Palestine Journal of Mathematics

Vol 13(4)(2024) , 1325–1333

CLEANNESS AND RELATED STRUCTURES IN BI-AMALGAMATION

M. Chhiti and L. Es-Salhi

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 13D05, 13D02.

Keywords and phrases: Bi-amalgamated algebras along ideals, weakly clean ring, n-clean ring, n-good ring, nil-good ring.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract *This paper establishes necessary and sufficient conditions for a bi-amalgamation to inherit the weakly clean (respectively, n-clean, n-good, nil-good) property. The new results compare to previous works carried on various settings of duplications and amalgamations, and capitalize on recent results on bi-amalgamations. Our results allow us to construct new and original examples of rings satisfying the above mentioned properties.*

1 Introduction

Throughout, all rings considered are commutative with unity and all modules are unital.
In 1977, W. K. Nicholson [16] introduced the concept of clean rings as a subclass of exchange rings. He defined a ring R to be clean if for every $a \in R$ there is u a unit in R and e an idempotent in R such that $a = u + e$, if this presentation is unique for every element, we call the ring uniquely clean. Over the last ten to fifteen years there has been an explosion of interest in this class of rings as well as the many generalizations and variations. Properties of rings related to the clean property have been largely expanded and researched. It is easily seen that a ring R is clean if and only if every element is a difference of a unit and an idempotent. Thus, it is natural to consider rings with the condition that every element is either a sum or a difference of a unit and an idempotent. These rings came into sight in the first palce in Anderson and Camillo [3], and were called weakly clean rings by Ahn and Anderson in [1]. In 2005, Xiao and Tong [18] introduced a generalization of clean rings, they defined a ring R to be n-clean if every element of R can be written as the sum of n units and an idempotent in R. Moreover, they proved that if R is a n-clean ring then so is the matrix ring $M_m(R)$ for any positive integer m. In 2005, P. Vàmos [17] introduced the concept of 2-good rings. He defined an element a in R to be 2-good if it can be expressed as the sum of two units in R, and defined a ring R to be 2-good if every element in R is 2-good. In general, a ring is n-good if every element can be written as the sum of n units, and these properties have distinct applications from those of clean. In 2016, P. Danchev [8] defined a property related to 2-good as follows: an element a in R is nil-good if $a = b + u$ where b is a nilpotent element of R and u is either 0 or a unit in R. The ring R is said to be nil-good if every element of R is nil-good.

Let A be a ring and let I be an ideal of A. In [11], the authors introduced the following subring:

$$A \bowtie I := \{(a, a + i) \mid a \in A, i \in I\},$$

of the direct product $A \times A$ and named it the amalgamated duplication of A along I. *This ring constuction arises as a sort of variant of the Nagata idealization. Among motivations for studying duplications of rings alongs ideals, it is worth noting that in [9] the author showed that the amalgamated duplication of an algebroid curve along a regular multiplicative ideal always*

yields an algebroid curve.
As it is immediately seen, whenever I is a nonzero ideal of A, the ring $A \bowtie I$ is not an integral domain. A new ring construction, which is more general than the amalgamated duplication and can be an integral domain, was introduced and studied by D'Anna, Finocchiaro and Fontana in [10, 13] as the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

When $A = B$ and $f = id_A$, the amalgamated algebras $A \bowtie^{id_A} I$ is the classical amalgamated duplication. Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([10, Examples 2.5 and 2.6]). On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by Anderson in [2] and motivated by a classical construction due to Dorroh [12], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [10, Section 2]. Also, the authors consider the iteration of the amalgamation process, giving some geometrical applications of it.

A new class of rings, which covers that of amalgamated algebras as a particular case, was recently introduced and studied by Kabbaj, Louartiti and Tamekkante in [14]: given ring homomorphisms $f : A \to B$, $g : A \to C$ and given ideals J of B and J' of C such that $I_0 := f^{-1}(J) = g^{-1}(J')$, the subring

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') | a \in A, (j, j') \in J \times J'\}$$

of $B \times C$ is called the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g).
In [14], the authors investigated about the basic ring-theoritic properties of bi-amalgamations and provided, among the other things, a description of its prime ideal structures. In [6], we have studied the transfer of the clean-like properties to bi-amalgamated algebras.
The aim of this paper is to continue the investigation of the transfer of some clean-like conditions to bi-amalgamated algebras. The next section establishes necessary and sufficient conditions for a bi-amalgamation to inherit the weakly clean (resp. n-clean, resp. n-good, resp. nil-good) property. At this point, we recall the following definitions:

Definition 1.1. 1) An element $r \in R$ is said to be (uniquely) weakly clean if r can be written (uniquely) in the form $r = e + u$ or $r = e - u$ where $u \in U(R)$ and $e \in Idem(R)$. The ring R is said to be (uniquely) weakly clean if every element of R is (uniquely) weakly clean.
2) Let n be an integer, $n \geq 1$. An element $r \in R$ is said to be n-clean if r can be expressed as the sum of an idempotent and n units of R. The ring R is said to be n-clean if every element of R is n-clean.
3) An element $r \in R$ is said to be n-good if $r = u_1 + ... + u_n$ with $u_1, ..., u_n \in U(R)$. The ring R is said to be n-good if every element of R is n-good.
4) A ring R is called (uniquely) nil-good if every element $r \in R$ can be represented (uniquely) as $r = a + u$, where $a \in Nil(R)$ and $u \in U(R) \cup \{0\}$. Such an element r is called (uniquely) nil-good too.

Throughout, for a ring R, $Nil(R)$, $U(R)$, $Idem(R)$ and $Rad(R)$, will denote the ideal of all nilpotent elements of R, the mulitiplicative group of units of R, the set of all idempotents of R and the Jacobson radical of R, respectively.

2 Main resuls

We begin with the following result:

Proposition 2.1. *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two non-zero proper ideals of B and C, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. Assume that for every $u \in U(A)$, $e \in Idem(A)$ and $(j, j') \in J \times J'$, either $(f(u) + j, g(u) + j') \in U(B \times C)$ (for example, if $J \times J' \subseteq Rad(B \times C)$) or $(f(e) + j, g(e) + j') \in Idem(B \times C)$ (for example, if $J \times J' \subseteq Idem(B \times C)$). If A is a weakly clean ring, then $A \bowtie^{f,g} (J, J')$ is a weakly clean ring.*

Proof. Assume that A is weakly clean and let $a \in A$ and $(j, j') \in J \times J'$. Then $a = u + e$ or $a = u - e$ for some $(u, e) \in U(A) \times Idem(A)$. Let $r = (f(a) + j, g(a) + j') \in A \bowtie^{f,g} (J, J')$. Then if $(f(u) + j, g(u) + j') \in U(B \times C)$, r has the weakly clean decomposition $r = (f(u) + j, g(u) + j') + (f(e), g(e))$ or $r = (f(u) + j, g(u) + j') - (f(e), g(e))$. It is easy to show that $(f(u) + j, g(u) + j') \in U(A \bowtie^{f,g} (J, J'))$. Indeed, using the assumption, it follows that $(f(u) + j, g(u) + j') \in U(B \times C)$. Then, there exists $(p, q) \in B \times C$ such that $((f(u) + j)p, (g(u) + j')q) = (1, 1)$. Hence, $((f(u) + j)(f(u^{-1}) - pf(u^{-1})j), (g(u) + j')(g(u^{-1}) - qg(u^{-1})j')) = ((f(u) + j)f(u^{-1}) - (f(u) + j)pf(u^{-1})j, (g(u) + j')g(u^{-1}) - (g(u)+j')qg(u^{-1})j) = (f(u)f(u^{-1})+jf(u^{-1})-f(u^{-1})j, g(u)g(u^{-1})+j'g(u^{-1})-g(u^{-1})j') = (f(uu^{-1}), g(uu^{-1})) = (1, 1)$. Then, $(f(u)+j, g(u)+j')$ is invertible in $A \bowtie^{f,g} (J, J')$. Hence r is a sum or a difference of a unit and an idempotent element in $A \bowtie^{f,g} (J, J')$. If $(f(e)+j, g(e)+j') \in Idem(B\times C)$, r has the weakly clean decomposition $r = (f(u), g(u))+(f(e)+j, g(e)+j')$ or $r = (f(u), g(u)) - (f(e) + j, g(e) + j')$ and then r is a sum or a difference of a unit and an idempotent element in $A \bowtie^{f,g} (J, J')$. Consequently, $A \bowtie^{f,g} (J, J')$ is a weakly clean ring.

□

The second main result examines the transfer of the n-clean property to bi-amalgamations.

Proposition 2.2. *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two non-zero proper ideals of B and C, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. Assume that for every $u \in U(A)$, $e \in Idem(A)$ and $(j, j') \in J \times J'$, either $(f(u) + j, g(u) + j') \in U(B \times C)$ (for example, if $J \times J' \subseteq Rad(B \times C)$) or $(f(e) + j, g(e) + j') \in Idem(B \times C)$ (for example, if $J \times J' \subseteq Idem(B \times C)$). If A is n-clean, then $A \bowtie^{f,g} (J, J')$ is n-clean. The converse holds if f (or g) is injective.*

Proof. Let $r = (f(a) + j, g(a) + j') \in A \bowtie^{f,g} (J, J')$ and write $a = e + u_1 + \cdots + u_n$ where $u_1, u_2, \ldots, u_n \in U(A)$ and $e \in Idem(A)$. Then if $(f(u) + j, g(u) + j') \in U(B \times C)$, r has the n-clean decomposition $r = (f(e), g(e)) + (f(u_1), g(u_1)) + \cdots + (f(u_n) + j, g(u_n) + j')$ and if $(f(e) + j, g(e) + j') \in Idem(B \times C)$, r has the n-clean decomposition $r = (f(e) + j, g(e) + j')+(f(u_1), g(u_1))+\cdots+(f(u_n), g(u_n))$, as desired. Conversely, assume that $A \bowtie^{f,g} (J, J')$ is n-clean and f is injective. Let $a \in A$. we can write $(f(a), g(a)) = (f(e), g(e))+(f(u_1), g(u_1))+ \cdots+(f(u_n), g(u_n))$ where, $(f(e), g(e)) \in Idem(A \bowtie^{f,g} (J, J'))$ and $(f(u_1), g(u_1)), \ldots, (f(u_n), g(u_n))$ are units in $A \bowtie^{f,g} (J, J')$. Thus, there exists $v_1, \ldots, v_n \in U(A)$ such that $(f(u_i), g(u_i))(f(v_i), g(v_i)) = (1, 1)$ for some $i = 1, \ldots, n$. Which implies that $u_iv_i = 1$ since f is injective. Consequently, u_i is a unit for some $i = 1, \ldots, n$. Furthermore, $(f(e), g(e))$ is an idempotent, then $(f(e), g(e))^2 = (f(e), g(e))$. Which implies that $e^2 = e$ since f is injective. Thus, e is an idempotent. Finally, a is a n-clean element, writing that $a = e + u_1 + \cdots + u_n$. Consequently, A is n-clean.

□

For the special case of amalgamations, we obtain:

Corollary 2.3. *Let $f : A \to B$ and let J be an ideal of B. Assume that either $J \subseteq Rad(B)$ or $J \subseteq Idem(B)$. Then, $A \bowtie^f J$ is a n-clean ring if and anly if A is a n-clean ring.*

For duplications, we have:

Corollary 2.4. *Let A be a ring and I an ideal of A such that $I \subseteq Rad(A)$. Then, $A \bowtie I$ is n-clean if and only if A is n-clean.*

The next result transfers the n-good property from the ring A to bi-amalgamations.

Proposition 2.5. *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two non-zero proper ideals of B and C, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. Assume that for every $u \in U(A)$ and $(j, j') \in J \times J'$, $(f(u) + j, g(u) + j') \in U(B \times C)$ (for example, if $J \times J' \subseteq Rad(B \times C)$). If A is a n-good ring, then $A \bowtie^{f,g} (J, J')$ is a n-good ring. The converse holds if f (or g) is injective or $I_0 \subseteq Rad(A)$.*

Proof. Assume that A is n-good and consider $a \in A$ and $(j, j') \in J \times J'$. Then, $a = u_1 + ... + u_n$ for some $u_1, ..., u_n \in U(A)$. Therefore, $(f(a) + j, g(a) + j') = (f(u_1), g(u_1)) + ... + (f(u_{n-1}), g(u_{n-1})) + (f(u_n) + j, g(u_n) + j')$. It is easy to check that, $(f(u_t), g(u_t)) \in U(A \bowtie^{f,g} (J, J'))$ for $t = 1, ..., n-1$. Using the assumption, it follows that, $(f(u_n) + j, g(u_n) + j') \in U(A \bowtie^{f,g} (J, J'))$. Finally, $(f(a) + j, g(a) + j')$ is n-good and thus, $A \bowtie^{f,g} (J, J')$ is n-good, as desired. Conversely, assume that $A \bowtie^{f,g} (J, J')$ is a n-good ring. If $I_0 \subseteq Rad(A)$, then the proof follows directly from [14, Proposition 4.1 (3)] and [17, Lemma 2 (a)]. If f or g is injective, then the proof is similar to the second part of the proof of Proposition 2.2.

□

The proposition 2.5 above recovers the special case of amalgamated algebras and duplications, as recorded below.

Corollary 2.6. *Let $f : A \to B$ and let J be an ideal of B. Assume that $J \subseteq Rad(B)$. Then, $A \bowtie^f J$ is n-good if and anly if so is A.*

Corollary 2.7. *Let A be a ring and I an ideal of A such that $I \subseteq Rad(A)$. Then, $A \bowtie I$ is n-good if and only if A is n-good.*

In the following result, we show that the transfer of the n-good property can be made via the special ring $f(A) + J$ (resp. $g(A) + J'$) and under the assumption $J' \subseteq Rad(C)$ (resp. $J \subseteq Rad(B)$).

Theorem 2.8. *With the above notation we have:*

(i) If $A \bowtie^{f,g} (J, J')$ is a n-good ring, then so is $f(A) + J$ and $g(A) + J'$.

(ii) Assume that $J' \subseteq Rad(C)$. Then, $A \bowtie^{f,g} (J, J')$ is a n-good ring if and only if $f(A) + J$ is a n-good ring.

(iii) Assume that $J \subseteq Rad(B)$. Then, $A \bowtie^{f,g} (J, J')$ is a n-good ring if and only if $g(A) + J'$ is a n-good ring.

Proof. (1) Recall first that homomorphic images of n-good rings are n-good by [7, Proposition 2.5]. Assume that $A \bowtie^{f,g} (J, J')$ is a n-good ring, then so are $f(A) + J$ and $g(A) + J'$ since they are homomorphic images of $A \bowtie^{f,g} (J, J')$ by [14, Proposition 4.1 (2)].
(2) Assume that $f(A) + J$ is a n-good ring and $J' \subseteq Rad(C)$, then $A \bowtie^{f,g} (J, J')$ is a n-good ring using [14, Proposition 4.1 (2)] and [17, Lemma 2 (a)]. The converse follows from (1).
(3) Similar to (2). □

The condition assumed in Proposition 2.5 is necessary, a counter-example is given below.

Example 2.9. Let $A := \mathbb{Z}_5$, $B := \mathbb{Z}_5 \times \mathbb{Z}_6$, $C := \mathbb{Z}_5 \times \mathbb{Z}_7$, $J := 0 \times \mathbb{Z}_6$ and $J' := 0 \times \mathbb{Z}_7$. Consider the two ring homomorphisms defined by: $f(a) = (a, 0)$ and $g(a) = (a, 0)$ for all $a \in A$. Then,

(i) $\exists u\ U(A)$ and $(j, j') \in J \times J'$ such that $(f(u) + j, g(u) + j') \notin U(B \times C)$.

(ii) A is 4-good.

(iii) $A \bowtie^{f,g} (J, J')$ is not 4-good.

Proof. (1) $f(2) + (0, 2) = (2, 2) \notin U(\mathbb{Z}_5 \times \mathbb{Z}_6)$.
(2) It is easy to check that $\mathbb{Z}_5$ is 4-good.
(3) Recall that the class of n-good rings is closed under direct products by [17, Proposition 3]. Then, $\mathbb{Z}_5 \times \mathbb{Z}_6$ is not 4-good since we can easily check that $\mathbb{Z}_6$ is not a 4-good ring. Thus, $f(A) + J$ is not 4-good since $f(A) + J = \mathbb{Z}_5 \times \mathbb{Z}_6$. Hence, $A \bowtie^{f,g} (J, J')$ is not 4-good by Theorem 2.8.

□

The following examples illustrate Corollary 2.3 and Corollary 2.6.

Example 2.10. Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, ..., X_n\}$ a finite set of indeterminates over B. Set the subring $A + XB[[X]] := \{h \in B[[X]] \mid h(0) \in A\}$ of the ring of power series $B[[X]]$. Then, $A + XB[[X]]$ is n-clean (resp. n-good) if and only if A is n-clean (resp. n-good).

Proof. It follows from [10, Example 2.5] and Corollary 2.3 (resp. Corollary 2.6). □

Example 2.11. Let T be a ring and $J \subseteq Rad(T)$ an ideal of T and let D be a subring of T such that $J \cap D = (0)$. The ring $D + J$ is n-clean (resp. n-good) if and only if D is n-clean (resp. n-good).

Proof. It follows from [10, Proposition 5.1 (3)] and Corollary 2.3 (resp. Corollary 2.6). □

In the following result, we establish necessary and sufficient conditions for a bi-amalgamation to be nil-good.

Theorem 2.12. *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two non-zero proper ideals of B and C, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. Consider the following conditions:*

(a) $f(A) + J$ (resp. $g(A) + J'$) is nil-good and $J' \subseteq Nil(C)$ (resp. $J \subseteq Nil(B)$).
(b) $A \bowtie^{f,g} (J, J')$ is nil-good.
(c) $J \times J' \subseteq Nil(B \times C)$.
Then:

(1) (a) ⇒ (b) ⇒ (c).
(2) If A is nil-good, then the three conditions are equivalent.

Proof. (1) If $f(A) + J$ (resp. $g(A) + J'$) is a nil-good ring and $J' \subseteq Nil(C)$ (resp. $J \subseteq Nil(B)$) then, $A \bowtie^{f,g} (J, J')$ is nil-good, in accordance with [14, Proposition 4.1 (2)] and [8, Proposition 2.8]. This proves (a)⇒(b).
Assume that $A \bowtie^{f,g} (J, J')$ is nil-good and let $j \in J$. Then, $(j, 0) = (f(u) + t, g(u) + t') + (f(b) + s, g(b) + s')$ for some $(f(u) + t, g(u) + t') \in U(A \bowtie^{f,g} (J, J')) \cup \{(0, 0)\}$ and $(f(b) + s, g(b) + s') \in Nil(A \bowtie^{f,g} (J, J'))$. Then, two cases can be distinguished:
case 1: If $(f(u) + t, g(u) + t') \in U(A \bowtie^{f,g} (J, J'))$. Then $(g(u) + t') \in U(g(A) + J')$. Moreover, we have $g(u) + t' = -(g(b) + s')$. It follows that, $g(u) + t' \in Nil(C)$. As a consequence, $Nil(C) = C$ which is a contradiction.
case 2: If $(f(u) + t, g(u) + t') = (0, 0)$. Then, $(j, 0) = (f(b) + s, g(b) + s')$ implies that $j = f(b) + s$. Hence, $J \subseteq Nil(B)$ since $f(b) + s \in Nil(f(A) + J) \subseteq Nil(B)$. Similarly, we can show that $J' \subseteq Nil(C)$. This proves (b)⇒(c).
(2) Assume that A is nil-good. Then, according to [14, Proposition 4.1 (3)], $\frac{f(A)+J}{J}$ is nil-good. By using [8, Proposition 2.8], it follows that $f(A) + J$ is nil-good since $J \subseteq Nil(B)$. This proves (c)⇒(a). A similar argument shows that $g(A) + J'$ is nil-good and $J \subseteq Nil(B)$. □

Theorem 2.12 recovers the special case of amalgamated algebras, as recorded in the next corollary.

Corollary 2.13. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then, $A \bowtie^f J$ is nil-good if and only if so is A and $J \subseteq Nil(B)$.*

In the next result, the nil-goodness of $A \bowtie^{f,g} (J, J')$ depends to the choice of f and g.

Proposition 2.14. *With the notation of Theorem 2.12, we have:*
If $A \bowtie^{f,g} (J, J')$ is a nil-good ring then $f(A)+J$ and $g(A)+J'$ are nil-good rings. The converse is true provided that A is nil-good and $\frac{A}{I_0}$ is uniquely nil-good.

Proof. Necessity follows by a simple combination of [14, Proposition 4.1 (2)] and [8, Example 4]. For the sufficency, let $a \in A$ and $(j, j') \in J \times J'$. Since A, $f(A) + J$ and $g(A) + J'$ are nil-good rings, we can write $a = u + n$, $f(a) + j = f(x) + j_1 + f(y) + j_2$ and $g(a) + j' = g(x')+j'_1+g(y')+j'_2$ where $(u, n) \in (U(A)\cup\{0\})\times Nil(A)$, $(f(x)+j_1, f(y)+j_2) \in (U(f(A)+J)\cup\{0\})\times Nil(f(A)+J)$ and $(g(x')+j'_1, g(y')+j'_2) \in (U(g(A)+J')\cup\{0\})\times Nil(g(A)+J')$. It is clear that, $\overline{f(x)}, \overline{f(u)}) \in U(\frac{f(A)+J}{J}) \cup \{0\})$ and $\overline{f(y)}, \overline{f(e)}) \in Nil(\frac{f(A)+J}{J})$. Moreover, we have $\overline{f(a)} = \overline{f(u)} + \overline{f(n)} = \overline{f(x)} + \overline{f(y)}$ and $\overline{g(a)} = \overline{g(u)} + \overline{g(n)} = \overline{g(x')} + \overline{g(y')}$. Thus, $(\overline{f(u)}, \overline{g(u)}) = (\overline{f(x)}, \overline{g(x')}$ and $(\overline{f(n)}, \overline{g(n)}) = (\overline{f(y)}, \overline{g(y')})$ since $\frac{f(A)+J}{J} \cong \frac{g(A)+J'}{J'} \cong \frac{A}{I_0}$ is uniquely nil-good. Therefore, there is $\tilde{j}_1, \tilde{j}_2 \in J$ and $\bar{j}'_1, \bar{j}'_2 \in J'$ such that, $f(x) = f(u) + \tilde{j}_1$, $f(y) = f(n) + \tilde{j}_2$, $g(x') = g(u) + \bar{j}'_1$ and $g(y') = g(n) + \bar{j}'_2$. We have, $(f(a) + j, g(a) + j') = (f(u) + \tilde{j}_1 + j_1, g(u) + \bar{j}'_1 + j'_1) + (f(n) + \tilde{j}_2 + j_2, g(n) + \bar{j}'_2 + j'_2)$. It is easy to see that, $(f(n)+\tilde{j}_2+j_2, g(n)+\bar{j}'_2+j'_2)$ is a nilpotent element of $A \bowtie^{f,g} (J, J')$. Since $(f(u)+\tilde{j}_1+j_1, g(u)+\bar{j}'_1+j'_1) \in U(f(A)+J\times g(A)+J')\cup\{0, 0\}$. it suffices to show that $(f(u)+\tilde{j}_1+j_1, g(u)+\bar{j}'_1+j'_1) \in U(A \bowtie^{f,g} (J, J'))$. Indeed, $f(u)+\tilde{j}_1+j_1, g(u)+\bar{j}'_1+j'_1) \in U(f(A)+J\times g(A)+J')$ then there exists $(f(\alpha)+\tilde{j}, g(\beta)+\bar{j}$ such that $((f(u)+\tilde{j}_1+j_1)(f(\alpha)+\tilde{j}), (g(u)+\bar{j}'_1+j'_1)(g(\beta)+\bar{j}')) = (1, 1)$. Thus, $(\overline{f(u)f(\alpha)}, \overline{g(u)g(\beta)}) = (\bar{1}, \bar{1})$. Then, $(\overline{f(\alpha)}, \overline{g(\beta)}) = (\overline{f(u^{-1})}, \overline{g(u^{-1})})$. So, there exists $(\tilde{j}_0, \bar{j}_0) \in J \times J'$ such that $(f(\alpha), g(\alpha)) = (f(u^{-1}) + \tilde{j}_0, g(u^{-1}) + \bar{j}_0)$. Hence, $(f(u) + \tilde{j}_1 + j_1, g(u)+\bar{j}'_1+j'_1)((f(u^{-1})+\tilde{j}_0+\tilde{j}, g(u^{-1})+\bar{j}_0+\bar{j})) = (f(u)+\tilde{j}_1+j_1, g(u)+\bar{j}'_1+j'_1)(f(\alpha)+\tilde{j}, g(\beta)+\bar{j}) = (1, 1)$. It follows that, $(f(u)+\tilde{j}_1+j_1, g(u)+\bar{j}'_1+j'_1) \in U(A \bowtie^{f,g} (J, J'))$. Thus, $A \bowtie^{f,g} (J, J')$ is nil-good.

□

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A\times E$ with multiplication given by $(a, e)(c, d) = (ac, ad + ec)$.
Let $i : A \hookrightarrow R$ be the canonical embedding. After identifying E with $0 \propto E$, E becomes an ideal of B. According to [10, Remark 2.8], It is non straightforward but it is known that $A \propto E$ coincides with $A \bowtie^i E$.

Corollary 2.13 recovers the special case of trivial ring extension, as recorded below.

Corollary 2.15. *Let A be a ring, E an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then, R is nil-good if and only if so is A.*

Proof. Clearly, $J^2 := (0 \propto E)^2 = 0$. it follows that $J \subseteq Nil(B)$. Corollary 2.13 completes the proof.

□

In the following result, we will show that bi-amalgamations can inherit the nil-good property from the ring A.

Proposition 2.16. *Under the above notation, assume that f (or g) is injective. Then, the following conditions are equivalent:*

(i) *$A \bowtie^{f,g} (J, J')$ is nil-good.*

(ii) *A is nil-good and $J \times J' \subseteq Nil(B \times C)$.*

Proof. (1)⇒(2): Suppose that $A \bowtie^{f,g} (J, J')$ is nil-good and f is injective. Let $a \in A$. Then, $(f(a), g(a)) \in A \bowtie^{f,g} (J, J')$ and we can write $(f(a), g(a)) = (f(u), g(u)) + (f(v), g(v))$ where, $(f(u), g(u))$ is either a unit or zero and $(f(v), g(v))$ is a nilpotent. Firstly, assume that $(f(u), g(u))$ is either a unit. Thus, there exists $\alpha \in A$ such that, $(f(u), g(u))(f(\alpha), g(\alpha)) =$

$(1,1)$. Which implies that, $u\alpha = 1$ since f is injective. Consequently,u is a unit. Furthermore, $(f(v), g(v))$ is a nilpotent, then $(f(v), g(v))^n = (0,0)$ for some positive integer n. Which implies that $v^n = 0$ since f is injective. Thus, v is a nilpotent. Finally, a is a nil-good element, writing that $a = u + v$. Assuming now that, $(f(u), g(u))$ is equal zero, we obtain that, $(f(a), g(a))$ is a nilpotent, which gives that $a^m = 0$ for some positive integer, since f is injective. Consequently, a is a nilpotent writing that $a = a + 0$. So, A is a nil-good ring, as desired.
Next, if $A \bowtie^{f,g} (J, J')$ is nil-good, then by Theorem 2.12 we have, $J \times J' \subseteq Nil(B \times C)$.
(2)⇒(1): Assume that A is nil-good and $J \times J' \subseteq Nil(B \times C)$ and consider $a \in A$ and $(j, j') \in J \times J'$. Since A is nil-good, we can write $a = u + v$, where u is either a unit or zero and v is a nilpotent. Then, $(f(a)+j, g(a)+j') = (f(u), g(u))+(f(v)+j, g(v)+j')$. It is clear that if u is a unit, then $(f(u), g(u))$ is a unit in $A \bowtie^{f,g} (J, J')$ and if $u = 0$ then, $(f(u), g(u)) = (0, 0)$. On the other hand, since $(j, j') \in J \times J' \subseteq Nil(B \times C)$ then, $(f(v) + j, g(v) + j')$ is a nilpotent in $A \bowtie^{f,g} (J, J')$. Consequently, $(f(a) + j, g(a) + j')$ is nil-good. Thus, $A \bowtie^{f,g} (J, J')$ is a nil-good ring.

□

In [4], G. Călugăreanu and T. Y. Lam defined the class of so-called fine rings that are rings for which each non-zero element can be expressed as the sum of a unit and a nilpotent. Clearly, nil-good rings are fine. The converse is not true in general.
In the following example, we illustrate Proposition 2.16 and we provide a family of non fine nil-good rings which arise as bi-amalgamations.

Example 2.17. Let $(A_1, \mathfrak{m}_1)$ be a nil-good ring which is not fine with ${\mathfrak{m}_1}^2 = 0$. Let $(A, \mathfrak{m}) := (A_1 \propto E_1)$ be the trivial ring extension of A_1 by a nonzero $A_1/\mathfrak{m}_1$-vector space E_1. (For instance, $A_1 := \mathbb{Z}_4$, $\mathfrak{m}_1 := 2\mathbb{Z}_4$ and $E_1 := \mathbb{Z}_2$). Let $B := A \propto E$ be the trivial ring extension of A by a nonzero $A/\mathfrak{m}$-vector space E. Let

$$f: \begin{array}{ccc} A & \rightarrow & B \\ (a,e) & \mapsto & ((a,e),0) \end{array}$$

be an injective ring homomorphism and $J := \mathfrak{m} \propto E$ be the maximal ideal of B. Let $C := A_1$ and let

$$g: \begin{array}{ccc} A & \rightarrow & C \\ (a,e) & \mapsto & a \end{array}$$

be a surjective ring homomorphism and $J' := \mathfrak{m}_1$ be the maximal ideal of C. Then,

(i) $A \bowtie^{f,g} (J, J')$ is nil-good.

(ii) $A \bowtie^{f,g} (J, J')$ is not fine.

Proof. (1) We can easily check that, $J^2 = 0$ and $J'^2 = 0$. According to Corollary 2.15, A is nil-good. Hence by using Proposition 2.16, it follows that $A \bowtie^{f,g} (J, J')$ is nil-good.
(2) $A_1 = Z_4$ is not fine since 2 is not a fine element. Then, $g(A) + J'$ is not fine since $g(A) + J' = C = A_1$. Hence by using [14, Proposition 4.1 (2)] and [4, Theorem 2.3], it follows that $A \bowtie^{f,g} (J, J')$ is not fine.

□

The last main result gives a characterization for the bi-amalgamation to be nil-good (resp. weakly clean, resp. n-clean).

Theorem 2.18. *Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms and J and J' be two non-zero proper ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$. Set $\overline{A} = \frac{A}{Nil(A)}$, $\overline{B} = \frac{B}{Nil(B)}$, $\overline{C} = \frac{C}{Nil(C)}$, $\pi_B : B \rightarrow \overline{B}$, $\pi_C : C \rightarrow \overline{C}$, the canonical projections, $\overline{J} = \pi_B(J)$ and $\overline{J'} = \pi_C(J')$.*
Let $\overline{f} : \overline{A} \rightarrow \overline{B}$, $\overline{g} : \overline{A} \rightarrow \overline{C}$ be the ring homomorphisms defined by setting: $\overline{f}(\overline{a}) = \overline{f(a)}$ and $\overline{g}(\overline{a}) = \overline{g(a)}$. Then, $\overline{A} \bowtie^{\overline{f},\overline{g}} (\overline{J}, \overline{J'})$ is nil-good (resp. weakly clean, resp. n-clean) if and only if $A \bowtie^{f,g} (J, J')$ is nil-good (resp. weakly clean, resp. n-clean).

Proof. Clearly, we can see that $\overline{f}$ and $\overline{g}$ are well defined and they are ring homomorphisms. Consider the map:

$$\begin{array}{rccl} \psi: & A\bowtie^{f,g}(J,J')/Nilp(A\bowtie^{f,g}(J,J')) & \rightarrow & \overline{A}\bowtie^{\overline{f},\overline{g}}(\overline{J},\overline{J'}) \\ & \overline{(f(a)+j,g(a)+j')} & \mapsto & (\overline{f}(\overline{a})+\overline{j},\overline{g}(\overline{a})+\overline{j'}) \end{array}$$

It is easy to show that the map ψ is well defined. It is also easy to check that ψ is a ring homomorphism. Besides, if $(\overline{f}(\overline{a})+\overline{j},\overline{g}(\overline{a})+\overline{j'})=(\overline{0},\overline{0})$ then, $(\overline{f}(\overline{a})+\overline{j}=\overline{0}$ and $\overline{g}(\overline{a})+\overline{j'}=\overline{0}$. Consequently, $(f(a)+j,g(a)+j')\in Nilp(A\bowtie^{f,g}(J,J')$, which means that, $\overline{(f(a)+j,g(a)+j')}=\overline{0}$. It follows that, ψ is injective. Clearly, ψ is surjective by construction. Thus, it is a ring isomorphism. Consequently, the desired result is obtained directly from [8, Proposition 2.8] (resp. [1, Theorem 1.9 (1)], resp, [18, Corollary 2.7]).

□

The following result is a consequence of the previous theorem.

Corollary 2.19. *With the above notation, assume that f (or g) is injective and $J\times J'\subseteq Nil(B\times C)$. Then, $A\bowtie^{f,g}(J,J')$ is nil-good (resp. weakly clean, resp. n-clean) if and only if A is nil-good (resp. weakly clean, resp. n-clean).*

Proof. Maintaining the same notation of Theorem 2.18, $\overline{A}\bowtie^{\overline{f},\overline{g}}(\overline{J},\overline{J'})$ is nil-good (resp. weakly clean, resp. n-clean) if and only if $A\bowtie^{f,g}(J,J')$ is nil-good (resp. weakly clean, resp. n-clean). Clearly, $\overline{J}=(0)$ and $\overline{J'}=(0)$ since $J\times J'\subseteq Nil(B\times C)$. Moreover, $\overline{f}$ is injective since f is also injective. By using [14, Proposition 4.1 (3)], it follows that $\overline{A}\bowtie^{\overline{f},\overline{g}}(\overline{J},\overline{J'})$ is nil-good (resp. weakly clean, resp. n-clean) if and only if $\overline{A}$ is nil-good (resp. weakly clean, resp. n-clean). Consequently, we get the desired result using [8, Proposition 2.8] (resp. [3, Theorem 23 (3)], resp, [18, Corollary 2.7]).

□

For the special case of amalgamations, we obtain the following corollary.

Corollary 2.20. *Let $f:A\rightarrow B$ be a ring homomorphism and J an ideal of B. Assume that $J\subseteq Nil(B)$. Then $A\bowtie^{f}J$ is a nil-good (resp. weakly clean, resp. n-clean) ring if and only if A is a nil-good (resp. weakly clean, resp. n-clean) ring.*

References

[1] *Myung-Sook Ahn and D.D. Anderson,* Weakly clean rings and almost clean rings, *The Rocky Mountain Journal of Mathematics,* **36(3)**, *783–798, (2006).*

[2] *D.D. Anderson,* Commutative rings, *in: Jim Brewer, Sarah Glaz, William Heinzer, Bruce Olberding (Eds.), Multiplicative Ideal Theory in Commutative Algebra: A tribute to the work of Robert Gilmer, Springer, New York, pp. 1–20, (2006).*

[3] *D.D. Anderson and V.P. Camillo,* Commutative rings whose elements are a sum of a unit and an idempotent , *Comm. Algebra,* **30(7)**, *3327–3336, (2002).*

[4] *G. Călugăreanu and T. Y. Lam,* Fine rings, *J. Algebra and its Appl.,* **(15)**, *(2015).*

[5] *M. Chhiti, M. Jarrar, S. Kabbaj and N. Mahdou,* Prüfer conditions in an amalgamated duplication of a ring along an ideal, *Comm. Algebra,* **43(1)**, *249–261, (2015).*

[6] *M. Chhiti and L. Es-Salhi,* Clean-like properties in bi-amalgamation algebras, *São Paulo Journal of Mathematical Sciences, 1–8, (2022).*

[7] *A. Chin and S. Yassemi,* Cleanness and related structures in amalgamated duplication rings, *Journal of Algebra and Its Applications,* **11(06)**, *12501046, (2012).*

[8] *P. Danchev,* Nil-good unital rings, *International Journal of Algebra. J. Okayama Univ.,* **10(7)**, *239–252, (2006).*

[9] *M. D'Anna,* A construction of Gorenstein rings, *J. Algebra,* **306(2)**, *507–519, (2006).*

[10] *M. D'Anna, C. A. Finacchiaro, and M. Fontana,* Amalgamated algebras along an ideal, *Comm Algebra and Aplications, Walter De Gruyter, 241–252, (2009).*

[11] *M. D'Anna and M. Fontana,* An amalgamated duplication of a ring along an ideal: the basic properties*, Journal of Algebra and its Applications.* **6(3)***, 443–459, (2003).*

[12] *J.L. Dorroh,* Concerning adjunctions to algebras*, Bull. Amer. Math. Soc.,* **38***, 85–88, (1932).*

[13] *C. Finocciaro and M. Fontana,* Prüfer-like conditions on an amalgamated algebra along an ideal*, Houston J. Math.,* **40(1)***, 63–79, 1, 2, 3, 6, 7, 8, (2014).*

[14] *S. Kabbaj, K. Louartiti and M. Tamekkante,* Bi-amalgamated algebras along ideals*, J. Commut. Algebra, 65–87, (2017).*

[15] *T. Koşan, S. Sahinkaya and Y. Zhou,* On weakly clean rings*, Communications in Algebra,* **45(8)***, 3494–3502, (2017).*

[16] *W. K. Nicholson,* Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc.,* **229***, 278–279, (1977).*

[17] *P. Vàmos,* 2-Good rings*, Quart. J. Math. (Oxford),* **56***, 417–430, (2005).*

[18] *G. Xiao and W. Tong,* n-clean rings and weakly unit stable range rings*, Communications in Algebra,* **33:5***, 1501–1517, DOI:10.1081/AGB-200060531.*

Author information

M. Chhiti, Faculty of Economics and Social Sciences of Fez, University S.M. Ben Abdellah Fez, Morocco.
E-mail: `chhiti.med@hotmail.com`

L. Es-Salhi, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.
E-mail: `loubna.essalhi@usmba.ac.ma`

Received: 2023-12-27
Accepted: 2024-10-20