Palestine Journal of Mathematics

Vol 13(4)(2024) , 1299–1313

$*$-EINSTEIN AND $*$-QUASI-YAMABE METRICS ON $\mathsf{N}(\mathsf{k})$-CONTACT METRIC MANIFOLDS

J. Das, K. Halder and A. Bhattacharyya

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53C25, 53C40; Secondary 11F23.

Keywords and phrases: $N(k)$-contact metric manifolds, $*$-Ricci tensor, $*$-Einstein solitons, $*$-quasi-Yamabe solitons, torse-forming vector field, conformal Killing vector field.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Jhantu Das was very thankful to the Council of Scientific and Industrial Research, India (File no: 09/1156(0012)/2018-EMR-I) for their assistance.

Corresponding Author: J. Das

Abstract In the present paper, we introduce the notion of $*$-quasi-Yamabe soliton. Also, discuss the nature of the $*$-Einstein soliton and $*$-quasi-Yamabe soliton on $N(k)$-contact metric manifolds with different types of potential vector fields. It is shown that if a $(2n+1)$-dimensional $N(k)$-contact metric manifold M admits a $*$-Einstein soliton whose potential vector field F is pointwise collinear with the Reeb vector field ζ, then F is a constant multiple of ζ, and the soliton is steady. Moreover, it is shown that, under certain conditions, a $(2n+1)$-dimensional $N(k)$-contact metric manifold endowed with a $*$-quasi-Yamabe soliton becomes a Sasakian manifold, and the soliton reduces to an expanding $*$-Yamabe soliton.Next, we explore an application of the torse-forming vector field on a $N(k)$-contact metric manifold in terms of $*$-Einstein soliton. Finally, an illustrative example of a $N(k)$-contact metric manifold is discussed to verify some of our results.

1 Introduction

In 2002, T. Hamada [13] defined the $*$-Ricci tensor Ric^* on real hypersurfaces of non-flat complex space forms by

$$Ric^*(X_1, X_2) = g(Q^*X_1, X_2) = \frac{1}{2}[\mathrm{trace}(\phi \otimes R(X_1, \phi X_2))]$$

for any $X_1, X_2 \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M and R, Q^*, ϕ are the Riemannian curvature tensor, $*$-Ricci operator, and tensor field of type $(1,1)$, respectively. The $*$-scalar curvature of M is denoted by r^* and is given by $r^* = \mathrm{trace}(Q^*)$. Here $\otimes$ represents the tensor product. A Riemannian manifold (M, g) is called $*$-Ricci flat if its Ric^* vanishes identically. Over the years, several notion related to the $*$-Ricci tensor were initiated. In [15], the authors initiated the notion of $*$-Ricci soliton and widely studied by many authors [10, 11, 19, 20] and others.

In 2016, G. Catino and L. Mazzieri [6] initiated the Einstein soliton as a self-similar solution to the Einstein flow equation given by

$$\frac{\partial}{\partial t}(g(t)) = -2\left(Ric(t) - \frac{r(t)}{2}g(t)\right),$$

where Ric is the $(0,2)$ symmetric Ricci tensor, r is the scalar curvature and g is the Riemannian metric on a smooth manifold M and t is the time variable.

A Riemannian metric g defined on a smooth manifold M is called an Einstein soliton if there exists a real constant τ and a smooth vector field F on M, such that

$$\mathrm{Ric}(X_1, X_2) + \frac{1}{2}(\mathcal{L}_F g)(X_1, X_2) + (\tau - \frac{r}{2})g(X_1, X_2) = 0 \tag{1.1}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$, where $\mathcal{L}_F g$ is the Lie derivative of the metric g along the potential vector field $F \in \mathfrak{X}(M)$. The vector field F on M plays vital roles in determining the nature of the soliton. An Einstein soliton is said to be shrinking if $\tau < 0$, steady if $\tau = 0$, and expanding if $\tau > 0$.

Very Recently, S. Roy et al. [18] initiated the concept of $*$-Einstein soliton which can be defined as

$$\mathrm{Ric}^*(X_1, X_2) + \frac{1}{2}(\mathcal{L}_F g)(X_1, X_2) + (\tau - \frac{r^*}{2})g(X_1, X_2) = 0 \tag{1.2}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$, τ being a real constant, provided Ric^* is a symmetric $*$-Ricci tensor. Moreover, if the potential vector field F is the gradient of a smooth function $f : M \to \mathbb{R}$, then the soliton (1.2) is called a $*$-gradient Einstein soliton. Here $\mathbb{R}$ represents the set of real numbers. Therefore, $*$-gradient Einstein soliton is given by

$$\mathrm{Ric}^*(X_1, X_2) + \nabla\nabla f(X_1, X_2) + (\tau - \frac{r^*}{2})g(X_1, X_2) = 0 \tag{1.3}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$, where ∇ is the Riemannian connection on M.

The notion of Yamabe flow was first initiated by Hamilton [14] to construct Yamabe metrics on compact Riemannian manifold of dimension greater than or equal to three. The Yamabe soliton as a self-similar solution to the Yamabe flow equation given by

$$\frac{\partial}{\partial t}(g(t)) = -r(t)g(t), \qquad g(0) = g_0,$$

where r is the scalar curvature of M.

A Riemannian metric g defined on a complete Riemannian manifold M of dimension $n(\geq 3)$ is called a Yamabe soliton if it obeys

$$\frac{1}{2}(\mathcal{L}_F g)(X_1, X_2) = (r - \lambda)g(X_1, X_2) \tag{1.4}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$. A Yamabe soliton is said to be shrinking if $\lambda > 0$, steady if $\lambda = 0$, and expanding if $\lambda < 0$.

In 2021, S. Roy et al. [19] introduced the notion of $*$-Yamabe soliton as follows:

$$\frac{1}{2}(\mathcal{L}_F g)(X_1, X_2) = (r^* - \lambda)g(X_1, X_2) \tag{1.5}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$, where r^* is the $*$-scalar curvature of M and $\lambda \in \mathbb{R}$.

In 2018, B. Y. Chen and S. Deshmukh [8] extended the notion of Yamabe soliton to quasi-Yamabe soliton. According to [8], the metric g satisfies the equation

$$\frac{1}{2}(\mathcal{L}_F g)(X_1, X_2) = (r - \lambda)g(X_1, X_2) + \sigma F^\flat(X_1)F^\flat(X_2) \tag{1.6}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$, where $\sigma : M \to \mathbb{R}$ is a smooth function, $F^\flat$ is the dual 1-form of F and $\lambda \in \mathbb{R}$.

Motivated by the above studies, we develop the notion of $*$-quasi-Yamabe soliton as:

Definition 1.1. A Riemannian or pseudo-Riemannian manifold (M, g) of dimension greater than or equal to three is said to admit $*$-quasi-Yamabe soliton if it satisfies

$$\frac{1}{2}(\mathcal{L}_F g)(X_1, X_2) = (r^* - \lambda)g(X_1, X_2) + \sigma F^\flat(X_1)F^\flat(X_2) \tag{1.7}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$, where $\sigma : M \to \mathbb{R}$ is a smooth function and $F^\flat$ is the dual 1-form of $F \in \mathfrak{X}(M)$, where r^* is the $*$-scalar curvature of M defined by above. This notion is denoted by (g, F, λ, σ). Furthermore, if $\sigma = 0$, then the $*$-quasi-Yamabe soliton (g, F, λ, σ) reduces to the $*$-Yamabe soliton (g, F, λ).

On the other hand, a nowhere-vanishing smooth vector field ν on a Riemannian or pseudo-Riemannian manifold (M, g) is called torse-forming [24] if it obeys the equation

$$\nabla_{X_1}\nu = \psi X_1 + \theta(X_1)\nu \tag{1.8}$$

for any $X_1 \in \mathfrak{X}(M)$, where ∇ is the Levi-Civita connection on M and $\psi : M \to \mathbb{R}$ is a smooth function and θ is a 1-form. It should be noticed that for special values of the function ψ and the 1-form θ in (1.8), we find the following:

- $\nu \in \mathfrak{X}(M)$ is called concircular [25], if $\theta \equiv 0$ in (1.8),
- $\nu \in \mathfrak{X}(M)$ is called concurrent [21, 26], if $\theta \equiv 0$ and $\psi = 1$ in (1.8),
- $\nu \in \mathfrak{X}(M)$ is called recurrent, if $\psi = 0$ in (1.8),
- $\nu \in \mathfrak{X}(M)$ is called parallel, if $\theta = \psi = 0$ in (1.8).

In [7], B. Y. Chen initiated a new smooth vector field called torqued vector field. If $\nu \in \mathfrak{X}(M)$ satisfies the equation (1.8) with $\theta(\nu) = 0$, then ν is called torqued vector field. In the case of torqued vector field, the function ψ is known as the torqued function on M and the 1-form θ is the torqued form of ν.

The paper is organized as follows: After the brief introduction, we discuss some fundamental definitions related to $N(k)$-contact metric manifolds and curvature formulas, which are contained in Section 2. Section 3 is devoted to the study of $*$-Einstein solitons on $N(k)$-contact metric manifolds with different kinds potential vector fields on M. Section 4 deals with the study of $N(k)$-contact metric manifolds whose metric g satisfies $*$-quasi-Yamabe soliton. In Section 5, we have discussed some properties of potential vector fields on $N(k)$-contact metric manifold admitting $*$-Einstein soliton. Finally, we present an example of three-dimensional $N(k)$-contact metric manifolds and validate some of our results.

2 Preliminaries

A smooth manifold M of dimension $(2\mathsf{n}+1)$ is said to have an almost contact structure if it admits a $(1,1)$ tensor field ϕ, a Reeb vector field ζ, and a 1-form η on M such that

$$\phi^2(\mathsf{X}_1) = -\mathsf{X}_1 + \eta(\mathsf{X}_1)\zeta, \qquad \eta(\zeta) = 1 \tag{2.1}$$

for any $\mathsf{X}_1 \in \mathfrak{X}(\mathsf{M})$. An immediate consequence of the relations (2.1) is that

$$\phi(\zeta) = 0, \quad \eta(\phi\mathsf{X}_1) = 0. \tag{2.2}$$

If M with an almost contact structure (ϕ,ζ,η) admits a Riemannian metric g such that

$$\mathsf{g}(\phi\mathsf{X}_1,\phi\mathsf{X}_2) = \mathsf{g}(\mathsf{X}_1,\mathsf{X}_2) - \eta(\mathsf{X}_1)\eta(\mathsf{X}_2), \quad \mathsf{g}(\mathsf{X}_1,\zeta) = \eta(\mathsf{X}_1) \tag{2.3}$$

for any $\mathsf{X}_1,\mathsf{X}_2 \in \mathfrak{X}(\mathsf{M})$, then $(\phi,\zeta,\eta,\mathsf{g})$ is called an almost contact metric structure and is denoted by $(\mathsf{M},\phi,\zeta,\eta,\mathsf{g})$. From (2.3), it follows that

$$\mathsf{g}(\phi\mathsf{X}_1,\mathsf{X}_2) + \mathsf{g}(\mathsf{X}_1,\phi\mathsf{X}_2) = 0. \tag{2.4}$$

On the other hand, D. E. Blair [1] defined the fundamental 2-form Φ associated with the structure $(\mathsf{M},\phi,\zeta,\eta,\mathsf{g})$ as follows:

$$\Phi(\mathsf{X}_1,\mathsf{X}_2) = \mathsf{g}(\mathsf{X}_1,\phi\mathsf{X}_2)$$

for any $\mathsf{X}_1,\mathsf{X}_2 \in \mathfrak{X}(\mathsf{M})$. Furthermore, an almost contact metric manifold $(\mathsf{M},\phi,\zeta,\eta,\mathsf{g})$ becomes a contact metric manifold if

$$\Phi(\mathsf{X}_1,\mathsf{X}_2) = \mathsf{d}\eta(\mathsf{X}_1,\mathsf{X}_2), \tag{2.5}$$

where d stands for the exterior differentiation. On a contact metric manifold, the $(1,1)$-tensor field h is defined as $\mathsf{h} = \frac{1}{2}\pounds_\zeta\phi$, where $\pounds_\zeta$ is the Lie derivative operator along ζ. The tensor field h is symmetric and satisfies

$$\mathsf{h}\phi + \phi\mathsf{h} = 0, \quad \mathrm{trace}(\mathsf{h}) = \mathrm{trace}(\phi\mathsf{h}) = 0, \quad \mathsf{h}\zeta = 0. \tag{2.6}$$

Also, we have

$$\nabla_{\mathsf{X}_1}\zeta = -\phi\mathsf{X}_1 - \phi\mathsf{h}\mathsf{X}_1 \tag{2.7}$$

for any $\mathsf{X}_1 \in \mathfrak{X}(\mathsf{M})$, where ∇ is the Levi-Civita connection of g on M.

In [5], Blair et al. defined the notion of (k,μ)-nullity distribution on contact metric manifold as follows:

$$\mathsf{N}(\mathsf{k},\mu) = \{\mathsf{X}_3 \in \mathsf{T}(\mathsf{M}) : \mathsf{R}(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3 = (\mathsf{k}\mathsf{I} + \mu\mathsf{h})[\mathsf{g}(\mathsf{X}_2,\mathsf{X}_3)\mathsf{X}_1 - \mathsf{g}(\mathsf{X}_1,\mathsf{X}_3)\mathsf{X}_2]\} \tag{2.8}$$

for any $\mathsf{X}_1,\mathsf{X}_2,\mathsf{X}_3 \in \mathfrak{X}(\mathsf{M})$, where $(\mathsf{k},\mu) \in \mathbb{R}^2$, I is an identity map. If the Reeb vector field ζ

belongs to (k,μ)-nullity distribution $\mathsf{N}(\mathsf{k},\mu)$, then we call a contact metric manifold as (k,μ)-contact metric manifold. Also, the contact metric manifold M is called $\mathsf{N}(\mathsf{k})$-contact metric manifold [22] if it satisfies (2.8) with $\mu=0$. In the case of $\mathsf{N}(\mathsf{k})$-contact metric manifold, the k-nullity distribution $\mathsf{N}(\mathsf{k})$ is given by [22]:

$$\mathsf{N}(\mathsf{k})=\{X_3\in T(M):R(X_1,X_2)X_3=k[g(X_2,X_3)X_1-g(X_1,X_3)X_2]\}$$

for any $X_1,X_2,X_3\in\mathfrak{X}(M)$. Further, if $k=1$, then a $N(k)$-contact metric manifold M is Sasakian. Also, if $k=0$, then the manifold M is locally isometric to $\mathbb{E}^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$ [2, 4].

For a $(2n+1)$-dimensional $N(k)$-contact metric manifold M the following relations also hold [1, 3]:

$$h^2=(k-1)\phi^2, \tag{2.9}$$

$$R(X_1,X_2)\zeta=k\{\eta(X_2)X_1-\eta(X_1)X_2\}, \tag{2.10}$$

$$R(\zeta,X_1)X_2=k\{g(X_1,X_2)\zeta-\eta(X_2)X_1\}, \tag{2.11}$$

$$(\nabla_{X_1}\eta)X_2=g(X_1+hX_1,\phi X_2), \tag{2.12}$$

$$(\nabla_{X_1}\phi)X_2=g(X_1+hX_1,X_2)\zeta-\eta(X_2)(X_1+hX_1), \tag{2.13}$$

$$\begin{aligned}Ric(X_1,X_2)&=2(n-1)\{g(X_1,X_2)+g(hX_1,X_2)\}\\&+2\{nk-n+1\}\eta(X_1)\eta(X_2),\quad n\geq 1\end{aligned} \tag{2.14}$$

for any $X_1,X_2\in\mathfrak{X}(M)$, where R is the curvature tensor of type $(1,3)$, Ric is the symmetric Ricci tensor of type $(0,2)$, Q is the Ricci operator, and it is defined as $Ric(X_1,X_2)=g(QX_1,X_2)$ for any $X_1,X_2\in\mathfrak{X}(M)$. For more details about the $N(k)$-contact metric manifolds, we cite [12, 16, 17] and the references therein.

Definition 2.1. On a $(2n+1)$-dimensional Riemannian manifold (M,g), a smooth vector field F is said to be a conformal Killing vector field on M [27, 28] if it obeys the equation

$$(\pounds_F g)(X_1,X_2)=2\gamma g(X_1,X_2) \tag{2.15}$$

for any $X_1,X_2\in\mathfrak{X}(M)$, where $\gamma:M\to\mathbb{R}$ is a smooth function. The function γ is also known as conformal coefficient. Moreover, the conformal Killing vector field F is called proper if γ is not constant. Also, the conformal Killing vector field F is called homothetic if γ is constant and F is called a proper homothetic vector field if γ is non-zero constant. Finally, the vector field F is called Killing if it satisfies (2.15) with $\gamma=0$.

Definition 2.2. A smooth vector field F on a contact metric manifold M is said to be an infinitesimal contact transformation [23] if it preserves the contact form η, i.e., there exists a smooth function $\rho:M\to\mathbb{R}$ that satisfies

$$(\pounds_F\eta)(X_1)=\rho\eta(X_1) \tag{2.16}$$

for any $X_1\in\mathfrak{X}(M)$, where $\pounds_F\eta$ denotes the Lie derivative of η by F. In particular, if ρ vanishes identically in (2.16), then the smooth vector field F is said to be a strict infinitesimal contact transformation.

3 Main results

This section is devoted to the study of $N(k)$-contact metric manifold admitting a $*$-Einstein soliton. To produce our prime theorems, we need the following Lemma:

Lemma 3.1. *([11]) On a $(2n+1)$-dimensional $N(k)$-contact metric manifold M, the $*$-Ricci tensor Ric^* is given by*

$$Ric^*(X_1, X_2) = -k\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\} \tag{3.1}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Taking $X_1 = X_2 = e_i$ in (3.1), where $\{e_i\}_{i=1}^{2n+1}$ is an orthonormal basis of the tangent space at each point of M and summing over $1 \leq i \leq (2n+1)$ we get

$$r^* = -2nk. \tag{3.2}$$

Theorem 3.2. *Let M be a $(2n+1)$-dimensional $N(k)$-contact metric manifold admitting a $*$-Einstein soliton (g, F, τ) whose non-zero potential vector field F is pointwise collinear with the Reeb vector field ζ. Then,*

(i) *The vector field F is constant multiple of ζ.*
(ii) *The $*$-Einstein soliton (g, F, τ) is steady.*
(iii) *The manifold M is $*$-Ricci flat.*
(iv) *The the manifold M is locally isometric to $\mathbb{E}^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*
(v) *The vector field F is a strict infinitesimal contact transformation.*

Proof. Let the non-zero potential vector field F be pointwise collinear with the Reeb vector field ζ. That is, $F = c\zeta$, where $c : M \to \mathbb{R}$ is a non-zero smooth function. Then from (1.2) and (3.2), we have

$$(\mathcal{L}_{c\zeta} g)(X_1, X_2) + 2Ric^*(X_1, X_2) + 2(\tau + nk)g(X_1, X_2) = 0 \tag{3.3}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Now, from the definition of Lie derivative and from (2.7), we have

$$\begin{aligned}(\mathcal{L}_{c\zeta} g)(X_1, X_2) &= cg(\nabla_{X_1}\zeta, X_2) + cg(X_1, \nabla_{X_2}\zeta) + X_1(c)\eta(X_2) + X_2(c)\eta(X_1) \\ &= 2cg(hX_1, \phi X_2) + X_1(c)\eta(X_2) + X_2(c)\eta(X_1).\end{aligned} \tag{3.4}$$

Therefore, with the help of (3.1) and (3.4), equation (3.3) becomes

$$\begin{aligned}&2cg(hX_1, \phi X_2) + X_1(c)\eta(X_2) + X_2(c)\eta(X_1) - 2k\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\} \\ &+2(\tau + nk)g(X_1, X_2) = 0.\end{aligned} \tag{3.5}$$

Replacing X_2 by ζ in (3.5) yields

$$X_1(c) = -\{2(\tau + nk) + \zeta(c)\}\eta(X_1). \tag{3.6}$$

Again replacing ζ instead of X_1 and X_2 in (3.5) we get

$$\zeta(c) = -(\tau + nk). \tag{3.7}$$

Take a local orthonormal basis $\{e_s\}_{s=1}^{2n+1}$ on a $(2n+1)$-dimensional $N(k)$-contact metric manifold. Then setting $X_1 = X_2 = e_s$ in (3.5) and summing over $1 \leq s \leq (2n+1)$, we obtain

$$\zeta(c) = -(\tau + nk)(2n+1) + 2nk. \tag{3.8}$$

Equating (3.7) with (3.8) we arrive at

$$\tau = -k(n-1). \tag{3.9}$$

Further, with the help of (3.7), the equation (3.6) becomes

$$X_1(c) = -(\tau + nk)\eta(X_1)$$

and hence we have

$$d(c) = -(\tau + nk)\eta, \tag{3.10}$$

where d stands for the exterior derivative operator.

Taking exterior derivative of (3.10) and using Poincare lemma $d^2 \equiv 0$, we obtain $\tau + nk = 0$. Thus we conclude from (3.10) that $d(c) = 0$, which implies that c is constant and therefore F is a constant multiple of ζ. This proves (i).

On substituting $\tau + nk = 0$ in (3.9) leads to $k = 0$, which eventually implies that $\tau = 0$ and hence the soliton is steady. This proves (ii).

Now using $k = 0$ in the identity (3.1) we get $Ric^* = 0$ and hence the manifold M is ∗-Ricci flat, and this proves (iii).

Furthermore, the manifold M is locally isometric to $\mathbb{E}^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ as $k = 0$ and hence the part (iv) of the theorem 3.2 is proved.

Again replacing X_2 by ζ in (3.3) and using the equation (3.1) and the fact that $\tau + nk = 0$, we obtain

$$(\pounds_F g)(X_1, \zeta) = 0$$

and hence

$$(\pounds_F \eta)(X_1) = g(X_1, \pounds_F \zeta) \tag{3.11}$$

for any $X_1 \in \mathfrak{X}(M)$.

Since $F = c\zeta$ and c is a constant it can be easily evaluated that $\pounds_F \zeta = 0$. Thus from (3.11) finally we have $(\pounds_F \eta)(X_1) = 0$ for any $X_1 \in \mathfrak{X}(M)$. Hence from definition 2.2, it follows that the potential vector field F is a strict infinitesimal contact transformation. This result ends the proof of Theorem 3.2. □

Theorem 3.3. *Let M be a $(2n+1)$-dimensional $N(k)$-contact metric manifold admitting a ∗-Einstein soliton (g, F, τ). If the potential vector field F is orthogonal to the Reeb vector field ζ, then the ∗-Einstein soliton is steady if and only if the manifold M is locally isometric to $\mathbb{E}^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

Proof. Let (g, F, τ) be a ∗-Einstein soliton on a $(2n+1)$-dimensional $N(k)$-contact metric manifold M, where F is orthogonal vector field and orthogonal to ζ. Then from (1.2) and (3.1), we have

$$(\pounds_F g)(X_1, X_2) - 2k\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\} + 2(\tau + nk)g(X_1, X_2) = 0 \tag{3.12}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Replacing ζ instead of X_1 and X_2 in (3.12) and using (2.1), we get

$$(\pounds_F g)(\zeta,\zeta) + 2(\tau + nk) = 0. \tag{3.13}$$

On the other hand, as $\nabla_\zeta \zeta = 0$ we deduce that

$$(\pounds_F g)(\zeta,\zeta) = 2g(\nabla_\zeta F, \zeta) = 2\nabla_\zeta(g(F,\zeta)) = 0. \tag{3.14}$$

With the help of (3.14) and from (3.13) we arrive at

$$\tau = -nk. \tag{3.15}$$

Hence the proof. □

In view of (3.15), we can state the following:

Corollary 3.4. *Let* M *be a* $(2n+1)$*-dimensional* $N(k)$*-contact metric manifold admitting a* $*$*-Einstein soliton* (g, F, τ) *whose the potential vector field* F *is orthogonal to the Reeb vector field* ζ*. If* $k = 1$*, i.e., the manifold* M *is Sasakian, then the* $*$*-Einstein soliton is shrinking.*

Theorem 3.5. *Let* M *be a* $(2n+1)$*-dimensional* $N(k)$*-contact metric manifold admitting a* $*$*-Einstein soliton* (g, F, τ)*. If the potential vector field* F *is an infinitesimal contact transformation, then the manifold* M *is locally isometric to* $\mathbb{E}^{n+1}(0) \times S^n(4)$ *for* $n > 1$ *and flat for* $n = 1$*. Furthermore, the* $*$*-Einstein soliton is steady and the potential vector field* F *is Killing.*

Proof. Let (g, F, τ) be a $*$-Einstein soliton on a $(2n+1)$-dimensional $N(k)$-contact metric manifold M, where F is an infinitesimal contact transformation. Then from (1.2) and (3.1), we have

$$(\pounds_F g)(X_1, X_2) - 2k\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\} + 2(\tau + nk)g(X_1, X_2) = 0 \tag{3.16}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Replacing X_1 and X_2 by ζ in (3.16) and using (2.1), we get

$$g(\pounds_F \zeta, \zeta) = (\tau + nk). \tag{3.17}$$

Again replacing X_2 by ζ in (3.16), then recalling (2.16) infers that

$$\pounds_F \zeta = (\rho + 2\tau + 2nk)\zeta. \tag{3.18}$$

Feeding (3.18) in (3.17) we have

$$\rho = -(\tau + nk). \tag{3.19}$$

This implies that ρ is constant.

On the other hand, as $\pounds_F$ and d commutes, from (2.16) we deduce that

$$\pounds_F d\eta = d(\rho\eta) = (d\rho) \wedge \eta + \rho(d\eta)$$

and hence

$$\pounds_F d\eta = \rho(d\eta). \tag{3.20}$$

Since a volume form, ω is closed, so, from Cartan's formula we have

$$\pounds_F\omega = \mathrm{div}(F)\omega. \tag{3.21}$$

Taking the Lie-derivative of the volume form $\omega = \eta \wedge (d\eta)^n$ along F and using (3.20) and (3.21) we obtain

$$\mathrm{div}(F) = (n+1)\rho. \tag{3.22}$$

Now integrating the forgoing equation over M and then applying Divergence theorem, we find

$$\rho = 0$$

and hence

$$\mathrm{div}(F) = 0. \tag{3.23}$$

Recalling (3.16), (3.19) and the fact that $\rho = 0$, one can easily obtain

$$(\pounds_F g)(X_1, X_2) - 2k\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\} = 0 \tag{3.24}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Taking contraction of (3.24) over X_1 and X_2 we get

$$\mathrm{div}(F) = 2nk. \tag{3.25}$$

This eventually implies that $k = 0$ as $\mathrm{div}(F) = 0$. Therefore, the manifold M is locally isometric to $\mathbb{E}^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$. Also, from (3.24), we have $(\pounds_F g)(X_1, X_2) = 0$ for any $X_1, X_2 \in \mathfrak{X}(M)$. This shows that F is Killing. On taking $\rho = k = 0$ in (3.19), we obtain $\tau = 0$. Thus, the $*$-Einstein soliton is steady. This is the desired result. □

Theorem 3.6. *Let M be a $(2n+1)$-dimensional $N(k)$-contact metric manifold admitting a $*$-Einstein soliton (g, F, τ). If the potential vector field F is the gradient of a smooth function ψ defined on M, then the Laplacian equation satisfied by ψ is*

$$\Delta(\psi) = 2nk - (\tau + nk)(2n+1).$$

Proof. Let (g, F, τ) be a $*$-Einstein soliton on a $(2n+1)$-dimensional $N(k)$-contact metric manifold M, where $F = \mathrm{grad}(\psi)$. Then from (1.2) and (3.1), we have

$$(\pounds_F g)(X_1, X_2) - 2k\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\} + 2(\tau + nk)g(X_1, X_2) = 0 \tag{3.26}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Now consider an orthonormal basis $\{e_s\}_{s=1}^{2n+1}$ on a $(2n+1)$-dimensional $N(k)$-contact metric manifold. Then setting $X_1 = X_2 = e_s$ in (3.26) and summing over $1 \leq s \leq (2n+1)$ we get

$$\mathrm{div}(F) - 2nk + (\tau + nk)(2n+1) = 0. \tag{3.27}$$

Since $F = \mathrm{grad}(\psi)$, equation (3.27) becomes

$$\Delta(\psi) = 2nk - (\tau + nk)(2n+1), \tag{3.28}$$

where Δ is the Laplacian operator. Hence the proof. □

Also, if we consider F as solenoidal i.e., $\operatorname{div}(\mathsf{F}) = 0$, then from (3.27) we have

$$\tau = \frac{nk(1-2n)}{2n+1}. \tag{3.29}$$

Again if $\tau = \frac{nk(1-2n)}{2n+1}$, then it follows from (3.27) that $\operatorname{div}(\mathsf{F}) = 0$, which means F is solenoidal. This leads to the following:

Theorem 3.7. *If a $(2n+1)$-dimensional $N(k)$-contact metric manifold M admits a $*$-Einstein soliton (g, F, τ), then the potential vector field F is solenoidal if and only if $\tau = \frac{nk(1-2n)}{2n+1}$.*

Theorem 3.8. *Let M be a $(2n+1)$-dimensional $N(k)$-contact metric manifold admitting a $*$-Einstein soliton (g, F, τ), where F is a conformal Killing vector field on M. Then F is a homothetic vector field on M, and the manifold M is $*$-Ricci flat. Furthermore, F is*
(i) *proper homothetic vector field if $\tau \neq 0$.*
(ii) *Killing vector field if and only if the $*$-Einstein soliton (g, F, τ) is steady.*

Proof. Let (g, F, τ) be a $*$-Einstein soliton on a $(2n+1)$-dimensional $N(k)$-contact metric manifold M, where F is a conformal Killing vector field on M. Then from (1.2), (3.1) and (2.15) we derive

$$\{\gamma - k + \tau + nk\}g(X_1, X_2) + k\eta(X_1)\eta(X_2) = 0 \tag{3.30}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Now replacing X_2 by ζ in (3.30) and using (2.1) we get

$$\{\gamma + \tau + nk\}\eta(X_2) = 0. \tag{3.31}$$

Since the equation (3.31) holds for all $X_2 \in \mathfrak{X}(M)$, we have

$$\gamma = -(\tau + nk). \tag{3.32}$$

Putting this value of γ in (3.30) we get

$$k\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\} = 0 \tag{3.33}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

In view of (2.3) and (2.5), the equation (3.33) becomes

$$kd\eta(\phi X_1, X_2) = 0. \tag{3.34}$$

This implies that $k = 0$ as $d\eta \neq 0$. So, from (3.32), we have $\gamma = -\tau$ and hence $\gamma = constant$. Also, from (3.1) it follows that $Ric^* = 0$. This reflects that the manifold M is $*$-Ricci flat. Again in the sense of the definition (2.1), the vector field F is proper homothetic vector field if $\tau \neq 0$. Moreover, F is Killing vector field if and only if $\tau = 0$. This is the desired result. □

4 ∗-Quasi-Yamabe soliton on $\mathsf{N}(\mathsf{k})$-contact metric manifold

Theorem 4.1. *If a* $(2n+1)$*-dimensional* $N(k)$*-contact metric manifold* M *admits a* ∗*-quasi-Yamabe soliton* (g, F, λ, σ) *with the non-zero potential vector field* F *being pointwise collinear with the Reeb vector field* ζ*, then the following are satisfied:*

(i) *The* ∗*-quasi-Yamabe soliton reduces to the* ∗*-Yamabe soliton* (g, F, λ).
(ii) *The vector field* F *becomes a constant multiple of* ζ.
(iii) *The vector field* F *is a strict infinitesimal contact transformation.*
(iv) *The manifold* M *becomes a Sasakian manifold.*
(v) *The* ∗*-quasi-Yamabe soliton* (g, F, λ, σ) *is expanding.*

Proof. Let $F = \beta\zeta$, where $\beta : M \to \mathbb{R}$ is a non-zero smooth function. Then, from (1.7) and (3.2), we have

$$\frac{1}{2}(\pounds_{\beta\zeta} g)(X_1, X_2) = (r^* - \lambda)g(X_1, X_2) + \sigma\beta^2\eta(X_1)\eta(X_2) \tag{4.1}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Now,

$$(\pounds_{\beta\zeta} g)(X_1, X_2) = \beta\{g(\nabla_{X_1}\zeta, X_2) + g(X_1, \nabla_{X_2}\zeta)\} + X_1(\beta)\eta(X_2) + X_2(\beta)\eta(X_1)$$

which, in view of (2.7) and (2.4) becomes

$$(\pounds_{\beta\zeta} g)(X_1, X_2) = 2\beta g(hX_1, \phi X_2) + X_1(\beta)\eta(X_2) + X_2(\beta)\eta(X_1). \tag{4.2}$$

Therefore, from (4.1) and (4.2) we get

$$2\beta g(hX_1, \phi X_2) + X_1(\beta)\eta(X_2) + X_2(\beta)\eta(X_1) = 2(r^* - \lambda)g(X_1, X_2) + 2\sigma\beta^2\eta(X_1)\eta(X_2). \tag{4.3}$$

Replacing X_2 by ζ in (4.3) yields

$$X_1(\beta) = \{2(r^* - \lambda) + 2\sigma\beta^2 - \zeta(\beta)\}\eta(X_1). \tag{4.4}$$

Again replacing ζ instead of X_2 in (4.4) we get

$$\zeta(\beta) = (r^* - \lambda) + \sigma\beta^2. \tag{4.5}$$

Take a local orthonormal basis $\{e_s\}_{s=1}^{2n+1}$ on a $(2n+1)$-dimensional $N(k)$-contact metric manifold. Then setting $X_1 = X_2 = e_s$ in (4.3) and summing over $1 \le s \le (2n+1)$, we obtain

$$\zeta(\beta) = (r^* - \lambda)(2n+1) + \sigma\beta^2. \tag{4.6}$$

Comparing (4.5) with (4.6) we arrive at

$$r^* = \lambda. \tag{4.7}$$

Further, in view of (4.7) and (4.5), the equation (4.4) becomes

$$X_1(\beta) = \sigma\beta^2\eta(X_1)$$

and hence

$$\mathrm{d}(\beta) = \sigma\beta^2\eta. \tag{4.8}$$

Taking exterior derivative of (4.8) and using Poincare lemma $\mathrm{d}^2 \equiv 0$, we obtain $\sigma\beta^2 = 0$, which implies that $\sigma = 0$ and hence the $*$-quasi-Yamabe soliton reduces to the $*$-Yamabe soliton (g, F, λ). This proves (i).

Using $\sigma = 0$ in (4.8), we obtain $\mathrm{d}(\beta) = 0$, which implies that β is constant and therefore F is a constant multiple of ζ. This proves (ii).

On putting $r^* = \lambda$ in (4.1) and the fact that $\sigma = 0$ leads to

$$(\mathcal{L}_F g)(X_1, X_2) = 0. \tag{4.9}$$

Now replacing X_2 by ζ in (4.9), we obtain

$$(\mathcal{L}_F g)(X_1, \zeta) = 0$$

and hence

$$(\mathcal{L}_F \eta)(X_1) = g(X_1, \mathcal{L}_F\zeta) \tag{4.10}$$

for any $X_1 \in \mathfrak{X}(M)$.

Since the potential vector field F is a constant multiple of ζ, so, we have $\mathcal{L}_F\zeta = 0$. Thus from (4.10) we have $(\mathcal{L}_F\eta)(X_1) = 0$ for any $X_1 \in \mathfrak{X}(M)$. Hence, from definition 2.2, it follows that the potential vector field F is a strict infinitesimal contact transformation. This proves (iii) of Theorem 4.1.

From (4.2), (4.9) and the fact that β = non-zero constant, one has

$$g(hX_1, \phi X_2) = 0. \tag{4.11}$$

Replacing hX_1 instead of X_1 in (4.11) and making use of (2.9) and (2.1), we lead

$$(k-1)g(X_1, \phi X_2) = 0,$$

which in view of (2.5) becomes

$$(k-1)\mathrm{d}\eta(X_1, \phi X_2) = 0. \tag{4.12}$$

Since $\mathrm{d}\eta \neq 0$, we immediately have $k = 1$ and hence the manifold M is Sasakian. This prove (v).

Finally making use of (3.2) in (4.7) gives

$$\lambda = -2nk. \tag{4.13}$$

Using $k = 1$ in (4.13) yields $\lambda = -2n$, and therefore the $*$-quasi-Yamabe soliton is expanding. This result ends the proof of Theorem 4.1. □

5 Application of torse-forming vector field on $N(k)$-contact metric manifold admitting $*$-Einstein soliton

Let (g, ν, τ) be a $*$-Einstein soliton on a $(2n+1)$-dimensional $N(k)$-contact metric manifold M, where ν is a torse-forming vector field on M. Then, we have from (1.2) and (3.1) that

$$(\mathcal{L}_\nu g)(X_1, X_2) - 2k\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\} + 2(\tau + nk)g(X_1, X_2) = 0 \tag{5.1}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$, where $\mathcal{L}_\nu$ denotes the Lie derivative in the direction of ν.

Also, from (1.8) we derive that

$$(\mathcal{L}_\nu g)(X_1, X_2) = 2\psi g(X_1, X_2) + \theta(X_1)g(\nu, X_2) + \theta(X_2)g(\nu, X_1) \tag{5.2}$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Therefore, from (5.1) and (5.2), we have

$$\begin{aligned} 2\{(\psi + \tau + nk - k)g(X_1, X_2) + k\eta(X_1)\eta(X_2)\} + \theta(X_1)g(\nu, X_2) \\ +\theta(X_2)g(\nu, X_1) = 0. \end{aligned} \tag{5.3}$$

Contracting the forgoing equation over X_1 and X_2, we infer

$$(\psi + \tau + nk - k)(2n + 1) + k + \theta(\nu) = 0$$

and therefore

$$\tau = k(1 - n) - \psi - \frac{k + \theta(\nu)}{2n + 1}. \tag{5.4}$$

This leads to the following:

Theorem 5.1. *Let* M *be a* $(2n + 1)$*-dimensional* $N(k)$*-contact metric manifold admitting a ∗-Einstein soliton* (g, ν, τ)*. If the potential vector field* ν *is a torse-forming vector field on* M*, then* $\tau = k(1 - n) - \psi - \frac{k+\theta(\nu)}{2n+1}$ *and the ∗-Einstein soliton* (g, ν, τ) *is expanding, steady, or shrinking according to whether* $k(1 - n) - \psi - \frac{k+\theta(\nu)}{2n+1} \gtreqless 0$.

Also, it is obvious from (5.4) that

$$\tau = \begin{cases} k(1-n) - \psi - \frac{k}{2n+1}, & \text{if} \quad \theta \equiv 0 \\ k(1-n) - 1 - \frac{k}{2n+1}, & \text{if} \quad \theta \equiv 0 \text{ and } \psi = 1 \\ k(1-n) - \frac{k+\theta(\nu)}{2n+1}, & \text{if} \quad \psi = 0 \\ k(1-n) - \frac{k}{2n+1}, & \text{if} \quad \theta = \psi = 0 \\ k(1-n) - \psi - \frac{k}{2n+1}, & \text{if} \quad \theta(\nu) = 0. \end{cases}$$

This leads to the following:

Corollary 5.2. *Let* M *be a* $(2n + 1)$*-dimensional* $N(k)$*-contact metric manifold admitting a ∗-Einstein soliton* (g, ν, τ)*, where* ν *is a torse-forming vector field defined on* M*. Then if* ν *is*

(i) *concircular, then* $\tau = k(1 - n) - \psi - \frac{k}{2n+1}$ *and the ∗-Einstein soliton* (g, ν, τ) *is expanding, steady, or shrinking according as* $k(1 - n) - \psi - \frac{k}{2n+1} \gtreqless 0$.
(ii) *concurrent, then* $\tau = k(1 - n) - 1 - \frac{k}{2n+1}$ *and the ∗-Einstein soliton* (g, ν, τ) *is expanding, steady, or shrinking according as* $k(1 - n) - 1 - \frac{k}{2n+1} \gtreqless 0$.
(iii) *recurrent, then* $\tau = k(1 - n) - \frac{k+\theta(\nu)}{2n+1}$ *and the ∗-Einstein soliton* (g, ν, τ) *is expanding, steady,*

or shrinking according as $\mathsf{k}(1-\mathsf{n}) - \frac{\mathsf{k}+\theta(\nu)}{2\mathsf{n}+1} \gtreqless 0$.
(iv) *parallel, then* $\tau = \mathsf{k}(1-\mathsf{n}) - \frac{\mathsf{k}}{2\mathsf{n}+1}$ *and the* $*$*-Einstein soliton* (g, ν, τ) *is expanding, steady, or shrinking according as* $\mathsf{k}(1-\mathsf{n}) - \frac{\mathsf{k}}{2\mathsf{n}+1} \gtreqless 0$.
(v) *torqued, then* $\tau = \mathsf{k}(1-\mathsf{n}) - \psi - \frac{\mathsf{k}}{2\mathsf{n}+1}$ *and the* $*$*-Einstein soliton* (g, ν, τ) *is expanding, steady, or shrinking according as* $\mathsf{k}(1-\mathsf{n}) - \psi - \frac{\mathsf{k}}{2\mathsf{n}+1} \gtreqless 0$.

6 Example

Here we give an example of a $*$-Einstein soliton on a 3-dimensional $\mathsf{N}(1-\alpha^2)$-contact metric manifold M as constructed in [9]. In this example we can calculate the components of $*$-Ricci tensor as follows

$$\mathsf{Ric}^*(\mathsf{e}_1, \mathsf{e}_1) = 0, \qquad \mathsf{Ric}^*(\mathsf{e}_2, \mathsf{e}_2) = \mathsf{Ric}^*(\mathsf{e}_3, \mathsf{e}_3) = -(1-\alpha^2).$$

Therefore in view of the above values of $*$-Ricci tensor, we have

$$\mathsf{r}^* = -2(1-\alpha^2).$$

Also we can easily calculate Lie derivative of g along e_1 as

$$(\pounds_{\mathsf{e}_1}\mathsf{g})(\mathsf{X}_1, \mathsf{X}_2) = 0 \qquad \forall\ \mathsf{X}_1, \mathsf{X}_2 \in \{\mathsf{e}_i : i = 1, 2, 3\}.$$

For $\alpha = 1$, the curvature tensor R vanishes and also $\mathsf{Ric}^* = 0$. Now tracing the equation (1.2) we get $\tau = 0$. Thus for this value of τ the data $(\mathsf{g}, \mathsf{e}_1, \tau)$ defines a $*$-Einstein soliton on this 3-dimensional $\mathsf{N}(0)$-contact metric manifold M. Moreover we can easily see that e_1 is a Killing vector field and hence the Theorem 3.5 and also the Theorem 3.8 are verified.

Again if $\mathsf{e}_1 = \zeta$, then from (1.7) and considering $\alpha = 1$ we obtain

$$\lambda = \sigma = 0.$$

Hence for this values of λ and σ the data $(\mathsf{g}, \mathsf{e}_1, \lambda, \sigma)$ defines a $*$-quasi-Yamabe soliton on this 3-dimensional $\mathsf{N}(0)$-contact metric manifold M and $*$-quasi-Yamabe soliton $(\mathsf{g}, \mathsf{e}_1, \lambda, \sigma)$ reduces to a Yamabe soliton as $\sigma = 0$. Hence the Theorem 4.1 is verified.

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Author information

J. Das, Department of Mathematics, Sidho-Kanho-Birsha University, India.
E-mail: dasjhantu54@gmail.com

K. Halder, Department of Mathematics, Sidho-Kanho-Birsha University, India.
E-mail: drkalyanhalder@gmail.com

A. Bhattacharyya, Department of Mathematics, Jadavpur University, India.
E-mail: bhattachar1968@yahoo.co.in

Received: 2023-12-31
Accepted: 2024-04-15