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COMPUTING THE TOTAL EDGE IRREGULARITY STRENGTHS OF ODD AND EVEN STAIRCASE GRAPHS

Yeni Susanti and Muhammad Arinal Haq

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Corresponding Author: Yeni Susanti

Abstract In this paper we determine the exact value of total edge irregularity strength of odd staircase graphs and even staircase graphs.

1 Introduction

Graph labelling plays an important role in the development of graph theory nowadays. A labelling of a graph, also called as valuation, is a function that assigns usually positive or non-negative integers to the graph elements subject to a certain condition. The set of all vertices alone, the set of all edges alone and the set of all vertices and altogether all edges are the most common sets taken as domain of labelling. Whenever a labelling has the set of all vertices and all edges as its domain, then the labelling is said to be total. From the most complete recent survey on labelling by Gallian [5] we know that there are various kinds of labelling on graphs. One of well-known labellings is edge irregular total labelling proposed by Bača et.all [4] as follows. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a connected simple undirected graph with non empty vertex set V_Γ and edge set E_Γ. Bača et al. [4] considered the weight $wt_\alpha(ab)$ of edge ab under a total m-labelling $\alpha : V_\Gamma \cup E_\Gamma \to \{1, 2, ..., m\}$ defined by

$$wt_\alpha(ab) = \alpha(a) + \alpha(ab) + \alpha(b)$$

for each $ab \in E_\Gamma$. The total labelling α is called an edge irregular total m-labelling if for any two different edges ab and $a'b'$, the weights $wt_\alpha(ab)$ and $wt_\alpha(a'b')$ are not the same. The minimum m such that Γ can be labelled by an edge irregular total m-labelling, denoted by $tes(\Gamma)$, is called the total edge irregularity strength of graph Γ. In general, the total edge irregularity strength of a given graph is not easy to obtain. The following result gives a helpful hint on the lower bound of the total edge irregularity strength of arbitrary graph.

Theorem 1.1. [4] *Let Γ be any graph of $|E_\Gamma|$ edges. If the maximum vertex degree of Γ is Δ_Γ, then*

$$tes(\Gamma) \geq max\left\{\left\lceil\frac{|E_\Gamma|+2}{3}\right\rceil, \left\lceil\frac{\Delta_\Gamma+1}{2}\right\rceil\right\}.$$

Apart from the above theorem, the following conjecture on the exact value of $tes(\Gamma)$ for any graph Γ was presented by Ivanco and Jendrol in [6]

Conjecture 1.2. [6] For arbitrary graph Γ of $|E_\Gamma|$ edges and of maximum vertex degree Δ_Γ, it follows that

$$tes(\Gamma) = max\left\{\left\lceil\frac{|E_\Gamma|+2}{3}\right\rceil, \left\lceil\frac{\Delta_\Gamma+1}{2}\right\rceil\right\}.$$

For some classes of graphs, including complete graphs, complete bipartite graphs and trees, it has been proved that the conjecture is true. The total edge irregularity strength of any tree was given by Ivančo and Jendrol ([6]) while the for complete graphs and complete bipartite graphs were presented by Jendrol, et al. [7]. The total edge irregularity strength of some graphs has been reported. For instance, in [1], [2], it is given the total edge irregularity strength of categorical product of two paths and strong product of two paths, respectively. While, in [3] and [8] it is presented the total edge irregularity strength for hexagonal grid graphs, and polar grid graphs, respectively. Siddiqui et al. in [9] and [10] give the total edge irregularity strength for accordion graphs, and for disjoint union of sun graphs. In [11] it is reported the total edge irregularity strength of some staircase graphs including mirror-staircase and double staircase graphs, which grid numbers differed by one from each level to the next level. Following this work, in preprint version given in [12] we continue investigating the staircase graphs which grid numbers differed by two from each level to the next level. And for this modification, we add term odd and even in mentioning the staircase graphs under investigation. Moreover, we give the exact value of their total edge irregularity strengths. And this paper is an improvement version of the preprint version.

2 Main Results

In this section we present the total edge irregularity strength of several classes of graphs. For the first discussion we consider odd staircase graphs. Let us denote the odd staircase graph of level $s \geq 1$ by OSC_s (see Figure 1). For this graph, we have

$$V_{OSC_s} = \{a_{p,q} | p = 0, 1, 2, \ldots, 2s-1, q = \lfloor \frac{p}{2} \rfloor, \ldots, s\}$$

as the vertex set and E_{OSC_s} which consists all edges given on the table below

edges	p	q
$a_{p,q}a_{p+1,q}$	0	$0, 1, 2, \ldots, s$
$a_{p,q}a_{p+1,q}$	$1, \ldots, 2s-2$	$\lceil \frac{p}{2} \rceil, \ldots, s$
$a_{p,q}a_{p,q+1}$	$0, 1$	$0, 1, 2, \ldots, s-1$
$a_{p,q}a_{p,q+1}$	$2, \ldots, 2s-1$	$\lfloor \frac{p}{2} \rfloor, \ldots, s-1.$

It is routine that $|V_{OSC_s}| = s^2 + 3s$ and $|E_{OSC_s}| = 2s^2 + 3s - 1$. The following theorem gives the exact value of tes of OSC_s.

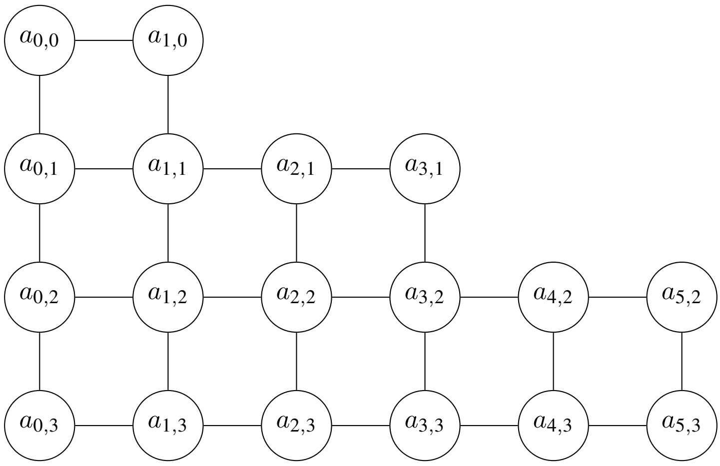

Figure 1: Odd Staircase Graph OSC_3

Theorem 2.1. *Let OSC_s be the odd staircase graph of $s \geq 1$ level. Then the total edge irregularity strength of OSC_s is*

$$tes(OSC_s) = \left\lceil \frac{2s^2 + 3s + 1}{3} \right\rceil.$$

Proof. Obviously, the maximum degree of the odd staircase graph is 2 for $s = 1$ and 4 for otherwise. Thus, by Theorem (1.1), we have

$$tes(OSC_s) \geq \left\lceil \frac{2s^2+3s+1}{3} \right\rceil.$$

To prove the upper bound, i.e. $tes(OSC_s) \leq \left\lceil \frac{2s^2+3s+1}{3} \right\rceil$, we are constructing a total edge irregularity m-labelling with $m = \left\lceil \frac{2s^2+3s+1}{3} \right\rceil$. Before we give the labelling, we determine the largest positive integer t such that

$$t^2 \leq \left\lceil \frac{2s^2+3s+1}{3} \right\rceil - 1.$$

(On Table 1 it is given several s's and t's.) Now we define a labelling

$$\alpha_1 : V_{OSC_s} \cup E_{OSC_s} \to \left\{1, 2, \ldots, \left\lceil \frac{2s^2+3s+1}{3} \right\rceil \right\}$$

as follows

label of edges and vertices	p and q
$\alpha_1(a_{p,q}) = q^2 + 1$	$p = 0, 1, \ldots, 2q+1$ $q = 0, 1, 2, \ldots, t$
$\alpha_1(a_{p,q}a_{p+1,q}) = p + q + 1$	$p = 0, 1, \ldots, 2q$ $q = 0, \ldots, t-1$
$\alpha_1(a_{p,q}a_{p+1,q}) = p + q + 1$	$p = 0, 1, \ldots, 2t-2$ $q = t$
$\alpha_1(a_{p,q}a_{p,q+1}) = p + q + 1$	$p = 0, 1, \ldots, 2q+1$ $q = 0, \ldots, t-1.$

If $t = s$, then the labelling is done. If $t < s$, then we continue to assign labels as the following

label of edges and vertices	p and q
$\alpha_1(a_{p,q}a_{p+1,q}) = p + q + 1$	$p = 2t-1, 2t$ $q = t$
$\alpha_1(a_{p,q}) = \lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, \ldots, 2q+1$ $q = t+1, \ldots, s-1$
$\alpha_1(a_{p,q}) = \lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, \ldots, 2s-1$ $q = s$
$\alpha_1(a_{p,q}a_{p,q+1}) = p + t^2 + 3t + 3 - \lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, \ldots, 2t+1$ $q = t$
$\alpha_1(a_{p,q}a_{p+1,q}) = p + (t+k-1)(2t+2k+3) + 6 - 2\lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, \ldots, 2q+1$ $q = t+k$ $k = 1, \ldots, s-t-1$
$\alpha_1(a_{p,q}a_{p,q+1}) = p + (t+k)(2t+2k+3) + 4 - 2\lceil \frac{2s^2+3s+1}{3} \rceil$	$p = 0, 1, \ldots, 2(t+k)+1$ $q = t+k$ $k = 1, \ldots, s-t-1$
$\alpha_1(a_{p,q}a_{p+1,q}) = p + 2s^2 + s + 3 - 2\lceil \frac{2s^2+3s+1}{3} \rceil$	$q = s$ $p = 0, 1, \ldots, 2s-2.$

From the above assignment, we obtain the following edge weights:

weights	p and q
$wt_{\alpha_1}(a_{p,q}a_{p+1,q}) = p + 2q^2 + q + 3$	$p = 0, 1, \ldots, 2q$ $q = 0, 1, 2, \ldots, t$
$wt_{\alpha_1}(a_{p,q}a_{p,q+1}) = p + 2q^2 + 3q + 4$	$p = 0, 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t$
$wt_{\alpha_1}(a_{p,q}a_{p+1,q}) = p + (t + k - 1)(2t + 2k + 1) + 4$	$p = 0, \ldots, 2q$ $q = t + k$ $k = 1, \ldots, s - t - 1$
$wt_{\alpha_1}(a_{p,q}a_{p,q+1}) = p + (t + k)(2t + 2k + 3) + 4$	$p = 0, 1, \ldots, 2(t + k) + 1$ $q = t + k$ $k = 1, \ldots, s - t - 1$
$wt_{\alpha_1}(a_{p,q}a_{p+1,q}) = p + 2s^2 + s + 3$	$q = s$ $p = 0, 1, \ldots, 2s - 2.$

The fact that the weights are all different can be verified as a routine. □

Table 1 Several s's and t's such that $t^2 \leq \lceil \frac{2s^2+3s+1}{3} \rceil - 1$

s	t		s	t		s	t		s	t		s	t		s	t
1	1		6	5		11	9		16	13		21	17		26	21
2	2		7	6		12	10		17	14		22	18		27	22
3	3		8	7		13	11		18	15		23	19		28	23
4	3		9	7		14	12		19	16		24	20		29	24
5	4		10	8		15	12		20	16		25	21		30	25

Below we give the tes for the second graph, namely the even staircase graphs ESC_s of level $s \geq 1$. Let ESC_s be the even staircase graph of level $s \geq 1$ (see Figure 2). Let the vertex set be

$$V_{ESC_s} = \{a_{p,q} | p = 0, 1, 2, \ldots, 2s \text{ and } q = \lfloor \frac{p-1}{2} \rfloor, \ldots, s\}.$$

We have that E_{ESC_s} consists of all edges as shown below

edges	p	q
$a_{p,q}a_{p+1,q}$	$0, 1, \ldots, 2s - 1$	$\lfloor \frac{p}{2} \rfloor, \ldots, s$
$a_{p,q}a_{p,q+1}$	0	$0, 1, 2, \ldots, s - 1$
$a_{p,q}a_{p,q+1}$	$1, 2, \ldots, 2s$	$\lfloor \frac{p-1}{2} \rfloor, \ldots, s - 1.$

By a simple counting it can be shown that $|V_{ESC_s}| = s^2 + 4s + 1$ and $|E_{ESC_s}| = 2s^2 + 5s$. In the following theorem the exact value of $tes(ESC_s)$ is given.

Theorem 2.2. *For any $s \geq 1$, let ESC_s be the odd staircase graph of s level. Then the total edge irregularity strength of ESC_s is*

$$tes(ESC_s) = \left\lceil \frac{2s^2 + 5s + 2}{3} \right\rceil.$$

Proof. Obviously, the maximum degree of ESC_s is 2 for $s = 1$ and is equal to 4 otherwise. Thus we have

$$tes(ESC_s) \geq \left\lceil \frac{2s^2 + 5s + 2}{3} \right\rceil$$

by Theorem (1.1). For the upper bound, we prove that $tes(ESC_s) \leq \left\lceil \frac{2s^2+5s+2}{3} \right\rceil$, by showing that there exists a total edge irregularity m-labelling with $m = \left\lceil \frac{2s^2+5s+2}{3} \right\rceil$. For the first step, we

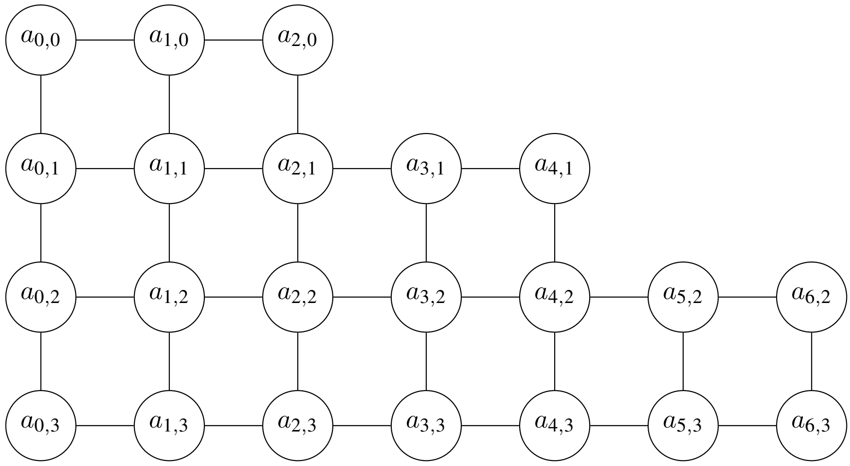

Figure 2: Even Staircase Graph ESC_3

determine the biggest positive integer t such that

$$t(t+1) \leq \left\lceil \frac{2s^2+5s+2}{3} \right\rceil - 1.$$

(Several s's and t's are listed on Table 2.) We then construct a total m-labelling

$$\alpha_2 : V_{ESC_s} \cup E_{ESC_s} \to \left\{1, 2, \ldots, \left\lceil \frac{2s^2+5s+2}{3} \right\rceil \right\}$$

with $m = \left\lceil \frac{2s^2+5s+2}{3} \right\rceil$ in the following way:

edges and vertices label	p and q
$\alpha_2(a_{p,q}) = q^2+q+1$	$p = 0, 1, \ldots, 2q+2$ $q = 0, 1, \ldots, t$
$\alpha_2(a_{p,q}a_{p+1,q}) = p+q+1$	$p = 0, 1, \ldots, 2q+1$ $q = 0, \ldots, t$
$\alpha_2(a_{p,q}a_{p,q+1}) = p+q+1$	$p = 0, 1, \ldots, 2q+2$ $q = 0, \ldots, t-1.$

We stop the process whenever $t = s$. For the case $t < s$, we continue with the following assignment

edges and vertices label	p and q
$\alpha_2(a_{p,q}a_{p+1,q}) = p+q+1$	$p = 2t, 2t+1$ $q = t$
$\alpha_2(a_{p,q}) = \lceil \frac{2s^2+5s+2}{3} \rceil$	$p = 0, 1, \ldots, 2q+2$ $q = t+1, \ldots, s-1$
$\alpha_2(a_{p,q}) = \lceil \frac{2s^2+5s+2}{3} \rceil$	$p = 0, 1 \ldots, 2$ $q = s$
$\alpha_2(a_{p,q}a_{p,q+1}) = p+t^2+4t+4-\lceil \frac{2s^2+5s+2}{3} \rceil$	$p = 0, 1, \ldots, 2t+2$ $q = t$

$\alpha_2(a_{p,q}a_{p+1,q}) = p + (t+k-1)(2t+2k+5) + 8 - 2\lceil\frac{2s^2+5s+2}{3}\rceil$	$p = 0, \ldots, 2q+1$ $q = t+k$ $k = 1, \ldots, s-t-1$
$\alpha_2(a_{p,q}a_{p,q+1}) = p + (t+k)(2t+2k+5) + 5 - 2\lceil\frac{2s^2+5s+2}{3}\rceil$	$p = 0, 1, \ldots, 2(t+k+1)$ $q = t+k$ $k = 1, \ldots, s-t-1$
$\alpha_2(a_{p,q}a_{p+1,q}) = p + (s-1)(2s+5) + 8 - 2\lceil\frac{2s^2+5s+2}{3}\rceil$	$q = s$ $p = 0, 1, \ldots, 2s-1.$

Table 2 Several s's and t's such that $t(t+1) \leq \lceil\frac{2s^2+5s+2}{3}\rceil - 1$

s	t		s	t		s	t		s	t		s	t		s	t
1	1		6	5		11	9		16	13		21	17		26	21
2	2		7	6		12	10		17	14		22	18		27	22
3	2		8	7		13	11		18	15		23	19		28	23
4	3		9	7		14	11		19	16		24	20		29	24
5	4		10	8		15	12		20	16		25	20		30	25

We then have the weights of the edges as follows:

weights	p and q
$wt_{\alpha_2}(a_{p,q}a_{p+1,q}) = p + 2q^2 + 3q + 3$	$p = 0, 1, \ldots, 2q+1$ $q = 0, 1, \ldots, t$
$wt_{\alpha_2}(a_{p,q}a_{p,q+1}) = p + 2q^2 + 5q + 5$	$p = 0, 1, \ldots, 2q+2$ $q = 1, 2, \ldots, t$
$wt_{\alpha_2}(a_{p,q}a_{p+1,q}) = p + (t+k-1)(2t+2k+5) + 8$	$p = 0, 1, \ldots, 2q+1$ $q = t+k$ $k = 1, \ldots, s-t-1$
$wt_{\alpha_2}(a_{p,q}a_{p,q+1}) = p + (t+k)(2t+2k+5) + 5$	$p = 0, 1, \ldots, 2(t+k)+1$ $q = t+k$ $k = 1, \ldots, s-t-1$
$wt_{\alpha_2}(a_{p,q}a_{p+1,q}) = p + 2s^2 + 3s + 3$	$q = s$ $p = 0, 1, \ldots, 2s-1$

The weights constitute numbers from 3 up to $2s^2 + 5s + 2$ and all different. This completes the proof. □

For the third observation, we consider the double odd staircase graph of level $s \geq 1$ denoted by $DOSC_s$ (see Figure 3). We have

$$V_{DOSC_s} = \{l_{p,q} | p = 1, 2, \ldots, 2s-1, q = \lceil\frac{p-1}{2}\rceil, \ldots, s\} \cup \{r_{p,q} | p = 1, 2, \ldots, 2s-1,$$
$$q = \lceil\frac{p-1}{2}\rceil, \ldots, s\}$$

and E_{DOSC_s} which consists of edges as given below

edges	p	q
$l_{p,q}r_{p,q}$	1	$0, 1, 2, \ldots, s$
$l_{p+1,q}l_{p,q}$	$1, 2, \ldots, 2s-2$	$p, \ldots, 2s-1$
$r_{p,q}r_{p+1,q}$	$1, 2, \ldots, 2s-2$	$p, \ldots, 2s-1$
$l_{p,q}l_{p,q+1}$	$1, 2, \ldots, 2s-1$	$\left\lceil\frac{p-1}{2}\right\rceil, \ldots, s-1$
$r_{p,q}l_{p,q+1}$	$1, 2, \ldots, 2s-1$	$\left\lceil\frac{p-1}{2}\right\rceil, \ldots, s-1.$

By a routine counting we have that $|V_{DOSC_s}| = 2s^2 + 4s - 2$ and $|E_{DOSC_s}| = 4s^2 + 3s - 3$.

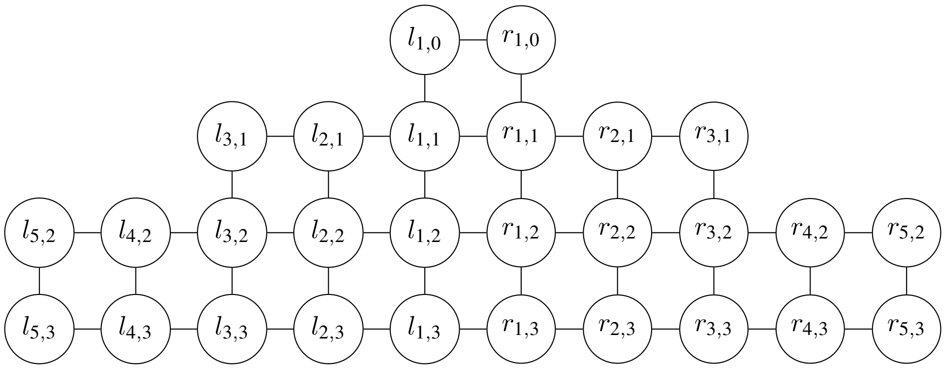

Figure 3: Double Odd Staircase Graph $DOSC_3$

In the following theorem we give the exact value of tes of $DOSC_s$ for any $s \geq 1$.

Theorem 2.3. *Let $DOSC_s$ be the double odd staircase graph of $s \geq 1$ level. Then the total edge irregularity strength of $DOSC_s$ is*

$$tes(DOSC_s) = \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil.$$

Proof. It is easy to observe that the maximum degree of the double odd staircase graph is 2 or 4, for $s = 1$ an $s \geq 2$, respectively. Therefore, by Theorem 1.1, we have

$$tes(DOSC_s) \geq \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil.$$

To complete the prove it is sufficient to show that $tes(DOSC_s) \leq \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil$ by defining a total edge irregularity m-labelling with $m = \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil$. Prior, we determine the largest positive integer t such that

$$2t^2 - t + 1 \leq \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil.$$

Table 3 Several s's and t's such that $2t^2 - t + 1 \leq \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil$

s	t		s	t		s	t		s	t		s	t
1	1		6	5		11	9		16	13		21	17
2	2		7	6		12	10		17	14		22	18
3	2		8	7		13	11		18	15		23	19
4	3		9	7		14	11		19	16		24	20
5	4		10	8		15	12		20	16		25	20

We define a total m-labelling

$$\alpha_3 : V_{DOSC_s} \cup E_{DOSC_s} \rightarrow \left\{1, 2, \ldots, \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil \right\}$$

with $m = \left\lceil \frac{4s^2 + 3s - 1}{3} \right\rceil$ by the following definition

edges and vertices label	p and q
$\alpha_3(l_{p,q}r_{p,q}) = 1$	$p = 1$ $q = 0$
$\alpha_3(l_{1,q}r_{1,q}) = 3q + 1$	$1 \leq q \leq s$
$\alpha_3(l_{p,q}) = 2q^2 - q + 1$	$p = 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t$
$\alpha_3(r_{p,q}) = 2q^2 - q + 1$	$p = 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t$
$\alpha_3(l_{p,q}l_{p+1,q}) = -p + 3q + 1$	$p = 1, \ldots, 2q$ $q = 1, 2, \ldots, t - 1$
$\alpha_3(l_{p,q}l_{p+1,q}) = -p + 3q + 1$	$p = 1, \ldots, 2t - 2$ $q = t$
$\alpha_3(l_{p,q}l_{p,q+1}) = -p + 3q + 2$	$p = 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t - 1$
$\alpha_3(r_{p,q}r_{p+1,q}) = p + 3q + 1$	$p = 1, \ldots, 2q$ $q = 1, 2, \ldots, t - 1$
$\alpha_3(r_{p,q}r_{p+1,q}) = p + 3q + 1$	$p = 1, \ldots, 2t - 2$ $q = t$
$\alpha_3(r_{p,q}r_{p,q+1}) = p + 3q + 1$	$p = 1, \ldots, 2q + 1$ $q = 1, 2, \ldots, t - 1.$

The labelling is complete in the case $t = s$. If $t < s$, we continue the labelling as follows.

edges and vertices label	p and q
$\alpha_3(l_{p,q}l_{p+1,q}) = -p + 3q + 1$	$p = 2t - 1, 2t$ $q = t$
$\alpha_3(r_{p,q}r_{p+1,q}) = p + 3q + 1$	$p = 2t - 1, 2t$ $q = t$
$\alpha_3(l_{p,q}) = \left\lceil \dfrac{4q^2 + 3q - 1}{3} \right\rceil$	$p = 1, 2, \ldots, 2q + 1$ $q = t + 1, \ldots, s$
$\alpha_3(r_{p,q}) = \left\lceil \dfrac{4q^2 + 3q - 1}{3} \right\rceil$	$p = 1, 2, \ldots, 2q + 1$ $q = t + 1, \ldots, s$
$\alpha_3(l_{p,q}l_{p,q+1}) = 2t^2 + 6t + 4 - i - \left\lceil \dfrac{4q^2 + 3q - 1}{3} \right\rceil$	$p = 1, \ldots, 2p + 1$ $q = t$
$\alpha_3(l_{p,q}l_{p+1,q}) = 4(t + k)^2 + (t + k) - p + 3 - 2\left\lceil \dfrac{4q^2 + 3q - 1}{3} \right\rceil$	$p = 1, \ldots, 2q$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$\alpha_3(l_{p,q}l_{p,q+1}) = 4(t + k)^2 + 5(t + k) - p + 5 - 2\left\lceil \dfrac{4q^2 + 3q - 1}{3} \right\rceil$	$p = 1, \ldots, 2q + 1$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$\alpha_3(l_{p,q}l_{p+1,q}) = 4s^2 + s - p + 1 - 2\left\lceil \dfrac{4q^2 + 3q - 1}{3} \right\rceil$	$p = 1, \ldots, 2q - 2$ $q = s$

$\alpha_3(r_{p,q}r_{p,q+1}) = 2t^2 + 2t + 2 + p - \left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p = 1, \ldots, 2q+1$ $q = t$
$\alpha_3(r_{p,q}r_{p+1,q}) = 4(t+k)^2 + (t+k) + p + 3 - 2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p = -(2q+1), \ldots, 2q$ $q = t+k$ $k = 1, 2, \ldots, s-t-1$
$\alpha_3(r_{p,q}r_{p,q+1}) = 4(t+k)^2 + 5(t+k) + p + 4 - 2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p = 1, \ldots, 2q+1$ $q = t+k$ $k = 1, 2, \ldots, s-t-1$
$\alpha_3(r_{p,q}r_{p+1,q}) = 4s^2 + s + p + 1 - 2\left\lceil \frac{4q^2+3q-1}{3} \right\rceil$	$p = 1, \ldots, 2q-2$ $q = s.$

We obtain edge weights as follows:

weight	p and q
$wt_{\alpha_3}(l_{p,q}r_{p,q}) = 3$	$p = 1$ $q = 0$
$wt_{\alpha_3}(l_{p,q}r_{1,q}) = 4q^2 + q + 3$	$1 \le q \le s$
$wt_{\alpha_3}(l_{p,q}l_{p+1,q}) = 4q^2 + q - p + 3$	$p = 1, 2, \ldots, 2q$ $q = 1, 2, \ldots, t$
$wt_{\alpha_3}(l_{p,q}l_{p,q+1}) = 4q^2 + 5q - p + 5$	$p = 1, 2, \ldots, 2q+1$ $q = 1, 2, \ldots, t$
$wt_{\alpha_3}(l_{p,q}l_{p+1,q}) = 4(m+k)^2 + (m+k) - p + 3$	$p = 1, \ldots, 2q$ $q = t+k$ $k = 1, 2, \ldots, s-t-1$
$wt_{\alpha_3}(l_{p,q}l_{p,q+1}) = 4(t+k)^2 + 5(t+k) - p + 5$	$p = 1, \ldots, 2q+1$ $q = t+k$ $k = 1, 2, \ldots, s-t-1$
$wt_{\alpha_3}(l_{p,q}l_{p+1,q}) = 4s^2 + s - p + 1$	$p = 1, \ldots, 2q-2$ $q = s$
$wt_{\alpha_3}(r_{p,q}r_{p+1,q}) = 4q^2 + q + p + 3$	$p = 1, 2, \ldots, 2q$ $q = 1, 2, \ldots, t$
$wt_{\alpha_3}(r_{p,q}r_{p,q+1}) = 4q^2 + 5q + p + 4$	$p = 1, 2, \ldots, 2q$ $q = 1, 2, \ldots, t$
$wt_{\alpha_3}(r_{p,q}r_{p+1,q}) = 4(t+k)^2 + (t+k) + p + 3$	$p = 1, \ldots, 2q$ $q = t+k$ $k = 1, 2, \ldots, s-t-1$
$wt_{\alpha_3}(r_{p,q}r_{p,q+1}) = 4(t+k)^2 + 5(t+k) + p + 4$	$p = 1, \ldots, 2q+1$ $q = t+k$ $k = 1, 2, \ldots, s-t-1$ $k \ge 1$
$wt_{\alpha_3}(r_{p,q}r_{p+1,q}) = 4s^2 + s + p + 1$	$p = 1, \ldots, 2q-2$ $q = s.$

It is a routine to verify that all weights are distinct. Therefore the theorem is confirmed to be true. □

We now come to the last graph to observe, i.e. the mirror odd staircase graph of level $s \geq 1$ denoted by $MOSC_s$ (see Figure 4). We have

$$\begin{aligned} V_{MOSC_s} =& \{a_{p,q} | p = -1, 0, 1, q = 0, 1, 2, \ldots, s\} \cup \{a_{p,q} | p = 2, \ldots, 2s-1, q = \left\lceil \frac{p-1}{2} \right\rceil, \ldots, s\} \\ & \cup \{a_{p,q} | p = -2, \ldots, -(2s-1), q = -\left\lceil \frac{p-1}{2} \right\rceil, \ldots, s\} \end{aligned}$$

and E_{MOSC_s} which consists of edges as given below

edges	p	q
$a_{p,q}a_{p+1,q}$	$-1, 0$	$0, 1, 2, \ldots, s$
$a_{p,q}a_{p+1,q}$	$1, 2, \ldots, 2s-2$	$\left\lceil \frac{p}{2} \right\rceil, \ldots, s$
$a_{p,q}a_{p+1,q}$	$-(2s-1), \ldots, -2$	$-\left\lceil \frac{p}{2} \right\rceil, \ldots, s$
$a_{p,q}a_{p,q+1}$	$-1, 0, 1$	$0, 1, \ldots, s-1$
$a_{p,q}a_{p,q+1}$	$2, \ldots, 2s-1$	$\left\lceil \frac{p-1}{2} \right\rceil, \ldots, s-1$
$a_{p,q}a_{p,q+1}$	$-(2s-1), \ldots, -2$	$-\left\lceil \frac{p-1}{2} \right\rceil, \ldots, s-1.$

It is easy to check that $|V_{MOSC_s}| = 2s^2 + 5s - 1$ and $|E_{MOSC_s}| = 4s^2 + 5s - 2$.

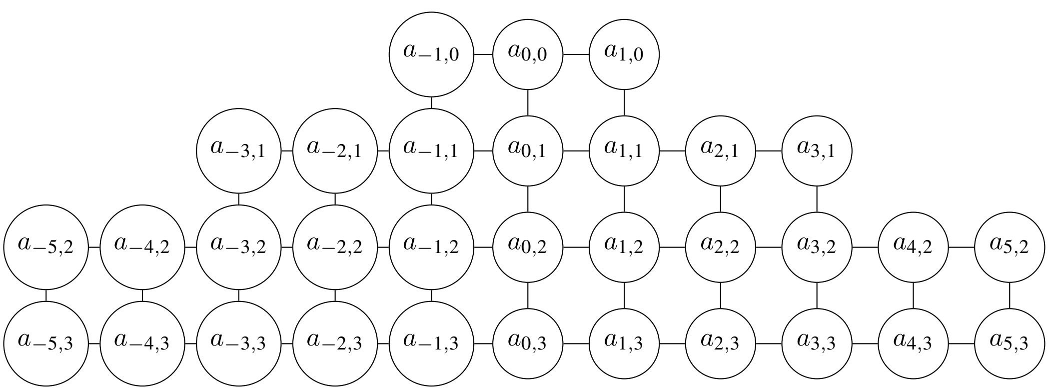

Figure 4: Mirror Odd Staircase Graph $MOSC_3$

Theorem 2.4. *Let for any $s \geq 1$, $MOSC_s$ be the mirror odd staircase graph of s level. Then the total edge irregularity strength of $MOSC_s$ is*

$$tes(MOSC_s) = \left\lceil \frac{4s^2 + 5s}{3} \right\rceil.$$

Proof. It is clear that the maximum degree of $MOSC_3$ is 3 for $s = 1$ and 4 for $s \neq 1$. Thus, we obtain

$$tes(MOSC_s) \geq \left\lceil \frac{4s^2 + 5s}{3} \right\rceil.$$

For completing the proof we show that $tes(MOSC_s) \leq \left\lceil \frac{4s^2 + 5s}{3} \right\rceil$ by constructing a total edge irregularity m-labelling with $m = \left\lceil \frac{4s^2 + 5s}{3} \right\rceil$. Similarly to the previous graphs, before we define the labelling, we determine the largest positive integer t such that

$$2t^2 + 1 \leq \left\lceil \frac{4s^2 + 5s}{3} \right\rceil.$$

Table 4 Several s's and t's such that $2t^2 + 1 \leq \left\lceil \frac{4s^2+5s}{3} \right\rceil$

s	t		s	t		s	t		s	t		s	t
1	1		6	5		11	9		16	13		21	17
2	2		7	6		12	10		17	14		22	18
3	2		8	7		13	11		18	15		23	19
4	3		9	7		14	11		19	16		24	20
5	4		10	8		15	12		20	16		25	20

We define a total m-labelling

$$\alpha_4 : V_{MOSC_s} \cup E_{MOSC_s} \rightarrow \left\{1, 2, \ldots, \left\lceil \frac{4s^2+5s}{3} \right\rceil\right\}$$

with $m = \left\lceil \frac{4s^2+5s}{3} \right\rceil$ in the following manner:

edges and vertices label	p and q
$\alpha_4(a_{p,0}) = 1$	$q = -1, 0, 1$
$\alpha_4(a_{p,q}) = 2q^2 + 1$	$p = -(2q+1), \ldots, 2q+1$ $q = 1, 2, \ldots, t$
$\alpha_4(a_{p,q}a_{p+1,q}) = p + 3q + 2$	$p = -(2q+1), \ldots, 2q$ $q = 0, \ldots, t-1$
$\alpha_4(a_{p,q}a_{p+1,q}) = p + 3q + 2$	$p = -(2q+1), \ldots, 2q-2$ $q = t$
$\alpha_4(a_{p,q}a_{p,q+1}) = p + 3q + 2$	$p = -(2q+1), \ldots, 2q+1$ $q = 0, \ldots, t-1.$

We stop the labelling whenever $t = s$. For the case $t < s$, then we continue with the following labels

edges and vertices label	p and q
$\alpha_4(a_{p,q}a_{p+1,q}) = p + 3q + 2$	$p = 2t-1, 2t$ $q = t$
$\alpha_4(a_{p,q}) = \left\lceil \frac{4s^2+5s}{3} \right\rceil$	$p = -(2q+1), \ldots, 2q+1$ $q = t+1, \ldots, s$
$\alpha_4(a_{p,q}a_{p,q+1}) = p + (2t+3)(t+2) - 1 - \left\lceil \frac{4s^2+5s}{3} \right\rceil$	$p = -(2q+1), \ldots, 2q+1$ $q = t$
$\alpha_4(a_{p,q}a_{p+1,q}) = p + 4(t+k)^2 + 3(t+k) + 4 - 2\left\lceil \frac{4s^2+5s}{3} \right\rceil$	$p = -(2q+1), \ldots, 2q$ $q = t+k$ $k = 1, 2, \ldots, s-t-1$
$\alpha_4(a_{p,q}a_{p,q+1}) = p + 4(t+k)^2 + 7(t+k) + 6 - 2\left\lceil \frac{4s^2+5s}{3} \right\rceil$	$p = -(2q+1), \ldots, 2q+1$ $q = t+k$ $k = 1, 2, \ldots, s-t-1$
$\alpha_4(a_{p,q}a_{p+1,q}) = p + 4s^2 + 3s + 2 - 2\left\lceil \frac{4s^2+5s}{3} \right\rceil$	$p = -(2q-1), \ldots, 2q-2$ $q = s.$

From the definition of α_4 we obtain the weight of all edges as follows:

weight	p and q
$wt_{\alpha_4}(a_{p,q}a_{p+1,q}) = p + 4$	$p = -1, 0$ $q = 0$
$wt_{\alpha_4}(a_{p,q}a_{p+1,q}) = p + 4q^2 + 3q + 4$	$p = -(2q+1), \ldots, 2q$ $q = 1, 2, \ldots, t$
$wt_{\alpha_4}(a_{p,q}a_{p+1,q}) = p + 4(t+k)^2 + 3(t+k) + 4$	$p = -(2q+1), \ldots, 2q$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$wt_{\alpha_4}(a_{p,q}a_{p,q+1}) = p + 6$	$p = -1, 0, 1$ $q = 0$
$wt_{\alpha_4}(a_{p,q}a_{p,q+1}) = p + 4q^2 + 7q + 6$	$p = -(2q+1), \ldots, 2q+1$ $q = 1, 2, \ldots, t$
$wt_{\alpha_4}(a_{p,q}a_{p,q+1}) = p + 4(t+k)^2 + 7(t+k) + 6$	$p = -(2q+1), \ldots, 2q+1$ $q = t + k$ $k = 1, 2, \ldots, s - t - 1$
$wt_{\alpha_4}(a_{p,q}a_{p+1,q}) = p + 4s^2 + 3s + 2$	$p = -(2q-1), \ldots, 2s-2$ $q = s.$

It can be verified in a routine way that the weights of all edges in $E(MOSC_s)$ are all different. Hence the theorem is proved. □

3 Conclusion remarks

From Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4. we conclude that the $tes(\Gamma)$ for $\Gamma = OSC_s, ESC_S, DOSC_s, MOSC_s$, is equal to $\left\lceil \frac{|E_\Gamma|+2}{3} \right\rceil$. These results obviously support the conjecture of Ivanco and Jendrol [6].

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Author information

Yeni Susanti, Department of Mathematics, Universitas Gadjah Mada, Indonesia.
E-mail: `yeni_math@ugm.ac.id`

Muhammad Arinal Haq, Department of Mathematics, Universitas Gadjah Mada, Indonesia.
E-mail: `muhammmad.arinal.haq@mail.ugm.ac.id`

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