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ON THE THEORETICAL INVESTIGATION OF MAXIMUM AND MINIMUM VALENCY RADIO LABELINGS OF THE POLY-SILICATE AND POLY-OXIDE NETWORKS

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Abstract Let δ and Δ be the minimum and maximum valency (degree) of a simple connected graph $G(V, E)$. A mapping $\sigma : V(G) \to \{0, 1, 2 \dots\}$ is called a *minimum valency radio labeling,* if it satisfies the inequality $d(x, z) + |\sigma(x) - \sigma(z)| \geq 1 + \delta(G)$ for all $x, z \in V(G)$. The span of a minimum valency radio labeling σ is the largest number in the range of σ and it's denoted by $r_\delta(\sigma)$. The *minimum valency radio number of G*, denoted by $r_\delta(G)$, is the minimum span taken over all minimum valency radio labelings σ of G. If we replace Δ instead of δ in the definition of minimum valency radio labeling, the labeling technique formulated is called the maximum *valency radio number of G*, denoted by $r_\Delta(G)$. In this paper, we have investigated the maximum and minimum valency radio numbers of certain chemical structures such as poly-silicates and poly-oxides.

1 Introduction

Today, due to various unexpected situations in our world, the survival of life has completely reformed and raised affording to the area of telecommunication. By the progressive usage of the electromagnetic waves, especially the radio waves, today the radio communication have reached the nook and corner of the Earth. Resulting the discovery of electromagnetic waves by Hertz in1887 [1], Griggs [2] exploit the notion of distance and frequency difference between transmitters, presented a labeling technique termed distance two labeling. This method is used to make best use of the bandwidth for Amplitude modulation (AM) radio stations with null channel co-channel interference. Numerous research papers were published by different research studies in this area; see [3, 4, 5, 6, 7]. Further, in 2001, Chartrand et.al [8] was motivated by distance two labeling technique and anticipated a labeling technique called radio labeling which is used to maximize the number of channels in a specified geographical area for Frequency Modulation (FM) radio stations. For the previous two decades, plentiful results were published for both graphs and interconnection networks; see [9, 10, 11, 12, 13].

More recently, Yenoke [14] was motivated by the applications of maximum and minimum valency in traffic congestion [15] and broadcasting problems [16], introduced a new labeling technique called maximum and minimum valency radio labelings. The formal graph theoretical definition is as follows: Let δ and Δ be the minimum and maximum valency (degree) of a simple connected graph $G(V, E)$. A mapping $\sigma : V(G) \to \{0, 1, 2 \dots\}$ is called a minimum valency radio labeling, if it satisfies the inequality $d(x, z) + |\sigma(x) - \sigma(z)| \geq 1 + \delta(G)$ for all $x, z \in V(G)$. The span of a minimum valency radio labeling σ is the largest number in the range of σ and it's denoted by $r_\delta(\sigma)$. The *minimum valency radio number of G*, denoted by $r_\delta(G)$, is the minimum span taken over all minimum valency radio labelings σ of G. If we replace Δ instead of δ in the definition of minimum valency radio labeling, the problem obtained is called the maximum *valency radio number of G*, denoted by $r_\Delta(G)$. This problem highlights the importance of finding the maximum number of channels in a stipulated bandwidth based on the parameters (i) frequency difference between the transmitters (ii) the distance between the transmitters (iii) the maximum valency and minimum valency of communication networks. Compared to the radio number problem, instead of concentrating on the diameter of the geographical area, here we are focusing on the congestion of the graph [17].

Yenoke [14] proved the following general results for any simple connected graph. (i) $\psi(G) \leq r_\delta(G) \leq r_\Delta(G)$, where $\psi(G)$is the chromatic number of a connected graph G. (ii) $r_\Delta(G) = r_\delta(G)$, whenever G is a r-regular graph. (iii) If we add any pendant edge e to the graph G, then $r_\delta(G+e) = \psi(G+e)$. In addition, for (i) the path p_n $(n > 2)$, $r_\Delta(G) = \lambda_{2,1}(G)$ and $r_\delta(G) = \psi(G)$, where $\lambda_{2,1}(G)$is the L(2,1) labeling number (ii) the complete binary tree $BT(k)$, $r_\Delta(G) = \lambda_{3,2,1}(G)$ and $r_\delta(G) = \psi(G)$, where $\lambda_{3,2,1}(G)$ is radio-3-chromatic number (iii) the complete bi-partite graph $K_{m,n}$, $r_\Delta(K_{m,n}) = (m-1)(n+m-1)+1$ and $r_\delta(K_{m,n}) = (n-1)(n+m-1)+1$ (iv) the wheel graph W_{n+1}, $r_\Delta(W_{n+1}) = n(n-1)+1$ and $r_\delta(W_{n+1}) = n+2$ (v) the fan graph $F_n = P_n + K_1$, $r_\Delta(F_n) = n(n-1)+1$ and $r_\delta(G) = n+1$ (vi) the windmill graph K_n^m $(m > 1)$ is $r_\Delta(K_n^m) \leq m(m(n-1)-1)(n-1)+1$ and $r_\delta(K_n^m) \leq m(n-2)(n-1)+1$. Further, he obtained the upper bound for the maximum valency radio number of complete binary tree as 17.

In this research work, we have studied the properties of few poly-silicate and poly-oxide silicates. Next, we have determined the maximum and minimum valency radio labelings separately for certain chemical structures such as rectangular silicate and oxide networks, triangulane oxide networks, triangulane silicate networks, $m \times m$ silicate and oxide sheets. Finally, we have concluded this paper with a list of chemical structures yet to be found.

2 Poly-oxide and Poly-Silicate structures

There is an extensive discussion of the chemical structures such as poly-oxide and poly-silicate in [18, 19, 20]. Essentially, all silicates are made up of SiO4 tetrahedra and are created when sand and metal oxides or metal carbonates combine together. In chemistry, the silicon ion is represented by the centre node of the SiO4 (aluminium calcium oxide silicate) tetrahedron, while the oxygen ions are represented by the corner nodes. We designated the SiO4 tetrahedron's central vertex as the silicon node and its corner vertices as oxygen nodes in graph theory terminology. Different silicate structures are created depending on how these SiO4 tetrahedrons are arranged. Pyro silicates, orthosilicates, chain silicates, sheet silicates, and cyclic silicates are the terms given to them.

2.1 Rectangular Silicate and Oxide Networks

A rectangular silicate[20] denoted by $RSL(m,l)$ is constructed in such a way that it consists of m $(m > 2)$ row lines and l number of edges in a row line. It has $m(3l+1) - \left(\left\lceil\left(\frac{l-1}{2}\right)\left\lfloor\frac{m}{2}\right\rfloor + \left(\frac{l+1}{2}\right)\left\lfloor\frac{m-1}{2}\right\rfloor\right\rceil\right)$ vertices and $6ml$ edges. Also, the rectangular oxide [19] of m $(m > 2)$ row lines and l number of edges in a row line is denoted by $ROX(m,l)$. It has $m(2l+1) - \left(\left\lceil\left(\frac{l-1}{2}\right)\left\lfloor\frac{m}{2}\right\rfloor + \left(\frac{l+1}{2}\right)\left\lfloor\frac{m-1}{2}\right\rfloor\right\rceil\right)$ vertices and $3ml$ edges. The maximum and minimum degrees of $RSL(m,l)$ and $ROX(m,l)$ are $(6, 3)$ and $(4, 2)$ respectively. It is demonstrated in figure 2.

2.2 Triangulane Silicate and Oxide Networks

A triangulane oxide network [20] of m row lines are designated by $TOX(m)$ and is built so that the number of vertices in the k^{th} line from top to bottom is $2k+2$, $1 \leq k \leq m$. It has the same maximum and minimum degrees as in $ROX(m,l)$. The number of vertices and edges of $TOX(m)$ are $\frac{3m^2+9m+2}{2}$ and $3m^2+6m$ respectively. It can be seen in figure 1. Similarly, the network constructed using the m row lines for silicate structure is called triangulane silicate networks. It is denoted by $TSL(m)$. See figure 3.

2.3 An $\mathbf{m} \times \mathbf{m}$ Silicate and Oxide sheets

An $m \times m$ oxide sheets are formed by two copies of triangulane oxide networks $TOX(m)$ by the fusion of the bottommost $m+1$ oxygen nodes. It is denoted by $(OX)_{m\times m}$. The number of vertices and edges of $(OX)_{m\times m}$ are $3m^2+8m+1$ and $6(m^2+2m)$ respectively. In the same manner, if we replace triangulane oxide by triangulane silicate, we can construct the $m \times m$ silicate sheet and it's denoted by $(SL)_{m\times m}$. It is visible in figure 4.

2.4 Silicate and Oxide Networks

Silicate networks have been created in different ways in the literature [19, 20]. A honeycomb network of dimension m yields one approach as follows: The honeycomb network is divided into its individual edges, the oxide ions are added to the new vertices, $6m$ new pendant edges are connected, one at a time, to the 2-degree silicon ions of $HC(m)$, and lastly oxygen ions are added to the pendent vertices. Finally, each silicon ion joins forces with three nearby oxygen ions to produce $SiO4$. The created network is known as the silicate network of dimension m and is represented by the symbol $SL\ (m)$. The $SL(m)$ network has $36m^2$ edges and $15m^2+3m$ vertices, respectively. The resulting network, which is known as an oxide network of dimension m and is denoted by $OX(m)$, is generated if we remove all the silicon vertices from the silicate network of dimension m. Vertex set and edge set of $OX(m)$ have cardinality values of $9m^2+3m$ and $18m^2$ respectively.

Result 1. The maximum and minimum valency (degree) of the silicate network of dimension m are $\Delta\left(SL(m)\right)=6$ and $\delta((SL(m))=3$ respectively.

Result 2. The oxide network of dimension m has a maximum valency (degree) of 4 and a minimum valency (degree) of 2, respectively.

Result 3. If $\Delta(G)$ or $\delta(G)$ equals to 2, then the corresponding valency radio number of the graph G is equal to the $L(2,1)$ labeling number of the graph G.

3 Main Results

In this section we have investigated the maximum and minimum valency radio numbers of poly-silicate and poly-oxide network such as rectangular silicate, rectangular oxide, triangulane silicate, triangulane oxide, $m \times m$ silicate and oxide sheets.

In this paper, we have named the vertices of $TSL(m)$ from top to bottom as follows: For any k lies between 1 and m, the k^{th}- row line vertices are named as $u_j^k,\ j=1,2\ldots 2k+2$ and the oxide vertices in between the k^{th} and $(k+1)^{th}$ row lines are named as $v_j^k,\ j=1,2\ldots k+1$. The silicate vertices which are below and adjacent to k^{th} row vertices are named as $w_j^k,\ j=1,2\ldots k+1$. Also, for $1\le k\le m-1$, the silicate vertices which are above and adjacent to $(k+1)^{th}$ row vertices are named as $s_j^k,\ j=1,2\ldots k+1$. The remaining two vertices, namely a silicate and an oxide vertex just above the first row are named as s_0^0 and v_0^0 respectively.

Theorem 3.1. Let $TOX(m)$ be a triangulane oxide network containing m row lines, then the maximum valency radio labeling of $TOX(m)$ satisfies $r_\Delta\left(TOX(m)\ \right)\le 45,\ m>3$.

Proof. First we define a labeling pattern σ from $V\left(TOX(m)\right)$ to the set of whole numbers as follows: For the case when $k=1,\ 5,\ 9\ \ldots 4\left(\left\lceil \frac{m}{4}\right\rceil-1\right)+1$, define $\sigma\left(u^k_{j+8(t-1)}\right)=4(j-1),\ j=1,\ 2,\ 3,\ 4, t=1,\ 2\ldots\left\lceil\frac{k}{4}\right\rceil$ and $\sigma\left(u^k_{j+8t-4}\right)=4(j-1)+2,\ j=1,\ 2,\ 3,\ 4, t=1,\ 2\ldots\left\lceil\frac{k-1}{4}\right\rceil$. Again, for the case $k=3,\ 7,\ 11\ \ldots 4\left(\left\lceil\frac{m}{4}\right\rceil-1\right)+3$, define $\sigma\left(u^k_{j+8(t-1)}\right)=4(j-1)+1,\ j=$

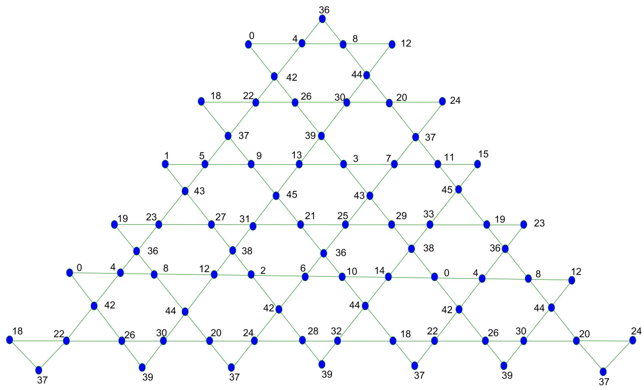

Figure 1. A triangulane oxide network $TOX(m)$ with $m = 6$ and a maximum valency radio labeling which attains the upper bound.

1, 2, 3, 4, $t = 1, 2 \ldots \lceil \frac{k}{4} \rceil$ and $\sigma\left(u^k_{j+8t-4}\right) = 4(j-1)+2,\ j = 1,\ 2,\ 3,\ 4,\ t = 1,\ 2 \ldots \lceil \frac{k}{4} \rceil$. Also, when $k = 2,\ 6,\ 10 \ldots 4\left(\lceil \frac{m}{4} \rceil - 1\right) + 2$, define $\sigma\left(u^k_{j+8(t-1)}\right) = 4(j-1)+18,\ j = 1,\ 2,\ 3,\ 4, t = 1,\ 2 \ldots \lceil \frac{k}{4} \rceil, \sigma\left(u^k_{j+8t-4}\right) = 4(j-1)+20,\ j = 1,\ 2,\ 3,\ 4, t = 1,\ 2 \ldots \lceil \frac{k-1}{4} \rceil$ and $\sigma\left(u^k_{j+8t-4}\right) = 20+4(j-1),\ j = 1, 2,\ \ t = \lceil \frac{k}{4} \rceil$. Likewise, when $k = 4,\ 8,\ 12 \ldots 4\lceil \frac{m}{4} \rceil$, define $\sigma\left(u^k_{j+8(t-1)}\right) = 4(j-1)+19,\ j = 1,\ 2,\ 3,\ 4, t = 1,\ 2 \ldots \lceil \frac{k}{4} \rceil, \sigma\left(u^k_{j+8t-4}\right) = 4(j-1)+21,\ j = 1,\ 2,\ 3,\ 4, t = 1,\ 2 \ldots \lceil \frac{k}{4} \rceil$ and $\sigma\left(u^k_{j+8t-4}\right) = 19+4(j-1),\ j = 1, 2,\ \ t = \lceil \frac{k+1}{4} \rceil$. In addition, the vertices in the set $\left\{v^k_j / j = 1, 2 \ldots k+1\right\}$ are labelled as follows: If $k \equiv 1(mod\ 4)$, then $\sigma\left(v^k_j\right) = \begin{cases} 42, & j \text{ is odd} \\ 44, & j \text{ is even} \end{cases}$ and $\sigma\left(v^k_j\right) = \begin{cases} 37, & j \text{ is odd} \\ 39, & j \text{ is even} \end{cases}$ whenever $k \equiv 2(mod\ 4)$. Likewise, if $k \equiv 3(mod\ 4)$, then $\sigma\left(v^k_j\right) = 43,\ j$ is odd and $\sigma\left(v^k_j\right) = 45,\ j$ is even. Last of all except the vertex $\sigma\left(v^0_0\right)$, label the vertices v^k_j for odd j and even j separately as 36 and 38 whenever $k \equiv 0(mod\ 4)$. Finally, label $\sigma\left(v^0_0\right)$ as 36. As, the maximum valency of $TOX(m)$ is 4, we must verify the inequality $d(x,z)+|\sigma(x)-\sigma(z)| \geq 5$ must satisfy for any pair of vertices in $TOX(m)$.

Case 1. Assume both the vertices are the row line vertices, then they are of the form $u^k_j,\ 1 \leq k \leq m,\ 1 \leq j \leq 2k+2$. That is, if $x = u^q_p$ and $z = u^t_s,\ 1 \leq q, t \leq m,\ 1 \leq p, s \leq 2k+2$, then the following possibilities arises.

Case 1.1. If $p = s$, then from the above defined labeling pattern, either $d\left(u^q_p, u^t_s\right) \geq 8$ and $\left|\sigma\left(u^q_p\right) - \sigma\left(u^t_s\right)\right| \geq 0$ or $d\left(u^q_p, u^t_s\right) \geq 2$ and $\left|\sigma\left(u^q_p\right) - \sigma\left(u^t_s\right)\right| \geq 4$. Hence, in both the chances, $d\left(u^q_p, u^t_s\right) + \left|\sigma\left(u^q_p\right) - \sigma\left(u^t_s\right)\right| \geq 5$.

Case 1.2. Supposing $p \neq s$, then x and z lie in two different row lines or in the same row line. Here, if $|p-s| = 1$, then $d\left(u^q_p, u^t_s\right) = 1$ and $\left|\sigma\left(u^q_p\right) - \sigma\left(u^t_s\right)\right| \geq 4$, otherwise, from the mapping either one of the following conditions holds and satisfies the required labeling condition. That is, $d\left(u^q_p, u^t_s\right) \geq 2$ and $\left|\sigma\left(u^q_p\right) - \sigma\left(u^t_s\right)\right| \geq 8$ or $d\left(u^q_p, u^t_s\right) \geq 8$ and $\left|\sigma\left(u^q_p\right) - \sigma\left(u^t_s\right)\right| \geq 0$ or $d\left(u^q_p, u^t_s\right) \geq 4$ and $\left|\sigma\left(u^q_p\right) - \sigma\left(u^t_s\right)\right| \geq 2$.

Case 2. Assume $x = v^q_p$ and $z = v^t_s$, where $1 \leq p, s \leq k+1$ and $1 \leq q, t \leq m$. If $p = s$, then either $\sigma\left(v^q_p\right) = \sigma\left(v^t_s\right)$ and $d\left(v^q_p, v^t_s\right) \geq 5$ or $\left|\sigma\left(v^q_p\right) - \sigma\left(v^t_s\right)\right| \geq 4$ and $d\left(v^q_p, v^t_s\right) \geq 2$. Otherwise, either $d\left(v^q_p, v^t_s\right) \geq 3$ and $\left|\sigma\left(v^q_p\right) - \sigma\left(v^t_s\right)\right| \geq 2$ or $d\left(v^q_p, v^t_s\right) \geq 8$. Hence, the required condition is verified in these possibilities.

Case 3. Suppose $x = u^q_p$ and $z = v^t_s$, where $1 \leq q, t \leq m,\ 1 \leq p \leq 2k+2,\ 1 \leq s \leq k+1$. Then for the least possibility of the vertices labelled with 33 and 36, $\left|\sigma\left(u^q_p\right) - \sigma\left(v^t_s\right)\right| = 2$

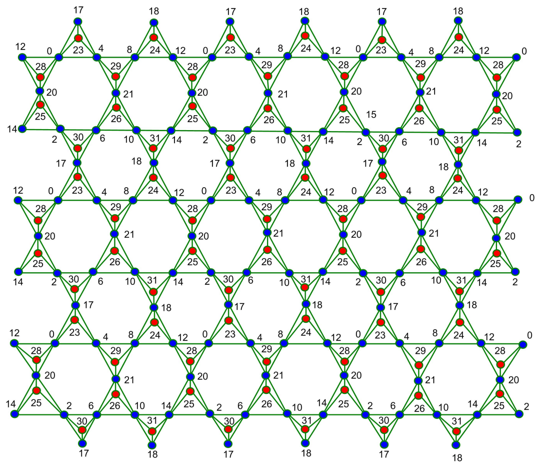

Figure 2. A rectangular silicate $RSL(6,\ 13)$ and its minimum valency radio labeling.

and $d\left(u_p^q, v_s^t\right) = 3$ Otherwise, the conditions are easily verified. Consequently, for any pair of vertices in $TOX(m)$, the maximum valency radio labeling condition is true for the mapping σ. Moreover, for $k \equiv 3(mod\, 4)$, the vertices $\sigma\left(v_j^k\right)$, whenever j is even, attains the maximum value 45. This concludes the proof of the theorem.

Theorem 3.2. The minimum valency radio labeling of a rectangular silicate $RSL(m, l)$ with minimum two row lines and atleast four edges in a row line satisfies $r_\delta\left(RSL(m, l)\ \right) \leq 31$.

Proof. First we name the vertices of $RSL(m, l)$ as follows: Let the $m(l+1)$ oxide vertices of the m row lines are names as as $\{u_j^k/\ j = 1, 2 \ldots l+1,\ \ k = 1, 2 \ldots m\}$. The oxide and silicate vertices just above the $(2k-1)^{th}, (\ k = 1, 2 \ldots \frac{m}{2})$ row lines are named as $\{v_j^k/j = 1, 2 \ldots \frac{l+1}{2} - 1\}$ and $\{x_j^k/j = 1, 2 \ldots \frac{l+1}{2} - 1\}$ respectively. Again, the silicate vertices and the rest of the oxide vertices just above the $(2k)^{th}$, $(\ k = 1, 2 \ldots \frac{m}{2})$ row lines are named as $\{y_j^k/j = 1, 2 \ldots \frac{l+1}{2}\}$ and $\{w_j^k/j = 1, 2 \ldots \frac{l+1}{2}\}$ respectively. Finally, the rest of the silicate vertices just below the $(2k-1)^{th}, (\ k = 1, 2 \ldots \frac{m}{2})$ row lines and $(2k)^{th}$, $(\ k = 1, 2 \ldots \frac{m}{2})$ row lines are named as $\{z_j^k/j = 1, 2 \ldots \frac{l+1}{2}\}$ and $\{s_j^k/j = 1, 2 \ldots \frac{l+1}{2} - 1\}$ respectively.

Next, we label the vertices of $RSL(m)$ with non-negative integers under the mapping σ as follows: For the set of vertices $\left\{u_j^k/j = 1, 2 \ldots l+1,\ \ k = 1, 2 \ldots m\right\}$, define $\sigma\left(u_{j+4(t-1)+1}^{2k-1}\right) = 4(j-1),\ j = 1,\ 2,\ 3,\ 4, t = 1,\ 2 \ldots \lceil \frac{l}{4} \rceil, k = 1, 2 \ldots \frac{m}{2}, \sigma\left(u_{j+4(t-1)+1}^{2k}\right) = 4(j-1)+2,\ j = 1,\ 2,\ 3,\ 4,\ t = 1,\ 2 \ldots \lceil \frac{l}{4} \rceil,\ k = 1, 2 \ldots \frac{m}{2},\ \ \sigma\left(u_1^k\right) = \begin{cases} 12, & k\ is\ odd \\ 14, & k\ is\ even \end{cases}$. Also, for the set of vertices $\{v_j^k/j = 1, 2 \ldots \frac{l+1}{2} - 1\}$, if j is odd, then $\sigma\left(v_j^k\right) = 17$ else $\sigma\left(v_j^k\right) = 18$. Again, for the vertices $\{w_j^k/j = 1, 2 \ldots \frac{l+1}{2}\}$, $\sigma\left(w_j^k\right) = 20$ or $\sigma\left(w_j^k\right) = 21$ according to j is odd or even. The silicate vertices, $\{x_j^k/j = 1, 2 \ldots \frac{l+1}{2} - 1\}$, $\{y_j^k/j = 1, 2 \ldots \frac{l+1}{2}\}$, $\{z_j^k/j = 1, 2 \ldots \frac{l+1}{2}\}$ and $\{s_j^k/j = 1, 2 \ldots \frac{l+1}{2} - 1\}$ are mapped as $\sigma\left(x_j^k\right) = \begin{cases} 23, & j\ odd \\ 24, & j\ even \end{cases}$, $\sigma\left(y_j^k\right) = \begin{cases} 25, & j\ odd \\ 26, & j\ even \end{cases}$, $\sigma\left(z_j^k\right) = \begin{cases} 28, & j\ odd \\ 29, & j\ even \end{cases}$ and $\sigma\left(s_j^k\right) = \begin{cases} 30, & j\ odd \\ 31, & j\ even \end{cases}$. It is visible in Figure 2.

Finally, we verify the given labeling is a valid minimum valency radio labeling by showing that $d(x, z) + |\sigma(x) - \sigma(z)| \geq 4\ \forall\ x, z \in V\left(RSL(m, l)\right)$. Choose two arbitrary vertices x and z in $RSL(m, l)$.

Case 1. Suppose x and z are oxide vertices. Assume $x = u_p^q$ and $z = u_s^t$, $1 \leq q, t \leq m$, $1 \leq p, s \leq l+1$, then either $d\left(u_p^q, u_s^t\right) \geq 4$ or $\left|\sigma\left(u_p^q\right) - \sigma\left(u_s^t\right)\right| \geq 4$ or $d\left(u_p^q, u_s^t\right) \geq 2$ and

$\left|\sigma\left(u_p^q\right)-\sigma\left(u_s^t\right)\right| \geq 2$. If $x = v_p^q$ and $z = v_s^t$, $1 \leq p,s \leq \frac{l+1}{2}-1$, $1 \leq q,t \leq m$ or $x = w_p^q$ and $z = w_s^t$, $1 \leq p,s \leq \frac{l+1}{2}$, $1 \leq q,t \leq m$, then either $d(x,z) \geq 4$ and $|\sigma(x)-\sigma(z)| \geq 0$ or $d(x,z) = 1$ and $|\sigma(x)-\sigma(z)| \geq 4$. Again if, $x \in \{v_j^k/j = 1,2\ldots\frac{l+1}{2}-1\}$ and $z \in \{w_j^k/j = 1,2..\frac{l+1}{2}\}$ then $d\left(v_j^k, w_j^k\right) \geq 2$ and $\left|\sigma\left(v_j^k\right)-\sigma\left(w_j^k\right)\right| \geq 2$. Also, if $x \in \{u_j^k/\ j = 1,2\ldots l+1,\ \ k = 1,2\ldots m\}$ and $z \in \{w_j^k/j = 1,2\ldots\frac{l+1}{2}\}$ then $\left|\sigma\left(u_j^k\right)-\sigma\left(w_j^k\right)\right| > 8$. Finally, if $x \in \{v_j^k/j = 1,2..\frac{l+1}{2}-1\}$ and $z \in \{u_j^k/\ j = 1,2\ldots l+1,\ \ k = 1,2\ldots m\}$, then $d\left(v_j^k, u_j^k\right) \geq 1$ and $\left|\sigma\left(v_j^k\right)-\sigma\left(u_j^k\right)\right| \geq 3$. Hence, $d(x,z)+|\sigma(x)-\sigma(z)| \geq 4$ is satisfied.

Case 2. Surmise that x and y are silicate vertices. If $x = x_p^q$ and $z = x_s^t$, $1 \leq p,s \leq \frac{l+1}{2}-1$, $1 \leq q,t \leq m$ or $x = y_p^q$ and $z = y_s^t$, $1 \leq p,s \leq \frac{l+1}{2}$, $1 \leq q,t \leq m$ or $x = z_p^q$ and $z = z_s^t$, $1 \leq p,s \leq \frac{l+1}{2}-1$, $1 \leq q,t \leq m$ or $x = s_p^q$ and $z = s_s^t$, $1 \leq p,s \leq \frac{l+1}{2}$, $1 \leq q,t \leq m$ then either $d(x,z) \geq 3$ and $|\sigma(x)-\sigma(z)| \geq 1$ or $d(x,z) > 4$. Again, if $x \in \left\{x_j^k/\ j = 1,2\ldots\frac{l+1}{2}-1\right\}$ and $z \in \{y_j^k/j = 1,2\ldots\frac{l+1}{2}\}$, then $d\left(x_j^k, y_j^k\right) \geq 3$ and $\left|\sigma\left(v_j^k\right)-\sigma\left(w_j^k\right)\right| \geq 1$. Similarly, if $x \in \left\{x_j^k/\ j = 1,2\ldots\frac{l+1}{2}-1\right\}$ and $z \in \left\{z_j^k/j = 1,2\ldots\frac{l+1}{2}-1\right\}$ or $x \in \left\{x_j^k/\ j = 1,2\ldots\frac{l+1}{2}-1\right\}$ and $z \in \left\{\frac{s_j^k}{j} = 1,2\ldots\frac{l+1}{2}\right\}$ or $x \in \left\{y_j^k/\ j = 1,2\ldots\frac{l+1}{2}\right\}$ and $z \in \left\{z_j^k/j = 1,2\ldots\frac{l+1}{2}-1\right\}$ or $x \in \left\{y_j^k/\ j = 1,2\ldots\frac{l+1}{2}\right\}$ and $z \in \left\{s_j^k/j = 1,2\ldots\frac{l+1}{2}\right\}$ or $x \in \left\{z_j^k/\ j = 1,2\ldots\frac{l+1}{2}-1\right\}$ and $z \in \left\{s_j^k/j = 1,2\ldots\frac{l+1}{2}\right\}$, then $d\left(x_j^k, y_j^k\right) \geq 3$ and $\left|\sigma\left(v_j^k\right)-\sigma\left(w_j^k\right)\right| \geq 1$. Thus, different possibilities in this case satisfies $d(x,z)+|\sigma(x)-\sigma(z)| \geq 4$.

Case 3. Assume that x is an oxide vertex and z is a silicate vertex. Except for the subcase that $x = w_p^q$, $1 \leq q \leq m$, $1 \leq p \leq \frac{l+1}{2}$ and $z = x_s^t$, $1 \leq s \leq \frac{l+1}{2}-1$, $1 \leq t \leq m$, all other possibilities trivially satisfy the condition, since $|\sigma(x)-\sigma(z)| > 4$. If $x \in \{w_j^k/j = 1,2...\frac{l+1}{2}\}$ and $z \in \left\{x_j^k/\ j = 1,2\ldots\frac{l+1}{2}-1\right\}$, then $\left|\sigma\left(w_p^q\right)-\sigma\left(x_s^t\right)\right| \geq 2$ and $d\left(x_j^k, y_j^k\right) \geq 2$. Thus, the minimum valency radio labeling condition is true for all cases in $RSL(m,l)$. Therefore, we have accomplished the result, $r_\delta\left(RSL(m,l)\ \right) \leq 31,\ m > 1,\ \ l > 4$.

Theorem 3.3. Suppose there are m row lines in a triangulane oxide network $TOX(m)$, then $r_\delta\left(TOX(m)\ \right) \leq 8$, $m > 2$.

Proof. First define the mapping σ for the vertex set $\left\{u_j^k/j = 1,2\ldots 2k+2,\quad k = 1,2\ldots m\right\}$ as $\sigma\left(u_{j+3(t-1)}^{2k-1}\right) = 2(j-1)$, $j = 1,\ 2,\ 3, t = 1,\ 2\ldots\left\lceil\frac{2k+2}{3}\right\rceil, k = 1,2\ldots\frac{m}{2}$ and $\sigma\left(u_{j+3(t-1)}^{2k}\right) = 2j-1,\quad j = 1,\ 2,\ 3,\ t = 1,\ 2\ldots\left\lceil\frac{2k+2}{3}\right\rceil,\ k = 1,2\ldots\frac{m}{2}$. Again, for the set of vertices $\left\{v_j^k/j = 1,2\ldots k+1,\ \ k = 1,2\ldots m\right\}$ in $TOX(m)$, define $\sigma\left(v_j^k\right) = \begin{cases} 7, & k\ is\ odd \\ 8, & k\ is\ even \end{cases}$. Also, the vertex v_0^0 is labelled as 6. As the verification part is similar to Theorem 3.2, we leave the verification part to the reader.

Theorem 3.4. For the triangulane silicate network $TSL(m)$, the minimum valency radio labeling satisfies $r_\delta\left(TSL(m,l)\ \right) \leq 31,\ m > 2$.

Proof. As the minimum valency of $TSL(m)$is 3, in order to achieve the desired upper bound, we have given a labeling pattern for oxide and silicate vertices separately. For the oxide vertices $\left\{\left\{u_j^k/j = 1,2\ldots 2k+2,\quad k = 1,2\ldots m\right\} \cup \left\{v_j^k/j = 1,2\ldots k+1,\ \ k = 1,2\ldots m\right\} \cup \left\{v_0^0\right\}\right\}$, we define $\sigma\left(u_{j+6(t-1)}^{2k-1}\right) = 3(j-1),\ \ j = 1,2,3,\ \ t = 1,2\ldots\left\lceil\frac{2k+2}{3}\right\rceil,\ \ k = 1,2\ldots\left\lceil\frac{m}{2}\right\rceil$, $\sigma\left(u_{j+6(t-1)+1}^{2k-1}\right) = 3(j-1)+1,\ \ j = 1,2,3,\ \ t = 1,2\ldots\left\lfloor\frac{2k+2}{3}\right\rfloor,\ \ k = 1,2\ldots\left\lceil\frac{m}{2}\right\rceil$ and $\sigma\left(u_{j+6(t-1)}^{2k}\right) = 3(j+2),\ j = 1,2,3,\ t = 1,2\ldots\left\lceil\frac{2k+2}{3}\right\rceil,\ k = 1,2\ldots\left\lfloor\frac{m}{2}\right\rfloor, \sigma\left(u_{j+6(t-1)+1}^{2k-1}\right) = 3(j+2)+1,\ \ j = 1,2,3,\ \ t = 1,2\ldots\left\lfloor\frac{2k+2}{3}\right\rfloor,\ \ k = 1,2\ldots\left\lfloor\frac{m}{2}\right\rfloor$. Again, for both k and j are odd, then $\sigma\left(v_j^k\right) = 27$, if k is odd and j is even, then $\sigma\left(v_j^k\right) = 28$, if both k and j are even then $\sigma\left(v_j^k\right) = 30$, otherwise $\sigma\left(v_j^k\right) = 31$, where $j = 1,\ 2\ldots k{+}1$. Lastly, the oxide vertex v_0^0 is label

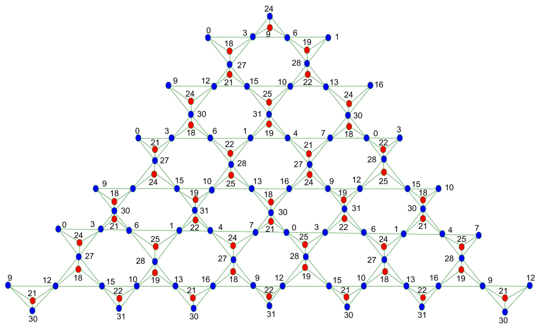

Figure 3. A minimum valency radio labeling of triangulane silicate network $TSL(6)$

as 24. Next, we consider the silicate vertex set $\left\{\{w_j^k/j=1,2\ldots k+1,\ k=1,2\ldots m\}\cup\{s_j^k/j=1,2\ldots k+1,\ k=1,2\ldots m\}\cup\{v_0^0\}\right\}$. For $k\ \equiv 1(mod\ 3)$, $\sigma\left(w_j^k\right)=\begin{cases}18, j\ is\ odd\\ 19,\ j\ is\ even\end{cases}$, if $k\ \equiv 2(mod\ 3)$, then $\sigma\left(w_j^k\right)=\begin{cases}24, j\ is\ odd\\ 25,\ j\ is\ even\end{cases}$, otherwise, $\sigma\left(w_j^k\right)=\begin{cases}21, j\ is\ odd\\ 22,\ j\ is\ even\end{cases}$. Likewise, for $k\ \equiv 1(mod\ 3)$, $\sigma\left(s_j^k\right)=\begin{cases}21, j\ is\ odd\\ 22,\ j\ is\ even\end{cases}$, if $k\ \equiv 2(mod\ 3)$ then $\sigma\left(s_j^k\right)=\begin{cases}18, j\ is\ odd\\ 19,\ j\ is\ even\end{cases}$, otherwise, $\sigma\left(s_j^k\right)=\begin{cases}24, j\ is\ odd\\ 25,\ j\ is\ even\end{cases}$. The left-out silicate vertex s_0^0 is labelled as 9. It can be seen through figure 3. The rest of the proof is left to the reader.

Theorem 3.5. For the rectangular oxide $ROX(m,l)$ with $m>1$ and $l>4$, the minimum valency radio number satisfies $r_\delta\left(ROX(m,l)\right)\le 8$.

Proof. First we name the vertices of $ROX(m,l)$ as in Theorem 3.4. Define a mapping $\sigma : V\left(ROX(m,l)\right)\to N\cup\{0\}$ as follows. $\sigma\left(u_{j+3(t-1)}^{2k-1}\right)=2(j-1),\ j=1,2,3,\ t=1,2\ldots\left\lceil\frac{l+1}{3}\right\rceil,\ k=1,2\ldots\frac{m}{2}, \sigma\left(u_{j+3(t-1)}^{2k}\right)=2(j-1)+1,\ j=1,2,3,\ t=1,2\ldots\left\lfloor\frac{l+1}{3}\right\rfloor,\ k=1,2\ldots\frac{m}{2}$. Also, for the vertices in the set $\{v_j^k/j=1,2\ldots\frac{l+1}{2}-1\}$ are labeled as 7. Rest of the vertices in the set $\{w_j^k/j=1,2\ldots\frac{l+1}{2}\}$ are labelled as 8. As in Theorem 3.1, it is easy to verify the labeling condition gets satisfied. Hence, $r_\delta\left(ROX(m,l)\right)\le 8$.

Theorem 3.6. Let $ROX(m,l)$ $(m>1,\ \ l>4)$ be a rectangular oxide network, where m denotes the number of row lines and l denotes the number of edges in a row line. Then, $r_\Delta\left(ROX(m,l)\right)\le 45$.

Proof. The following labeling pattern attains the proof of this theorem. Define σ from $V\left(ROX(m,l)\right)$ to non-negative integers as follows. For the subset set $\{u_j^k,\ j=1,2\ldots l+1,\ \ k=1,2\ldots m\}$ of $V\left(ROX(m,l)\right)$, define $\sigma\left(u_{j+5(t-1)}^{4k-3}\right)=4(j-1),\ j=1,2\ldots 5,\ \ t=1,2\ldots\left\lceil\frac{l+1}{5}\right\rceil,\ \ k=1,2\ldots\left\lceil\frac{m}{4}\right\rceil,\ \sigma\left(u_{j+5(t-1)}^{4k-1}\right)=4(j-1)+1,\ \ j=1,2\ldots 5,\ \ t=1,2\ldots\left\lceil\frac{l+1}{5}\right\rceil,\ \ k=1,2\ldots\left\lfloor\frac{m}{4}\right\rfloor,\ \sigma\left(u_{j+5(t-1)}^{4k-2}\right)=4(j-1)+16,\ \ j=1,2\ldots 5,\ \ t=1,2\ldots\left\lceil\frac{l+1}{5}\right\rceil,\ \ k=1,2\ldots\left\lceil\frac{m}{4}\right\rceil,\ \ \sigma\left(u_{j+5(t-1)}^{4k}\right)=4(j-1)+17,\ \ j=1,2\ldots 5,\ \ t=1,2\ldots\left\lceil\frac{l+1}{5}\right\rceil,\ \ k=1,2\ldots\left\lfloor\frac{m}{4}\right\rfloor$. Again, for the vertices in the set $\{v_j^k/j=1,2\ldots\frac{l+1}{2}-1\}$ are mapped as $\sigma\left(v_{2j-1}^{2k-1}\right)=37, j=1,2\ldots\left\lceil\frac{\frac{l+1}{2}-1}{2}\right\rceil, k=1,2\ldots\frac{m}{2}, \sigma\left(v_{2j}^{2k-1}\right)=39, j=1,2\ldots\left\lfloor\frac{\frac{l+1}{2}-1}{2}\right\rfloor, k=1,2\ldots\frac{m}{2},\ \sigma\left(v_{2j-1}^{2k}\right)=38, j=1,2\ldots\left\lceil\frac{\frac{l+1}{2}-1}{2}\right\rceil, k=1,2\ldots\frac{m}{2}-1$ and $\sigma\left(v_{2j}^{2k}\right)=40,\ j=1,2\ldots\left\lfloor\frac{\frac{l+1}{2}-1}{2}\right\rfloor,\ k=1,2\ldots\frac{m}{2}-1$. Moreover, for the vertex set

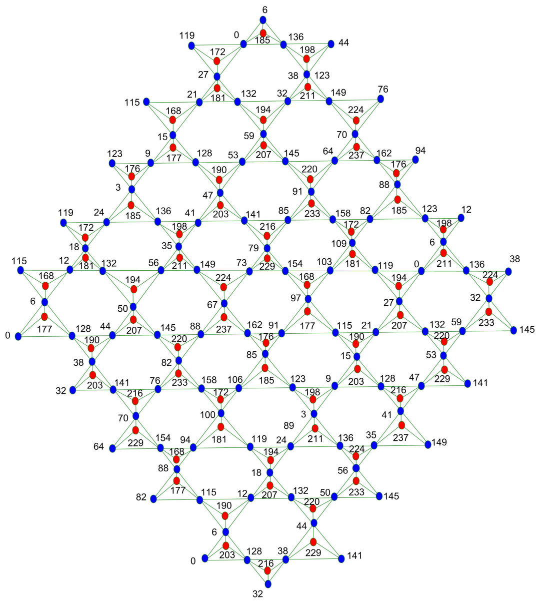

Figure 4. A maximum valency radio labeling of 6×6 silicate sheet which attains the bound of Theorem 3.7.

$\{w_j^k/j = 1, 2 \ldots \frac{l+1}{2}\}$, the vertices are labelled as 43 if both j and k are odd, 45 if j is odd and k is even, 44 if j is even and k is odd, 45 if both j and k are even. Next, we can follow the same steps as in Theorem 3.4, to attained the required result.

Theorem 3.7. Let $(SL)_{m\times m}$ be a $m \times m$ silicate sheet which contains twice the number of row lines as in $TOX(m)$. Then, the maximum valency radio labeling satisfies $r_\Delta\left((SL)_{m\times m}\right) \leq 237,\ m > 3$.

Proof. Let us address the oxide and silicate vertices of $(SL)_{m\times m}$ in the following manner. Start addressing the horizontal row line oxide vertices from the top leftmost to bottom rightmost as $\{u_j^k/\ j = 1, 2 \ldots 2m,\ \ k = 1, 2 \ldots m\}$. The rest of the oxide vertices in the vertical column lines are addressed from top to bottom as $\{v_1^1,\ v_1^2 \ldots v_1^m\} \cup \{v_j^k/k = 1, 2 \ldots m,\ j = 2, 3 \ldots m + 1\} \cup \{v_m^1,\ v_m^2 \ldots v_m^m\}$. Also, the silicate vertices are addressed from the topmost left to bottom right most as $\{s_1^1,\ s_2^1 \ldots s_m^1\} \cup \{s_j^k/j = 1, 2 \ldots m+1,\ k = 2, 3 \ldots 2m-1\} \cup \{s_1^{2m},\ s_2^{2m} \ldots s_m^{2m}\}$. Next, we define a labeling pattern which will fulfil the required maximum radio labeling condition.

For the set of vertices $\left\{u_j^k/j = 1, 2 \ldots 2m,\ \ k = 1, 2 \ldots m\right\}$, if exists, then assign the labels as $\sigma\left(u_{j+5(t-1)}^{4(k-1)+1}\right) = 6(j-1), j = 1, 2 \ldots 5,\ t = 1, 2 \ldots \left\lceil\frac{2m}{10}\right\rceil, k = 1, 2 \ldots \left\lceil\frac{m}{4}\right\rceil,\ \sigma\left(u_{j+5t}^{4(k-1)+1}\right) = 6(j-1) + 3, j = 1, 2 \ldots 5,\ t = 1, 2 \ldots \left\lceil\frac{2m-5}{10}\right\rceil, k = 1, 2 \ldots \left\lceil\frac{m}{4}\right\rceil,\ \sigma\left(u_{j+5(t-1)}^{4(k-1)+2}\right) = 6(j-1) + 32, j = 1, 2 \ldots 5,\ t = 1, 2 \ldots \left\lceil\frac{2m}{10}\right\rceil, k = 1, 2 \ldots \left\lceil\frac{m}{4}\right\rceil,\ \sigma\left(u_{j+5t}^{4(k-1)+2}\right) = 6(j-1) + 35, j = 1, 2 \ldots 5,\ t = 1, 2 \ldots \left\lceil\frac{2m-5}{10}\right\rceil, k = 1, 2 \ldots \left\lfloor\frac{m}{4}\right\rfloor,\ \sigma\left(u_{j+5(t-1)}^{4k-1}\right) = 6(j-1) + 64, j = 1, 2 \ldots 5,\ t = 1, 2 \ldots \left\lceil\frac{2m}{10}\right\rceil, k = 1, 2 \ldots \left\lceil\frac{m}{4}\right\rceil,\ \sigma\left(u_{j+5t}^{4(k-1)+2}\right) = 6(j-1)+67, j = 1, 2 \ldots 5,\ t = 1, 2 \ldots \left\lceil\frac{2m-5}{10}\right\rceil, k = 1, 2 \ldots \left\lfloor\frac{m}{4}\right\rfloor, \sigma\left(u_{j+5(t-1)}^{4k}\right) = 6(j-1)+82, j = 1, 2 \ldots 5,\ t = 1, 2 \ldots \left\lceil\frac{2m}{10}\right\rceil, k = 1, 2 \ldots \left\lceil\frac{m}{4}\right\rceil,\ \ \sigma\left(u_{j+5t}^{4k}\right) = 6(j-1) + 85, j = 1, 2 \ldots 5,\ t = 1, 2 \ldots \left\lceil\frac{2m-5}{10}\right\rceil, k = 1, 2 \ldots \left\lfloor\frac{m}{4}\right\rfloor$.

Likewise, for the set $\left\{v_j^k/k = 1, 2 \ldots m,\ j = 1, 2, 3 \ldots m + 1\right\}$ of oxide vertices, we define $\sigma\left(v_{j+3(t-1)}^{4(k-1)+1}\right) = 4(j-1)+115, j = 1, 2, 3,\ t = 1, 2 \ldots \left\lceil\frac{m+1}{3}\right\rceil,\ k = 1, 2 \ldots \left\lceil\frac{m}{4}\right\rceil,\ \sigma\left(v_{j+3(t-1)}^{4(k-1)+2}\right) = 4(j-1) + 128, j = 1, 2, 3,\ \ \ t = 1, 2 \ldots \left\lceil\frac{m+1}{3}\right\rceil, k = 1, 2 \ldots \left\lceil\frac{m}{4}\right\rceil, \sigma\left(v_{j+3(t-1)}^{4k-1}\right) = 4(j -$

1) + 141, $j = 1,2,3,\quad t = 1,2\ldots\left\lceil\frac{m+1}{3}\right\rceil, k = 1,2\ldots\left\lfloor\frac{m}{4}\right\rfloor$, $\sigma\left(v^{4k}_{j+3(t-1)}\right) = 4(j-1)+154, j = 1,2,3,\quad t = 1,2\ldots\left\lceil\frac{m+1}{3}\right\rceil, k = 1,2\ldots\left\lfloor\frac{m}{4}\right\rfloor$. Finally for the set of silicate vertices, $\{\{s^1_1,\ s^1_2\ldots s^1_m\}\cup\{s^k_j/j = 1,2\ldots m+1,\ k = 2,3\ldots 2m-1\}\cup\ \{s^{2m}_1,\ s^{2m}_2\ldots s^{2m}_m\}\}$, we have $\sigma\left(s^1_{j+3(t-1)}\right) = 4(j-1)+168, j = 1,2,3,\ t = 1,2\ldots\left\lceil\frac{m}{3}\right\rceil$, $\sigma\left(s^{6k+1}_{j+3(t-1)}\right) = 4(j-1)+168, j = 1,2,3,\ t = 1,2\ldots\left\lceil\frac{m+1}{3}\right\rceil,\ \ k = 1,\ 2\ldots\left\lceil\frac{2m-7}{6}\right\rceil, \sigma\left(s^{6(k-1)+2}_{j+3(t-1)}\right) = 4(j-1)+177, j = 1,2,3,\ t = 1,2\ldots\left\lceil\frac{m+1}{3}\right\rceil,\ \ k = 1,\ 2\ldots\left\lceil\frac{2m-2}{6}\right\rceil, \sigma\left(s^{6(k-1)+3}_{j+3(t-1)}\right) = 4(j-1)+190$, $j = 1,2,3,\ t = 1,2\ldots\left\lceil\frac{m+1}{3}\right\rceil,\ \ k = 1,\ 2\ldots\left\lceil\frac{2m-3}{6}\right\rceil,\ \sigma\left(s^{6(k-1)+4}_{j+3(t-1)}\right) = 4(j-1)+203$, $j = 1,2,3,\ t = 1,2\ldots\left\lceil\frac{m+1}{3}\right\rceil,\ \ k = 1,\ 2\ldots\left\lceil\frac{2m-4}{6}\right\rceil,\ \sigma\left(s^{6(k-1)+5}_{j+3(t-1)}\right) = 4(j-1)+216$, $j = 1,2,3,\ t = 1,2\ldots\left\lceil\frac{m+1}{3}\right\rceil,\ \ k = 1,\ 2\ldots\left\lceil\frac{2m-5}{6}\right\rceil,\ \sigma\left(s^{6k}_{j+3(t-1)}\right) = 4(j-1)+229$, $j = 1,2,3,\ t = 1,2\ldots\left\lceil\frac{m+1}{3}\right\rceil,\ \ k = 1,\ 2\ldots\left\lceil\frac{2m-6}{6}\right\rceil$. Also, for the rest of the silicate vertices $\{s^{2m}_1,\ s^{2m}_2\ldots s^{2m}_m\}\}$, label any one of the labeling patterns for silicate vertices based on the value $2m$. The mapping is visible in figure 4. As the proofs for the following results are parallel to the previous theorems, we have left the entire proof to the reader.

Theorem 3.8. Let G be an $m\times m$ oxide sheet $(OX)_{m\times m}$. Then, the minimum valency radio labeling of G satisfies $r_\delta(G\ )\leq 8,\ m>2$.

Theorem 3.9. For $m>2$, the minimum valency radio labeling of an $m\times m$ silicate sheet, satisfies $r_\delta\left((SL)_{m\times m}\ \right)\leq 31$.

Theorem 3.10. Let $OX(n)$ be the oxide network of dimension n, then the minimum valency radio number of $OX(n)$ satisfies $r_\delta\ (OX(n)\ )\leq 8$.

Proof. As, $\delta(OX(n)) = 2$, the minimum valency radio number is equal to the $L(2,1)$ labeling number. Kins et.al. [21] determined an upper bound for the $L(2,1)$ labeling number of oxide network as $\lambda_{2,1}(OX(n))\leq 8$. Hence, $r_\delta(OX(n))\leq 8$.

4 Conclusion remarks

In this paper we have investigated the upper bounds for the maximum and minimum valency radio labellings for certain categories of Poly-oxides and Poly-silicates structures such as triangulane oxide and silicate networks, rectangular oxide and silicate networks and $m\times m$ oxide and silicate sheets. The problem of finding the maximum valency radio labelling of silicate network, oxide network, triangulane silicate, rectangular silicate and $m\times m$ oxide sheets are still open. In addition, determining the minimum valency radio labelling of silicate network is also undetermined. Further this research work can be extended to other interconnection networks and graphs.

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Competing interests

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