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A generalization of the Bessel-Riesz's ultrahyperbolic kernel.

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Abstract The purpose of this article is to present a generalization of the well-known ultrahyperbolic kernel of Nozaki using weighted generalized functions associated with nondegenerate quadratic forms. Some of its properties related to the iterated ultrahyperbolic Bessel differential operator are presented; for example, that for a certain value of the parameter, it is an elementary solution.

1 Introduction and Preliminaries.

The importance of Marcel Riesz's works in the field of Fractional Calculus is well known, especially in the study of the Riemann-Liouville integral both in Euclidean and hyperbolic spaces. It is just necessary to see his extensive and interesting work [1] to understand it. In his desire to introduce the Riemann-Liouville integral in a space with the ultrahyperbolic metric, generalizing what was done by Riesz [1], Y. Nozaki [2] introduced his kernel, which allows him to define it via the convolution operation.

Indeed, given $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ...y_n)$ as two points in $\mathbb{R}^n$ with $x_i \geq 0$, $y_i \geq 0, i = 1, 2, ...n$; Riesz considered the hyperbolic or Lorenzian distance between these points given by

$$r_{xy} = \sqrt{(x_1 - y_1)^2 - (x_2 - y_2)^2 - ... - (x_n - y_n)^2}, \tag{1.1}$$

where x is taken as a fixed point and y as a variable point.

The set $r_{xy}^2 > 0$ and $x_1 - y_1 > 0$ is the retrograde light cone, while the condition $x_1 - y_1 < 0$ defines the direct cone. Under these conditions, the Riemann-Liouville integral of order α of the function $f(x)$ is introduced as

$$I^\alpha f(x) = \frac{1}{H_n(\alpha)} \int f(y) r_{xy}^{\alpha-n} dy \tag{1.2}$$

where

$$H_n(\alpha) = \pi^{\frac{n-2}{n}} 2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha+2-n}{2}\right) \tag{1.3}$$

and the integral converges for $\alpha > n - 2$. It can be seen that $I^\alpha f$ satisfies some important relationships, such as:

$$I^\alpha I^\beta = I^{\alpha+\beta}; \tag{1.4}$$

$$\Box I^{\alpha+2} = I^\alpha \tag{1.5}$$

where $\Box$ denotes the wave operator

$$\Box = \frac{\partial^2}{\partial x_1} - \frac{\partial^2}{\partial x_2} - ... - \frac{\partial^2}{\partial x_n}. \tag{1.6}$$

Nozaki [2], generalized results due to Riesz. To do this, given the points $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ...y_n)$ he considered the ultrahyperbolic distance

$$r_{xy}^2 = \sum_{i=1}^{p}(x_i - y_i)^2 - \sum_{i=p+1}^{p+q}(x_i - y_i)^2; \quad p + q = n, \tag{1.7}$$

where n is the dimension of the space.

Analogously to what was done by Riesz, considering $P = x$ as a fixed point and $Q = y$ as a variable point, he took the inverse cone with vertex at P defined by the relations as D^P

$$r_{PQ}^2 > 0, \;\; x_1 - y_1 > 0, \tag{1.8}$$

and defined $\phi(P, Q)$ to the kernel given by

$$\phi(P, Q) = \frac{r_{PQ}^{\alpha-n}}{K_n(\alpha)} = \frac{r_{+}^{\alpha-n}}{K_n(\alpha)}, \tag{1.9}$$

where

$$K_n(\alpha) = \pi^{\frac{n-1}{2}} \frac{\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{1.10}$$

It can be observed that if in (1.10) we consider $p = 1$, $K_n(\alpha)$ reduce to the hyperbolic given by (1.3).

According Remmark 1, page 76 of [2], we denote the convolution of $f(y)$ with the kernel $\phi_\alpha(P, Q)$ by $(f * \phi_\alpha)(P)$, we may write the Riemann-Liouville integral in the form

$$J^\alpha f(P) = (f * \phi_\alpha)(P). \tag{1.11}$$

Among other properties, it is proved (Theorem 3, [2]) that

$$\Box\,(f * \phi_{\alpha+2}) = f * \phi_\alpha, \tag{1.12}$$

where

$$\Box = \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{p+q}\frac{\partial^2}{\partial x_i^2}; \quad p + q = n. \tag{1.13}$$

The property (1.12) is analogous to (1.5).

The generalized functions associated with quadratic forms and denoted by r^λ, P_+^λ, P_-^λ, $(P + i0)^\lambda$, $(P - i0)^\lambda$, $\left(m^2 + P + i0\right)^\lambda$, $\left(m^2 + P - i0\right)^\lambda$ are important contributions due to Gelfand and Shilov [3] that allow to express the solutions of differential equations and also of potentials.

Let

$$P(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots x_{p+q}^2, \;\; p + q = n, \tag{1.14}$$

and let $\Gamma_+ = \{x \in \mathbb{R}^n, P(x) > 0, x_1 > 0\}$ and $\Gamma_- = \{x \in \mathbb{R}^n, P(x) > 0, x_1 < 0\}$. Let λ be a complex number. According to Gelfand [3], the generalized function P_+^λ is defined by means of the integral

$$\left(P_+^\lambda, \varphi\right) = \int_{P>0} P^\lambda(x)\varphi(x)dx, \tag{1.15}$$

this integral converges for $Re(\alpha) \geq 0$ and is an analytic function of λ. For the values $Re(\lambda) \leq 0$ the analytic continuation is used to extend the definition of $\left(P_+^\lambda, \varphi\right)$. Trione [5] considers the family of functions R introduced by Nozaki

$$R_\alpha\left(P(x)\right) = \begin{cases} \frac{P^{\frac{\alpha-n}{2}}(x)}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{1.16}$$

where α is a complex parameter, n the dimension of the space and $K_n(\alpha)$ is given by (1.10) and in a simple and synthetic way, that we will adopt in this work, she proves the properties that we gather below:

$$\square\, P^{\frac{\alpha+2-n}{2}} = \alpha(\alpha+2-n)P^{\frac{\alpha-n}{2}}, \tag{1.17}$$

$$K_n(\alpha+2) = \alpha(\alpha+2-n)K_n(\alpha), \tag{1.18}$$

$$\square R_{\alpha+2}(P) = R_\alpha(P), \tag{1.19}$$

$$R_{-2k}(P) = \square^k\delta, \quad k=0,1,2,... \tag{1.20}$$

$$R_0(P) = \delta, \tag{1.21}$$

$$\square^k R_{2k}(P) = \delta, \quad k=0,1,2,... \tag{1.22}$$

$$\square^k R_\alpha(P) = R_{\alpha-2k}(P), \tag{1.23}$$

where δ is the Dirac delta and $\square$ is the ultrahyperbolic operator given by (1.13).

Now let's define the Bessel ultrahyperbolic operator as

$$\square_\gamma^B = B_{x_1} + B_{x_2} + ... + B_{x_p} - B_{x_{p+1}} - ... - B_{x_{p+q}}, \quad p+q=n, \tag{1.24}$$

where

$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}; \quad \gamma_i > 0, \quad i=1,2,...,n. \tag{1.25}$$

H. Yildirim et. al. [6] demonstrated that the generalized functions R_{2k}^B is the unique elementary solution of the ultrahyperbolic Bessel operator (1.24), iterated k-times. Then,

$$(\square_\gamma^B)^k R_{2k}^B(x) = \delta, \tag{1.26}$$

where

$$R_{2k}^B(x) = \frac{\left(x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2\right)^{\frac{2k-n-2|\nu|}{2}}}{K_n(2k)}, \tag{1.27}$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|\nu|-1}{2}}\Gamma\left(\frac{2+2k-n-2|\nu|}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\Gamma(2k)}{\Gamma\left(\frac{2+2k-p-2|\nu|}{2}\right)\Gamma\left(\frac{p-2k}{2}\right)}. \tag{1.28}$$

Aguirre [7] demonstrated certain relations between the ultrahyperbolic Bessel operator iterated k-times and the $R_\alpha^B(x)$ kernel, and studied in particular the kernel R_0^B.

2 Elements of the theory of weighted generalized functions associated with quadratic forms

In this paragraph some elements of the theory of weighted generalized functions associated with quadratic forms introduced by E. Shishkina [8] are present.

For this, the space $\mathbb{R}_+^n$ is considered:

$$\mathbb{R}_+^n = \{x=(x_1,...,x_n);\, x_i>0,\, i=1,2,...,n\} \tag{2.1}$$

and Ω is an open set in $\mathbb{R}^n$, symmetric with respect to each hyperplane $x_i=0$, $i=1,2,...,n$. Let $\Omega_+ = \Omega\cap\mathbb{R}_+^n$ and $\overline{\Omega}_+ = \Omega\cap\overline{\mathbb{R}_+^n}$ be its closure, where

$$\overline{\mathbb{R}_+^n} = \{x=(x_1,...,x_n);\, x_i\geq 0,\, i=1,2,...,n\}. \tag{2.2}$$

Let

$$C_{ev}^{0,\infty}(\overline{\Omega}_+) = \left\{ \begin{array}{c} f\in C^\infty(\overline{\Omega}_+) \text{ with compact support, even with respect} \\ \text{to each variable } x_i,\ i=1,...,n \end{array} \right\} \tag{2.3}$$

A multi-index $\gamma = (\gamma_1, ..., \gamma_n)$ consists of fixed positive real numbers $\gamma_i > 0$, $i = 1, 2, ..., n$ and $|\gamma| = \gamma_1 + ... + \gamma_n$. Let $P = P(x)$ be given by (1.14) and let φ be a function in the space $C^{0,\infty}_{ev}(\overline{\Omega}_+)$, the weighted generalized function $P^{\lambda}_{\gamma,+}$ is defined by the integral

$$\left(P^{\lambda}_{\gamma,+}, \varphi\right) = \int_{\{P(x)>0\}^+} P(x)^{\lambda}\varphi(x)x^{\gamma}dx, \tag{2.4}$$

where $\{P(x) > 0\}^+ = \{x \in \mathbb{R}^n_+ : P(x) > 0\}$, $\lambda \in \mathbb{C}$ y $x^{\lambda} = x_1^{\lambda_1}x_2^{\lambda_2}...x_n^{\gamma_n}$.

The generalized convolution product is defined by the formula

$$(f * g)_{\gamma}(x) = \int_{\mathbb{R}^n_+} f(y)\left({}^{\gamma}T^y_x g\right)(x)y^{\gamma}dy, \quad f, g \in S_{ev} \tag{2.5}$$

where

$$S_{ev} = \left\{ f \in C^{\infty}_{ev} : \sup_{x\in\mathbb{R}^n_+} |x^{\alpha}D^{\beta}f(x)| < \infty, \forall\, \alpha, \beta \in \mathbb{Z}^n_+ \right\} \tag{2.6}$$

and ${}^{\gamma}T^y_x$ is the multidimensional generalized traslation given by

$$\left({}^{\gamma}T^y_x f\right)(x) = \left({}^{\gamma_1}T^{y_1}_{x_1}f, ...,{}^{\gamma_n} T^{y_n}_{x_n} f\right)(x) \tag{2.7}$$

and the unidimensional generalized traslations

$$\left({}^{\gamma_i}T^{y_i}_{x_i}f\right)(x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma_i}{2}\right)}\int_0^{\pi} f(x_1, ..., x_{i-1}, \sqrt{x_i^2 + \tau_i^2 - 2x_iy_i\cos\varphi_i}, x_{i+1}, ..., x_n)\sin^{\gamma_i-1}\varphi_i d\varphi_i. \tag{2.8}$$

For u a weighted generalized function belongs to S'_{ev} and $f \in S_{ev}$, we have

$$(u * f)_{\gamma}(x) = (u,{}^{\gamma} T^y_x f)(x). \tag{2.9}$$

In the development of this work the Hankel transform we will be used, which for a function $f \in L^{\gamma}_1$ is defined as

$$\mathbb{F}_{\gamma}[f](\xi) = \mathbb{F}_{\gamma}[f(x)](\xi) = \int_{\mathbb{R}^n_+} f(x)\mathbf{j}_{\gamma}(x;\xi)x^{\gamma}dx \tag{2.10}$$

where

$$\mathbf{j}(x, \xi) = \prod_{i=1}^{n} j_{\frac{n-1}{2}}(x_i, \xi_i), \quad \gamma_i > 0, \; i = 1, 2, ..., n; \tag{2.11}$$

and

$$j_{\nu}(r) = \frac{2^{\nu}\Gamma(\gamma + 1)}{r^{\nu}}J_{\gamma}(r); \tag{2.12}$$

$J_{\gamma}(r)$ is the Bessel function of the first kind of order ν.

Among the properties it verifies, we can point out:

$$\mathbb{F}_{\gamma}\delta_{\gamma}(x) = 1 \tag{2.13}$$

$$\mathbb{F}_{\gamma}[f * g](x) = \mathbb{F}_{\gamma}[f](x)\mathbb{F}_{\gamma}[g](x) \tag{2.14}$$

As Shishkina [8] stated, the weighted generalized function $P^{\lambda}_{\gamma,+}$ associated with quadratic forms are rised for finding fundamental solutions of iterated B-ultrahyperbolic differential equations, i.e.

$$\Box^k_B u = \delta_{\gamma}(x) \tag{2.15}$$

when $k \in \mathbb{N}$, $x \in \mathbb{R}^n$, $x_i > 0, i = 1, 2, ..., n$ and $\delta_{\gamma}(x)$ is the Dirac delta defined by

$$(\delta_{\gamma}, \varphi)_{\gamma} = \int_{\mathbb{R}^n_+} \delta(x)\varphi(x)x^{\gamma}dx = \varphi(0), \quad \varphi(x) \in S_{ev}. \tag{2.16}$$

Another interesting application is the construction of ultrahyperbolic Riesz potentials with the Bessel operator.

3 Ultrahyperbolic Bessel-Riesz kernel with weighted generalized function.

We start by defining the kernel that will be the main object of study will be defined and it will called Marcel Riesz's B-ultrahyperbolic kernel.

$$R^B_{\alpha,\gamma}\left(P_{\gamma,+}(x)\right) = \begin{cases} \dfrac{P_{\gamma,+}^{\frac{\alpha-n-|\gamma|}{2}}(x)}{\tilde{K}_{n,\gamma}(\alpha)} & \text{si } x \in \Gamma^+ \\ 0 & \text{si } x \notin \Gamma^+, \end{cases} \tag{3.1}$$

where $\Gamma^+ = \{x \in \mathbb{R}^n, P(x) > 0, x_i > 0, i = 1, 2, ..., n\}$ and $\alpha \in \mathbb{C}$,

$$\tilde{K}_{n,\gamma}(\alpha) = \frac{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{2+\alpha-n-|\gamma|}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\sqrt{\pi}\,\Gamma\left(\frac{2+\alpha-p-|\gamma'|}{2}\right)\Gamma\left(\frac{p+|\gamma'|-\alpha}{2}\right)} \tag{3.2}$$

where p is the number of positive terms of the quadratic form (1.14) and $\gamma' = (\gamma_1, ..., \gamma_p)$. It can be easily seen that if $|\gamma| = 0$, $R^B_{\alpha,\gamma}\left(P_{\gamma,+}(x)\right)$ coincides with (1.16) and in the case of being $p = 1$ and $|\gamma| = 0$ it turns out that (3.1) coincides with the kernel of M. Riesz ([1], p.31).

We also observe that the kernel defined by (3.1) is formally analogous to the one introduced and studied by Yildirim et. al. ([6], f. 10) and they share similar properties.

To study the action of the operator $\square^B_\gamma$ on the kernel given by (3.1) it is necessary to take into account that, from formula (23) of [8] it follows

$$\square^B_\gamma P^{\lambda+1}(x) = 4(\lambda+1)\left(\lambda + \frac{n+|\gamma|}{2}\right)P^\lambda(x). \tag{3.3}$$

Taking $\lambda = \frac{\alpha-n-|\gamma|}{2}$ we have

$$\alpha(2+\alpha-n-|\gamma|)\square^B_\gamma P^{\frac{\alpha-n-|\gamma|}{2}+1}(x) = P^{\frac{\alpha-n-|\gamma|}{2}}(x). \tag{3.4}$$

According to Shishkina (cf.[8], f.(28)) we have

$$\left(P_{\gamma,+}^{\frac{\alpha-n-|\gamma|}{2}}, \varphi\right)_\gamma = \frac{1}{\alpha(2+\alpha-n-|\gamma|)}\left(P_{\gamma,+}^{\frac{\alpha-n-|\gamma|}{2}+1}, \square^B_\gamma \varphi\right)_\gamma \tag{3.5}$$

then

$$\square^B_\gamma P_{\gamma,+}^{\frac{\alpha-n-|\gamma|}{2}+1} = \alpha(2+\alpha-n-|\gamma|)P_{\gamma,+}^{\frac{\alpha-n-|\gamma|}{2}}. \tag{3.6}$$

Remark 3.1. It can be seen that if $\gamma = (0, ..., 0)$, (3.6) coincides with (1.17).

Taking into account the same procedure performed by Trione in [5], to prove (1.18), after long but simple operations we obtain

$$\tilde{K}_{n,\gamma}(\alpha+2) = \alpha(\alpha+2-n-|\gamma|)\tilde{K}_{n,\gamma}(\alpha). \tag{3.7}$$

Therefore, from what has been exposed above, the following can be stated

Lemma 3.2. *Given $R^B_{\alpha,\gamma}$, the kernel defined by (3.1) and $\square^B_\gamma$ the operator given by (1.24). Then it is verified*

$$(\square^B_\gamma)^k R^B_{\alpha,\gamma}(P_\gamma(x)) = R^B_{\alpha-2\cdot k,\gamma}(P(x)), \quad k = 0, 1, 2, ... \tag{3.8}$$

Proof. From (3.1), (3.6) and (3.7), we have

$$\square^B_\gamma R^B_{\alpha+2,\gamma}(P_\gamma(x)) = R^B_{\alpha,\gamma}(P_\gamma(x)). \tag{3.9}$$

Applying $\square^B_\gamma$ on both side of (3.9)

$$\square^B_\gamma\left(\square^B_\gamma R^B_{\alpha+2,\gamma}(P_\gamma(x))\right) = \square^B_\gamma R^B_{\alpha,\gamma}(P_\gamma(x)) = R^B_{\alpha-2,\gamma}(P_\gamma(x)) \tag{3.10}$$

that is

$$(\square^B_\gamma)^2 R^B_{\alpha,\gamma}(P(x)) = R^B_{\alpha-4,\gamma}(P_\gamma(x)) = R^B_{\alpha-2\cdot 2,\gamma}(P_\gamma(x)), \tag{3.11}$$

and in general

$$(\square^B_\gamma)^k R^B_{\alpha,\gamma}(P_\gamma(x)) = R^B_{\alpha-2\cdot k,\gamma}(P(x)), \quad k = 0, 1, 2, ... \tag{3.12}$$

which is what was intended to be proved. □

Remark 3.3. It can be seen that if $\gamma=(0,...,0)$, (3.8) coincides with (1.23).

Lemma 3.4. *$R^B_{2k,\gamma}(P(x))$ is a convolutor in $D'(\mathbb{R}^n)$.*

Proof. We will prove that the kernel $R^B_{2k,\gamma}(P_\gamma(x))$ is a distribution whose support is the origin 0, i.e. it is a combination of δ. We start by considering $\tilde{K}_{n,\gamma}(\alpha)$ given by (3.2). Indeed, we rewrite

$$\tilde{K}_{n,\gamma}(\alpha)=\frac{\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{2+\gamma-n-|\gamma|}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\sqrt{\pi}\,\Gamma\left(\frac{2+\alpha-p-|\gamma'|}{2}\right)\Gamma\left(\frac{p+|\gamma'|-\alpha}{2}\right)}. \tag{3.13}$$

Taking into account the duplication formula of the Gamma function ([9], Theorem 2.10)

$$\Gamma(\alpha)=\frac{2^{\alpha-1}}{\pi^{1/2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right), \tag{3.14}$$

then

$$\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)=\frac{2^{\alpha-1}}{\pi^{1/2}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right), \tag{3.15}$$

and

$$\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)=\frac{\pi}{\cos\left(\frac{\pi\alpha}{2}\right)}. \tag{3.16}$$

While the product of the denominator of (3.13) can be written as

$$\Gamma\left(\frac{2+\alpha-p-|\gamma'|}{2}\right)\Gamma\left(\frac{p+|\gamma'|-\alpha}{2}\right)=\frac{\pi}{\sin\left[\pi\left(\frac{p+|\gamma'|-\alpha}{2}\right)\right]}. \tag{3.17}$$

The function $\Gamma\left(\frac{2+\gamma-n-|\gamma|}{2}\right)$ is holomorphic in $\alpha=-2k$, $n+|\gamma|$ odd and n odd. Then $|\gamma|$ must be even, and then:

(i) $|\gamma'|$ and $|\gamma''|$ are odd;

(ii) or $|\gamma'|$ and $|\gamma''|$ are even.

Therefore, in $\alpha=-2k$, taking into account ([9], Theorem 2.12) we have

$$\Gamma\left(1-\frac{2k+n+|\gamma|}{2}\right)=\frac{\pi}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)\sin\left[\pi\left(\frac{n+|\gamma|}{2}+k\right)\right]}. \tag{3.18}$$

On the other hand, according to Shishkina ([8], Theorem 39), and taking into account that the function $\Gamma(z)$ has a single pole at $z=-k$, k a nonegative integer with residues $Res_{z=-k}\Gamma(z)=\frac{(-1)^k}{k!}$ we have

$$\lim_{\alpha\to-2k}\frac{P_{\gamma,+}^{\frac{\alpha-n-|\gamma|}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)}\lim_{\alpha\to-2k}\left[\frac{\Gamma\left(\frac{2+\alpha-p-|\gamma'|}{2}\right)\Gamma\left(\frac{p+|\gamma'|-\alpha}{2}\right)}{\prod_{i=1}^n\Gamma\left(\frac{\alpha_i+1}{2}\right)\pi^{-1/2}\Gamma\left(\frac{2+\alpha-n-|\gamma|}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}\right]. \tag{3.19}$$

Finaly, from (3.14),(3.15), (3.16), (3.17), (3), (3.18) and (3.19) we have

$$R^B_{-2k,\gamma}(P_\gamma(x))=\lim_{\alpha\to-2k}\frac{P_{\gamma,+}^{\frac{\alpha-n-|\gamma|}{2}}}{\tilde{K}_{n,\gamma}(\alpha)}=(\square^B_\gamma)^k\delta_\gamma(x) \tag{3.20}$$

□

Remark 3.5. If in (3.20) we consider $\gamma=(0,0,...,0)$, we obtain the formula (1.20).

Remark 3.6. This formula (3.20) was obtained taking into consideration the hypotheses of Theorem 2, by Shishkina, [8].

Theorem 3.7. *$R^B_{2k,\gamma}(P_\gamma(x))$ is an elementary solution of the n-dimensional Bessel ultrahyperbolic differential operator iterated k-times.*

Proof. By Lemma 1, taking $\alpha = 2k$, we have

$$\left(\Box^B_\gamma\right)^k R^B_{2k,\gamma}(P_\gamma(x)) = R^B_{0,\gamma}(P(x)) \tag{3.21}$$

and taking in (3.20) $k = 0$, we get

$$R^B_{0,\gamma}(P_\gamma(x)) = \delta_\gamma. \tag{3.22}$$

Then, from (3.21) y (3.22) we have

$$\left(\Box^B_\gamma\right)^k R^B_{2k,\gamma}(P_\gamma(x)) = \delta_\gamma(x), \quad k = 0, 1, 2, ... \tag{3.23}$$

which is what we wanted to prove. □

4 Conclusion remarks.

We have considered and studied a family of kernels depending on a weighted generalized function associated with quadratic forms that generalize both elliptic, hyperbolic and ultrhyperbolic ones. It has also been demonstrated that it is possible to define the Riemann-Liouville integral in Lorentzian spaces via the generalized convolution operation. In addition, several properties of the introduced kernels have been proved, among them the one of being an elementary solution of the ultrahyperbolic Bessel operator. Finally, it has been shown that they turn out to be a convolutor in S'_{ev} for certain values of the parameter.

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