \sum_{i=0}^{l}\hat{p}_{l,i}(y,z)\left([i]_{p,q}\frac{h-z}{[l]_{p,q}}-t\right)_+$$

It is easy to follows from Mean Value Theorem,

$$(\mathcal{R}^{y,l}_{p,q}f)(y,z) = f^{(2,0)}(\xi,z)\int_0^{h-z}\mathcal{K}_{20}(y,z;t)dt, \quad \xi\in[0,h-z].$$

After some simple calculation, we get

$$(\mathcal{R}^{y,l}_{p,q}f)(y,z) = -\frac{y(h-y-z)}{2[l]_{p,q}}f^{(2,0)}(\xi,z),$$

where $\xi\in[0,h-z]$.
With the help of equation (2.5), *we get*

$$\left|(\mathcal{R}^{y,l}_{p,q}f)(y,z)\right| \leq \frac{h^2}{8[l]_{p,q}}\mathcal{M}_{20}f, \quad (y,z)\in\mathcal{T}_h. \qquad \square$$

Remark 2.6. From (2.5) it can be deduced that
- If $f(.,z)$ is a concave function then $\left(\mathcal{R}^{y,l}_{p,q}f\right)(y,z)\geq 0$, i.e. $\left(\mathcal{B}^{y,l}_{p,q}f\right)(y,z)\leq f(y,z)$.
- If $f(.,z)$ is a convex function then $\left(\mathcal{R}^{y,l}_{p,q}f\right)(y,z)\leq 0$, i.e. $\left(\mathcal{B}^{y,l}_{p,q}f\right)(y,z)\geq f(y,z)$,

for $y\in[0,h-z]$ and $z\in[0,h]$. $\square$

Remark 2.7. Based on the following approximation formula

$$f = \mathcal{B}_{p,q}^{z,n} f + \mathcal{R}_{p,q}^{z,n} f.$$

we can deduce the following results for the remainder $\mathcal{R}_{p,q}^{z,n} f$.

1. *If* $f(y,.) \in C[0, h-y]$, *then*

$$\Big|(\mathcal{R}_{p,q}^{z,n} f)(y,z)\Big| \le \left(1 + \frac{\sqrt{p^{n-1}}h}{2\delta\sqrt{[n]_{p,q}}}\right) M\big(f(y,.);\delta\big), \quad y \in [0,h].$$

and for $\delta = \frac{1}{\sqrt{[n]_{p,q}}}$ *and* $0 < q < p \le 1$, *we have*

$$\Big|(\mathcal{R}_{p,q}^{z,n} f)(y,z)\Big| \le \left(1 + \frac{h}{2}\right) M\left(f(y,.); \frac{1}{\sqrt{[n]_{p,q}}}\right), \quad y \in [0,h].$$

2. *If* $f(y,.) \in C^2[0,h]$ *and* $0 < q < p \le 1$, *then*

$$(\mathcal{R}_{p,q}^{z,n} f)(y,z) = -\frac{z(h-y-z)}{2[n]_{p,q}} f^{(0,2)}(y,\eta), \qquad \eta \in [0, h-y],$$

and

$$\Big|(\mathcal{R}_{p,q}^{z,n} f)(y,z)\Big| \le \frac{h^2}{8[n]_{p,q}} \mathcal{M}_{02} f, \quad (y,z) \in \mathcal{T}_h,$$

where

$$\mathcal{M}_{ij} f = \max_{\mathcal{T}_h} \big|f^{(i,j)}(y,z)\big|.$$

3 Product Operators

Let $\mathcal{P}_{p,q}^{ln} = \mathcal{B}_{p,q}^{y,l}\mathcal{B}_{p,q}^{z,n}$ *and* $\mathcal{Q}_{p,q}^{nl} = \mathcal{B}_{p,q}^{z,n}\mathcal{B}_{p,q}^{y,l}$ *be the products of the operators* $\mathcal{B}_{p,q}^{y,l}$ *and* $\mathcal{B}_{p,q}^{z,n}$. *The operator* $\mathcal{P}_{p,q}^{ln}$ *is given by*

$$(\mathcal{P}_{p,q}^{ln} f)(y,z) = \begin{cases} \sum_{i=0}^{l}\sum_{j=0}^{n} \hat{p}_{l,i}(y,z)\hat{q}_{n,j}\left(\frac{[i]_{p,q}(h-z)}{p^{i-l}[l]_{p,q}}, z\right) f\left(\frac{[i]_{p,q}(h-z)}{p^{i-l}[l]_{p,q}}, [j]_{p,q}\frac{(p^{i-l}[l]_{p,q}-[i]_{p,q})h+z[i]_{p,q}}{p^{j-n}p^{i-l}[l]_{p,q}[n]_{p,q}}\right), & (y,z) \in \mathcal{T}_h \setminus \{(0,h),(h,0)\}, \\ f(0,h), & (0,h) \in \mathcal{T}_h, \\ f(h,0), & (h,0) \in \mathcal{T}_h. \end{cases}$$

Theorem 3.1. *For the product operator* $\mathcal{P}_{p,q}^{ln}$ *the following relations hold:*
(i) $(\mathcal{P}_{p,q}^{ln} f)(y,0) = (\mathcal{B}_{p,q}^{y,l} f)(y,0)$,
(ii) $(\mathcal{P}_{p,q}^{ln} f)(0,z) = (\mathcal{B}_{p,q}^{z,n} f)(0,z)$,
(iii) $(\mathcal{P}_{p,q}^{ln} f)(y,h-y) = f(y,h-y), \qquad y,z \in [0,h]$.

Some simple calculation shows that the above relations are satisfied by the product operator. Moreover, it is clear from the property (i) *and* (ii) *that*
$(\mathcal{P}_{p,q}^{ln} f)(0,0) = f(0,0)$. □

Remark 3.2. The product operator $\mathcal{P}_{p,q}^{ln}$ interpolates the real valued function f at the vertex $(0,0)$ and on the hypotenuse $y+z=h$ of the triangle $\mathcal{T}_h$. □

We can define the product operator $\mathcal{Q}_{p,q}^{nl}$ *as follows*

$$(\mathcal{Q}_{p,q}^{nl} f)(y,z) = \begin{cases} \sum_{i=0}^{l}\sum_{j=0}^{n} \hat{p}_{l,i}(u, \frac{[j]_{p,q}(h-y)}{p^{j-n}[n]_{p,q}})\hat{q}_{n,j}(y,z) f\left([i]_{p,q}\frac{(p^{j-n}[n]_{p,q}-[j]_{p,q})h+y[j]_{p,q}}{p^{j-n}p^{i-l}[l]_{p,q}[n]_{p,q}}, [j]_{p,q}\frac{(h-y)}{p^{j-n}[n]_{p,q}}\right), & (y,z) \in \mathcal{T}_h \setminus \{(0,h),(h,0)\}, \\ f(0,h), & (0,h) \in \mathcal{T}_h, \\ f(h,0), & (h,0) \in \mathcal{T}_h. \end{cases}$$

Moreover, $(\mathcal{Q}_{p,q}^{nl}f)(y,z)$ *satisfies the following properties:*
(i) $(\mathcal{Q}_{p,q}^{nl}f)(y,0) = (\mathcal{B}_{p,q}^{y,l}f)(y,0)$,
(ii) $(\mathcal{Q}_{p,q}^{nl}f)(0,z) = (\mathcal{B}_{p,q}^{z,n}f)(0,z)$,
(iii) $(\mathcal{Q}_{p,q}^{nl}f)(h-z,z) = f(h-z,z), \quad y,z \in [0,h]$.

Based on the following approximation formula

$$f = \mathcal{P}_{p,q}^{ln}f + \mathcal{R}_{p,q}^{\mathcal{P}^{ln}}f,$$

we present some interesting results.

Theorem 3.3. *If* $f \in C(\mathcal{T}_h)$ *and* $0 < p < q \leq 1$, *then*

$$\left|\left(\mathcal{R}_{p,q}^{\mathcal{P}^{ln}}f\right)(y,z)\right| \leq (1+h)M\left(f; \frac{1}{\sqrt{[l]_{p,q}}}, \frac{1}{\sqrt{[n]_{p,q}}}\right), \quad (y,z) \in \mathcal{T}_h. \tag{3.1}$$

Proof. *Based on the approximation formula for the product operator* $\mathcal{P}_{p,q}^{ln}f$, *we have*

$$\begin{aligned}\left|(\mathcal{R}_{p,q}^{\mathcal{P}^{ln}}f)(y,z)\right| \leq \Bigg[&\frac{1}{\delta_1}\sum_{i=0}^{l}\sum_{j=0}^{n}\hat{p}_{l,i}(y,z)\hat{q}_{n,j}\left([i]_{p,q}\frac{(h-z)}{p^{i-l}[l]_{p,q}}, z\right)\left|y-[i]_{p,q}\frac{(h-z)}{p^{i-l}[l]_{p,q}}\right| \\ &+\frac{1}{\delta_2}\sum_{i=0}^{l}\sum_{j=0}^{n}\hat{p}_{l,i}(y,z)\hat{q}_{n,j}\left([i]_{p,q}\frac{(h-z)}{p^{i-l}[l]_{p,q}}, z\right)\left|z-[j]_{p,q}\frac{(p^{i-l}[l]_{p,q}-[i]_{p,q})h+[i]_{p,q}z}{p^{i-l}p^{j-n}[l]_{p,q}[n]_{p,q}}\right| \\ &+\sum_{i=0}^{l}\sum_{j=0}^{n}\hat{p}_{l,i}(y,z)\hat{q}_{n,j}\left([i]_{p,q}\frac{(h-z)}{p^{i-l}[l]_{p,q}}, z\right)\Bigg]M(f;\delta_1,\delta_2).\end{aligned}$$

After some simple computations, we have

$$\sum_{i=0}^{l}\sum_{j=0}^{n}\hat{p}_{l,i}(y,z)\hat{q}_{n,j}\left([i]_{p,q}\frac{(h-z)}{p^{i-l}[l]_{p,q}}, z\right)\left|y-[i]_{p,q}\frac{(h-z)}{[l]_{p,q}}\right| \leq \sqrt{\frac{y(h-y-z)p^{l-1}}{[l]_{p,q}}},$$

$$\sum_{i=0}^{l}\sum_{j=0}^{n}\hat{p}_{l,i}(y,z)\hat{q}_{n,j}\left([i]_{p,q}\frac{(h-z)}{p^{i-l}[l]_{p,q}}, z\right)\left|z-[j]_{p,q}\frac{(p^{i-l}[l]_{p,q}-[i]_{p,q})h+[i]_{p,q}z}{p^{i-l}p^{j-n}[l]_{p,q}[n]_{p,q}}\right| \leq \sqrt{\frac{z(h-y-z)p^{n-1}}{[n]_{p,q}}}.$$

Since

$$\sum_{i=0}^{l}\sum_{j=0}^{n}\hat{p}_{l,i}(y,z)\hat{q}_{n,j}\left([i]_{p,q}\frac{(h-z)}{p^{i-l}[l]_{p,q}}, z\right) = 1.$$

It follows that

$$\left|(\mathcal{R}_{p,q}^{\mathcal{P}^{ln}}f)(y,z)\right| \leq \left(\frac{1}{\delta_1}\sqrt{\frac{y(h-y-z)p^{l-1}}{[l]_{p,q}}}+\frac{1}{\delta_2}\sqrt{\frac{z(h-y-z)p^{n-1}}{[n]_{p,q}}}\ +\ 1\right)M(f;\delta_1,\delta_2).$$

Since

$$\frac{y(h-y-z)}{[l]_{p,q}} \leq \frac{h^2}{4[l]_{p,q}}, \qquad \frac{z(h-y-z)}{[n]_{p,q}} \leq \frac{h^2}{4[n]_{p,q}}, \qquad for\ all\ (y,z) \in \mathcal{T}_h.$$

For $0 < q < p \leq 1$ *, we have*

$$\left|(\mathcal{R}_{p,q}^{\mathcal{P}^{ln}}f)(y,z)\right| \leq \left(\frac{h}{2\delta_1\sqrt{[l]_{p,q}}}+\frac{h}{2\delta_2\sqrt{[n]_{p,q}}}\ +\ 1\right)M(f;\delta_1,\delta_2)$$

$$\left|(\mathcal{R}_{p,q}^{\mathcal{P}^{ln}}f)(y,z)\right| \leq (1+h)\,M\left(f;\frac{1}{\sqrt{[l]_{p,q}}},\frac{1}{\sqrt{[n]_{p,q}}}\right).$$

□

4 Boolean sum operators

Let $\mathcal{S}_{p,q}^{ln}$ *and* $\mathcal{T}_{p,q}^{nl}$ *denote the Boolean sums of the* (p,q)*-Bernstein operators* $\mathcal{B}_{p,q}^{y,l}$ *and* $\mathcal{B}_{p,q}^{z,n}$ *then we have:*

$$\begin{aligned}\mathcal{S}_{p,q}^{ln} &:= \mathcal{B}_{p,q}^{y,l} \oplus \mathcal{B}_{p,q}^{z,n} = \mathcal{B}_{p,q}^{y,l} + \mathcal{B}_{p,q}^{z,n} - \mathcal{B}_{p,q}^{y,l}\mathcal{B}_{p,q}^{z,n}, \\ \mathcal{T}_{p,q}^{nl} &:= \mathcal{B}_{p,q}^{z,n} \oplus \mathcal{B}_{p,q}^{y,l} = \mathcal{B}_{p,q}^{z,n} + \mathcal{B}_{p,q}^{y,l} - \mathcal{B}_{p,q}^{z,n}\mathcal{B}_{p,q}^{y,l}.\end{aligned}$$

Now we prove the following results.

Theorem 4.1. *For a real valued function f defined on $\mathcal{T}_h$, we have*

$$\mathcal{S}_{p,q}^{ln} f\Big|_{\partial \mathcal{T}_h} = f\Big|_{\partial \mathcal{T}_h}.$$

Proof. *By definition*

$$\mathcal{S}_{p,q}^{ln} f = (\mathcal{B}_{p,q}^{y,l} + \mathcal{B}_{p,q}^{z,n} - \mathcal{B}_{p,q}^{y,l}\mathcal{B}_{p,q}^{z,n}) f.$$

By the interpolation properties satisfied by $\mathcal{B}_{p,q}^{y,l}$, $\mathcal{B}_{p,q}^{z,n}$ and the properties (i)-(iii) of the operator $\mathcal{P}_{p,q}^{ln}$, we have

$$(\mathcal{S}_{p,q}^{mn} f)(y,0) = (\mathcal{B}_{p,q}^{y,l} f)(y,0) + f(y,0) - (\mathcal{B}_{p,q}^{y,l} f)(y,0) = f(y,0),$$

$$(\mathcal{S}_{p,q}^{ln} f)(0,z) = f(0,z) - (\mathcal{B}_{p,q}^{z,n} f)(0,z) + (\mathcal{B}_{p,q}^{z,n} f)(0,z) = f(0,z),$$

$$(\mathcal{S}_{p,q}^{ln} f)(y,h-y) = f(y,h-y) + f(y,h-y) - f(y,h-y) = f(y,h-y),$$

for all $y, z \in [0,h]$. □

Let $\mathcal{R}_{p,q}^{\mathcal{S}^{ln}} f$ denotes the remainder for approximation formula for the Boolean sum operator $\mathcal{S}_{p,q}^{ln}$ then we have

$$f = \mathcal{S}_{p,q}^{ln} f + \mathcal{R}_{p,q}^{\mathcal{S}^{ln}} f.$$

Theorem 4.2. *If $f \in C(\mathcal{T}_h)$ and $0 < q < p \leq 1$ then*

$$\left|\left(\mathcal{R}_{p,q}^{\mathcal{S}^{ln}} f\right)(y,z)\right| \leq \left(1+\frac{h}{2}\right) M\left(f(.,z); \frac{1}{\sqrt{[l]_{p,q}}}\right) + \left(1+\frac{h}{2}\right) M\left(f(y,.); \frac{1}{\sqrt{[n]_{p,q}}}\right)$$
$$+(1+h)\, M\left(f; \frac{1}{\sqrt{[l]_{p,q}}}, \frac{1}{\sqrt{[n]_{p,q}}}\right), \tag{4.1}$$

for all $(y,z) \in \mathcal{T}_h$.

Proof. *Considering the following relation*

$$f - \mathcal{S}_{p,q}^{ln} f = f - \mathcal{B}_{p,q}^{y,l} f + f - \mathcal{B}_{p,q}^{z,n} f - (f - \mathcal{P}_{p,q}^{ln} f),$$

we have

$$\left|(\mathcal{R}_{p,q}^{\mathcal{S}^{ln}} f)(y,z)\right| \leq \left|(\mathcal{R}_{p,q}^{y,l} f)(y,z)\right| + \left|(\mathcal{R}_{p,q}^{z,n} f)(y,z)\right| + \left|(\mathcal{R}_{p,q}^{\mathcal{P}^{ln}} f)(y,z)\right|.$$

Now using 2.4, 2, 3.1, theorem 4.2 is established. □

Remark 4.3. We can prove the analogous results for the remainders of the Product Operator and the boolean sum operator considering the following approximation formula

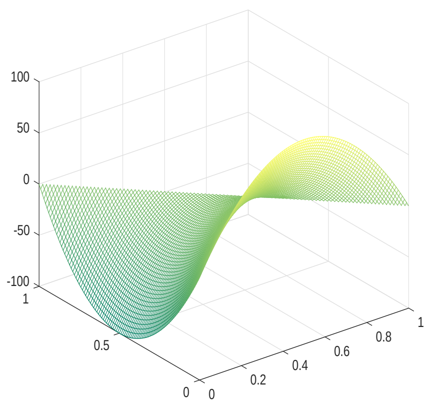
$$f = \mathcal{Q}_{p,q}^{nl} f + \mathcal{R}_{p,q}^{\mathcal{Q}^{nl}} f = \mathcal{B}_{p,q}^{z,n}\mathcal{B}_{p,q}^{y,l} f + \mathcal{R}_{p,q}^{\mathcal{Q}^{nl}} f$$

and

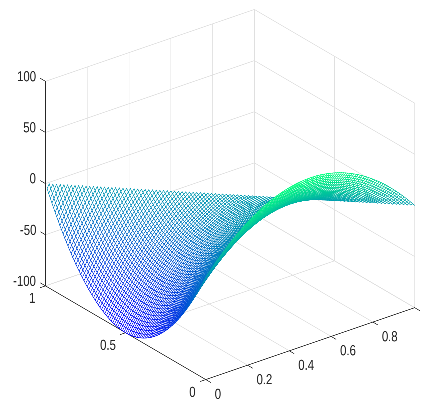
$$f = \mathcal{T}_{p,q}^{nl} f + \mathcal{R}_{p,q}^{\mathcal{T}^{nl}} f = (\mathcal{B}_{p,q}^{z,n} \oplus \mathcal{B}_{p,q}^{y,l}) f + \mathcal{R}_{p,q}^{\mathcal{T}^{nl}} f.$$ □

5 Graphical analysis

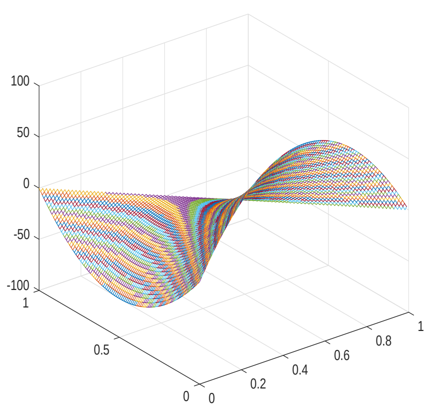
Let $f(y,z) = (-10+20z)^2 - (-10+20y)^2$ be a function on triangle $\mathcal{T}_h$, for analysing the operators graphically. The graphs of operators $\mathcal{B}_{p,q}^{y,l} f$, $\mathcal{B}_{p,q}^{z,n} f$, $\mathcal{P}_{p,q}^{ln} f$ and $\mathcal{S}_{p,q}^{ln} f$ for the values $p = 0.80$ and $q = .70$ are represented in figures 1b, 1c, 1d and 1e respectively. Also the graphs of operators $\mathcal{B}_{p,q}^{y,l} f$, $\mathcal{B}_{p,q}^{z,n} f$, $\mathcal{P}_{p,q}^{ln} f$ and $\mathcal{S}_{p,q}^{ln} f$ for the values $p = 0.90$ and $q = .80$ are represented in figures 2b, 2c, 2d and 2e respectively. All the graphs of operators in figure 1 are plotted for $l = n = 3$ and all the graphs of operators in figure 2 are plotted for $l = n = 4$. One can observe that the graphs of Boolean sum operators are more close to the graph of function and Boolean sum operators interpolate on all the bondary of triangle $\mathcal{T}_h$. Actually all other operators are defined for defining the Boolean sum operators. The parameter p and q will provide more flexibility.



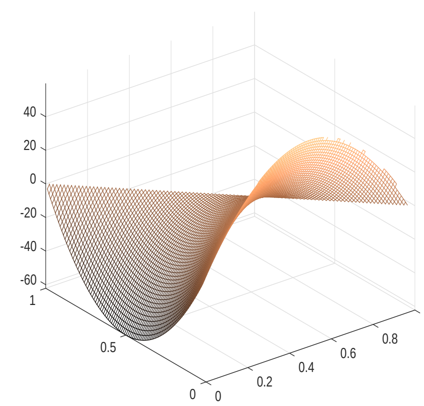
(a) $f = (-10 + 20z)^2 - (-10 + 20y)^2$



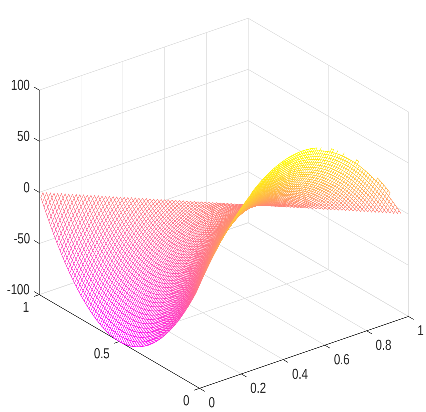
(b) $\mathcal{B}^{y,l}_{p,q} f \ for \ p = 0.80 \ and \ q = 0.70$



(c) $\mathcal{B}^{z,n}_{p,q} f \ for \ p = 0.80 \ and \ q = 0.70$

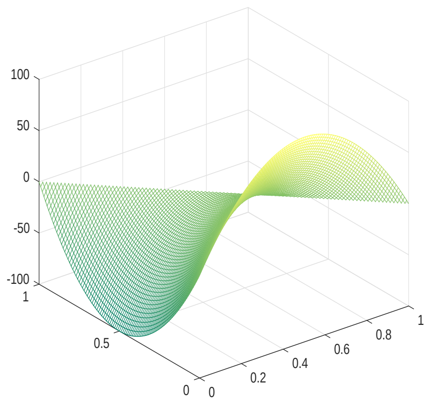


(d) $\mathcal{P}^{ln}_{p,q} f \ for \ p = 0.80 \ and \ q = 0.70$

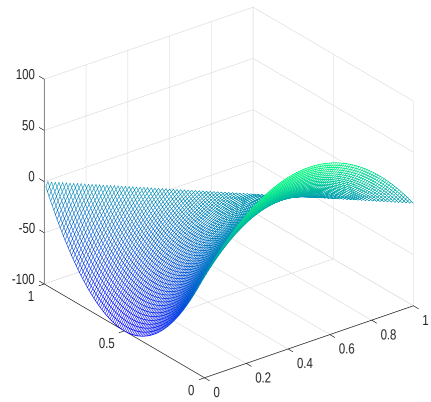


(e) $\mathcal{S}^{ln}_{p,q} f \ for \ p = 0.80 \ and \ q = 0.70$

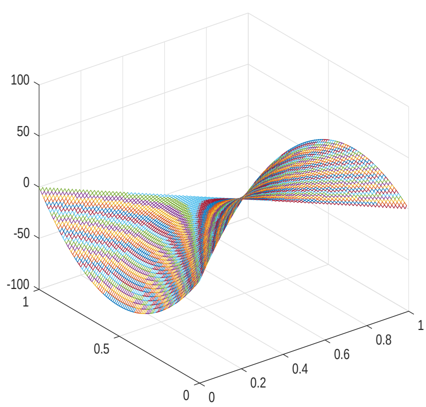
Figure 1: $\mathcal{B}^{y,l}_{p,q} f$, $\mathcal{B}^{z,n}_{p,q} f$, $\mathcal{P}^{ln}_{p,q} f$ and $\mathcal{S}^{ln}_{p,q} f$ approximate the function f for $p = 0.80$, $q = 0.70$ and $l = n = 3$.



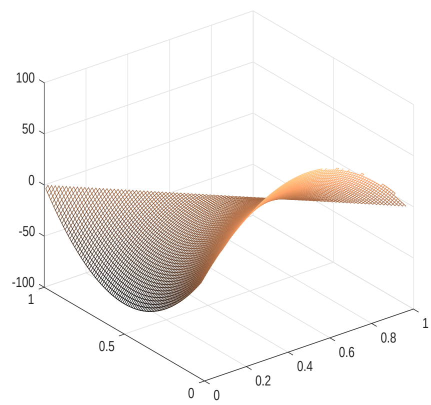
(a) $f = (-10 + 20z)^2 - (-10 + 20y)^2$



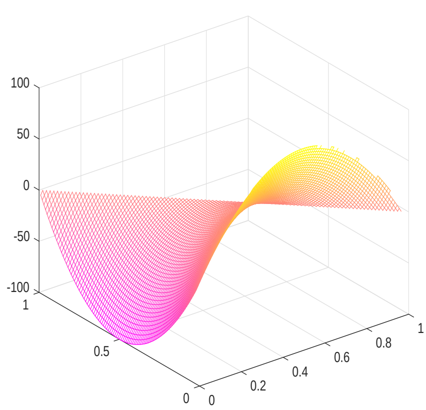
(b) $\mathcal{B}^{y,l}_{p,q} f \ for \ p = 0.90 \ and \ q = 0.80$



(c) $\mathcal{B}^{z,n}_{p,q} f \ for \ p = 0.90 \ and \ q = 0.80$



(d) $\mathcal{P}^{ln}_{p,q} f \ for \ p = 0.90 \ and \ q = 0.80$



(e) $\mathcal{S}^{ln}_{p,q} f \ for \ p = 0.90 \ and \ q = 0.80$

Figure 2: $\mathcal{B}^{y,l}_{p,q} f,\ \mathcal{B}^{z,n}_{p,q} f,\ \mathcal{P}^{ln}_{p,q} f$ and $\mathcal{S}^{ln}_{p,q} f$ approximate the function f for $p = 0.90,\ q = 0.80$ and $l = n = 4$.

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Received: 2023-12-29
Accepted: 2024-02-09