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ON CLOSEDNESS OF LINEAR COMBINATION OF CLOSED AND CONVEX SETS AND THE CANCELLATION PROPERTY

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Abstract In this note we study the connections between the cosedness of the algebraic sum $A + B$ and the closedness of a combination $sA + tB$ where $s, t \in \mathbb{R}$ (Theorem 2.7). In theorem 2.8 we give sufficient conditions for closedness of the set $sA+tB$. We give also some conditions when we can cancel the set B from the inclusion $A + B \subset B + C$ (Theorem 2.2)

1 Introduction

Let X be a real topological vector space. For $A, B \subset X$ and $\lambda \in \mathbb{R}$ we define a sets

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\}.$$

The first of the above two sets is called a *Minkowski sum of sets A and B* or a *algebraical sum of sets A and B*.
Since for any $\lambda \in \mathbb{R}, \lambda \neq 0$ the function $f_\lambda : X \to X$ is an homeomorphism therefore if A is closed set then so is the set λA. The situation with a closedness of algebraic sum $A + B$, where A and B are closed is more complicated.
For instance, if $g : (0, \infty) \to \mathbb{R}$ is lower semicontinous function, then by taking

$$A = \{(0, y) : y \in \mathbb{R}\}$$

and

$$B = \{(x, v) : x > 0, v \geq g(x)\} \subset \mathbb{R}^2,$$

we obtain a two closed sets such that the sum $A + B = (0, \infty) \times \mathbb{R} \subset \mathbb{R}^2$ is evidently not closed subset of $\mathbb{R}^2$. Moreover if the function g is convex then we have a two closed and convex sets which algebraic sum is not closed.
This example shows that the sum $A + B$ need not to be closed even if A and B are closed and convex subsets in finite dimensional topological vector space. But, it is well known that if A, B are closed subsets of topological vector space X and one of these sets is compact then the algebraic sum $A + B$ is closed. Also, it can be proved ([7]) that if X is a Banach space such that the algebraic sum of any two closed bounded and convex subsets of X is a closed set, then X is reflexive.
Closedness of the algebraic sum and the cancellation law were study in many works ([1] ,[2] , [4] , [5], [7] , [8] ,[9]) some related studies you can also find in ([3],[6]).
In this note we study the connections between the cosedness of the algebraic sum $A + B$ and the closedness of a combination $sA + tB$ where $s, t \in \mathbb{R}$. Futhermore we give some conditions when we can cancel the set B from the inclusion $A + B \subset B + C$.

A very simple observation shows that the closedness of the set $A + B$, does not imply, the closedness of the set $A - B = A + (-1)B$. To see this let us take

$$A = \left\{ n + \frac{1}{n+1} : n \in \mathbb{N} \right\}$$

and

$$B = \mathbb{N} \subset \mathbb{R}.$$

Then the sets $A, B, A + B$ are closed but $A - B$ is not closed. So in general the closednes of the set $A + B$ does not imply the closedness of the set $sA + tB$ for all $s, t \in \mathbb{R}$.

We can noticed an easy facts in the following propositions:

Proposition 1.1. *Let X be a topological vector space and $A, B \subset X$. The following conditions are equivalent:*

a) The sets $sA + tB$ are closed for all $s, t > 0$

b) The sets $A + uB$ are closed for all $u > 0$

c) The sets $uA + B$ are closed for all $u > 0$

d) The sets $sA + tB$ are closed for all $0 < s, t < 1$.

Proposition 1.2. *Let X be a topological vector space and $A, B \subset X$. The following conditions are equivalent:*

a) The sets $sA + tB$ are closed for all $s, t \in \mathbb{R} \setminus \{0\}$

b) The sets $A + uB$ are closed for all $u \neq 0$

c) The sets $uA + B$ are closed for all $u \neq 0$

d) The sets $sA + tB$ are closed for all $-1 < s, t < 1, s \neq 0, t \neq 0$.

Let us remind that for $A, B \subset X$ the set

$$A \dot{-} B = \bigcap_{x \in B} (A + \{-x\})$$

is called a *Minkowski-Pontryagin difference of the sets A and B.*
If A is closed or convex set, then so is the set $A \dot{-} B$, as an intersection of such sets. More properties of the set $A \dot{-} B$ can be found in [1].

The next proposition constitute the connections between the cancellation law and Minkowski-Pontryagin difference.

Proposition 1.3. *Let X be a vector space and let $A, B, C \subset X$. Then*

a) $C \subset (C + B) \dot{-} B$

b) The equality $C = (C + B) \dot{-} B$ is equivalent to the following cancellation property:

If $A + B \subset B + C$ then $A \subset C$.

Proof. We prove only b) since a) is obvious. For if let us assume that $C = (C + B) \dot{-} B$ and $A + B \subset B + C$ then

$$A \subset (A + B) \dot{-} B \subset (B + C) \dot{-} B = C.$$

On the other hand if the implication

$$\text{If } A + B \subset B + C \text{ then } A \subset C.$$

holds true. Then by taking any $u \in (C+B)\dot{-}B$ we have $\{u\} + B \subset B + C$ therefore by assumption $\{u\} \subset C$ and hence $(C+B)\dot{-}B = C$. □

It can be proved (see [8]) that if B, C are subsets of topological vector space such that B is bounded and C is closed and convex then $(C+B)\dot{-}B = C$. Further we obtain this fact as a part of corollary 2.5.

2 Closedness of linear combination of sets and the cancellation property

At the start of this section let us remind definition of the asymptotic cone of an subset of topological vector space.

Definition 2.1. Let X be a topological vector space and let $A \subset X$. The *asymptotic cone* of the set A is defined as

$$A_\infty = \left\{ x : x = \lim_{n\to\infty} (\beta_n a_n), \beta_n \to 0, \beta_n > 0, a_n \in A \right\}$$

Ii is easy to observe that if A is closed and convex then $A + A_\infty \subset A$.

Theorem 2.2. *Let X be a topological vector space and $B, C \subset X$. Assume that C is convex and*

$$\bigcap_{n=1}^{\infty} \left(C + \frac{1}{n} B \right) \subset C.$$

Then $(C+B)\dot{-}B = C$.

Proof. Without lost of generality, we may assume that $0 \in B$. Let $x \in (B+C)\dot{-}B$ hence $x + B \subset B + C$ and thus

$$2x + B \subset x + (x + B) \subset x + B + C \subset B + C + C = B + 2C$$

since C is a convex set. In a similar way as above, we can see that

$$nx + B \subset B + nC,$$

for all $n \in \mathbb{N}$.
Dividing the last inclusion by n we get that

$$x \in \left(x + \frac{1}{n} B \right) \subset C + \frac{1}{n} B$$

and thus

$$x \in \bigcap_{n=1}^{\infty} \left(C + \frac{1}{n} B \right)$$

hence by assumption $x \in C$. □

Lemma 2.3. *Let X be a topological vector space and $B, C \subset X$. If the set B is bounded and the set C is closed, then*

$$\bigcap_{n=1}^{\infty} \left(C + \frac{1}{n} B \right) \subset C.$$

Proof. Take any neighbourhood U of 0 in X. Since the set B is bounded therefore $\frac{1}{k}B \subset U$ for some $k \in \mathbb{N}$. Therefore

$$\bigcap_{n=1}^{\infty}\left(C + \frac{1}{n}B\right) \subset C + \frac{1}{k}B \subset C + U.$$

Hence

$$\bigcap_{n=1}^{\infty}\left(C + \frac{1}{n}B\right) \subset \overline{C} = C.$$

□

Lemma 2.4. *Let X be a topological vector space and $B, C \subset X$. If the set C is compact and $B_\infty = \{0\}$, then*

$$\bigcap_{n=1}^{\infty}\left(C + \frac{1}{n}B\right) \subset C.$$

Proof. If the set C is compact, then

$$\bigcap_{n=1}^{\infty}\left(C + \frac{1}{n}B\right) \subset C + \bigcap_{n=1}^{\infty}\left(\frac{1}{n}B\right) \subset C + B_\infty = C.$$

□

From theorem 2.2 and lemma 2.3 and lemma 2.4 we get the following

Corollary 2.5. *Let X be a topological vector space and $B, C \subset X$. Then*

a) If the set C is closed and convex and the set B is bounded then $(C + B)\dot{-}B = C$(see [8]).

b) If the set C is compact and convex and $B_\infty = \{0\}$, then $(C + B)\dot{-}B = C$.

Remark 2.6. Notice that in any infinite dimensional normed space there exists a unbounded set B such that $B_\infty = \{0\}$, (see [2]).

Theorem 2.7. *Let $A, B, A+B$ be a closed subsets of topological vector space X and $0 < s < 1$. If $A+sB = (A+B)\dot{-}(1-s)B$ then $A+sB$ is closed. Moreover if $A+sB$ is closed and convex and B is bounded and convex then $A+sB = (A+B)\dot{-}(1-s)B$.*

Proof. The proof follows immediately from equality $A + B = A + sB + (1 - s)B$. □

Now we prove the following theorem

Theorem 2.8. *Let X be a normed space and A, B be a closed and convex subsets X. Assume that the one of sets*

$$A_1 = \left\{\frac{x}{||x||} : x \in A\right\}, B_1 = \left\{\frac{x}{||x||} : x \in B\right\}$$

is sequentially compact in some linear Hausdorff topology τ on X which is weaker than the norm topology and the second of coresponding sets A, B is closed in this topology. If $(A\dot{-}A) \cap (-B\dot{-}B) = \{0\}$, then for all $s, t \in \mathbb{R}$ the set $sA + tB$ is closed.

Proof. First observe that A, B satisfies the assumptions of these theorem then for all $s \neq 0, t \neq 0$ (the case $s = 0$ or $t = 0$ is trivial) the sets sA and tB also satisfies assumptions of this theorem. Hence it is enough to prove that $A + B$ is a closed set. In our proof we assume that the set A_1 is seqentially compact in some linear topology weaker than norm topology and B is closed in this topology.
Let $z_n = a_n + b_n \in A + B, a_n \in A, b_n \in B$ and suppose that $z_n \to z_0 \in X$. We will show that $z_0 \in A + B$.

At first let us consider the case when the sequence a_n is norm bounded. Hence there exists a subsequence a_{n_k} such that

$$a_{n_k} = ||a_{n_k}|| \cdot \frac{a_{n_k}}{||a_{n_k}||} \to a_0 \in A.$$

Now fom the equality $b_{n_k} = z_{n_k} - a_{n_k} \to z_0 - a_0$ in topology τ, and since B is closed in τ therefore $z_0 - a_0 \in B$. But this implies that $z_0 = a_0 + (z_0 - a_0) \in A + B$.
Consider now the case when the sequence a_n is not norm bounded. We may assume that $||a_n|| \to \infty$. From the equality

$$\frac{z_n}{||a_n||} = \frac{a_n}{||a_n||} + \frac{b_n}{||a_n||}$$

the compactness of A_1 and convergence of z_n we conclude that there exists $x_0 \neq 0$ and subseqence a_{n_k} such that $\frac{a_{n_k}}{||a_{n_k}||} \to x_0 \in A \dot{-} A$ and $\frac{b_{n_k}}{||a_{n_k}||} \to -x_0 \in B \dot{-} B$ but this contradicts to the assumption that $(A \dot{-} A) \cap (-B \dot{-} B) = \{0\}$. □

Remark 2.9. In many situations as topology τ in the above theorem we may use the weak topology or weak-star topology.

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