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On r-Clean Algebras

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Abstract A ring R is said to be r-clean if each element is the sum of an idempotent and a regular element of R. An r-clean ring generalizes the concept of a clean ring by replacing units with regular elements. If R is a commutative ring, an R-algebra A is considered clean if each element is the sum of an idempotent and a unit of A. In this work, we define an r-clean algebra as a generalization of a clean algebra. Additionally, we define an r-endoclean module as a module whose endomorphism ring is r-clean. Furthermore, we present the relationship between the r-clean properties of the R-algebra A, its base ring, and the algebra itself as a module over the ring.

1 Introduction

Clean rings are among the most widely studied properties of rings. If R is an associative ring with identity, an element $a \in R$ is said to be clean if it can be expressed as the sum of an idempotent and a unit of R. A ring R is called clean if every element of R is clean. The concept of clean rings was first introduced by Nicholson [1]. Research related to this clean ring has been constantly evolving until now. Several researchers have proposed generalizations of the definition of a clean ring as presented in [2, 3, 4, 5]. In addition, a study on the ring endomorphisms of a module, which is a clean ring, was conducted by Nicholson et al. [6]. The concept of a clean endomorphism ring is used to define the notion of a clean module. Camillo and Khurana [7] defined clean modules as modules whose endomorphism rings are clean.

An element $a \in R$ is (von Neumann) regular if there exists an element $b \in R$ such that $a = aba$. If $x \in R$ is a unit, then there exists an inverse element $x^{-1} \in R$ such that $x = xx^{-1}x$, meaning x is a regular element. The concept of a clean element in a ring was generalized to r-clean by Ashrafi and Nasibi [2], where units are replaced by regular elements. According to [2], an element $a \in R$ is said to be r-clean if it can be expressed as the sum of an idempotent and a regular element of R. A ring R is called an r-clean ring if every element of R is r-clean. Some properties of r-clean rings are presented in [8]. Furthermore, motivated by the observation that the set of clean elements does not necessarily form an ideal, Chen and Chen [9] defined the concept of a clean ideal. As the clean ring was generalized to an r-clean ring, Yuwaningsih et al. [10] extended the notion of a clean ideal to the r-clean ideal. An ideal J of a ring R is called an r-clean ideal if each element can be expressed as the sum of an idempotent and a regular element of R.

According to [2] and [8], if $\{R_i\}_{i\in\Lambda}$ is a collection of r-clean rings, then the ring $\prod_{i\in\Lambda} R_i$ is also r-clean. For any commutative ring R, each R_i can be considered an R-module for every $i \in \Lambda$. Thus, the ring $\prod_{i\in\Lambda} R_i$ can also be viewed as an R-module. A structure that is both a ring and a module is known as an algebra. This concept motivated us to explore r-clean properties in R-algebras. The clean properties of R-algebras were previously investigated by Wijayanti [11]. Therefore, in this work, we generalize the clean property of R-algebras to the r-clean property of R-algebras.

Since many structures are involved in the composition of an R-algebra, research on the properties of r-clean R-algebras is of particular interest. This research aims to define r-clean R-algebras and investigate some of their properties. Previously, we defined a generalization of clean modules, called an r-endoclean module. Furthermore, we investigate the relationship between the r-clean properties of the R-algebra, its base ring, and the algebra as a module over the ring.

Throughout this article, R is assumed to be a commutative ring with identity unless stated otherwise. Moreover, we denote $Id(R)$ as the set of all idempotents of R, $Reg(R)$ as the set of all regular elements of R, and Λ as the index set.

2 The Definition of r-Clean Algebra and the r-Endoclean Module

We begin by defining r-clean R-algebras as follows.

Definition 2.1. An R-algebra A is called an r-clean algebra if, for each element $a \in A$, there exists an idempotent $e \in A$ and a regular element $r \in A$ such that $a = e + r$.

Let α be an endomorphism of R and R an r-clean R-algebra. Then, according to [2], the formal power series ring $R[[x]]$ and the skew power series ring $R[[x;\alpha]]$ are both r-clean R-algebras.

Example 2.2. Let X and Y be rings, and let M be a (Y, X)-bimodule. Assume that one of the following conditions holds:

(i) X and Y are clean.

(ii) One of the rings, X or Y, is clean, and the other is r-clean.

According to [8], the formal triangular matrix ring $T = \begin{bmatrix} X & 0 \\ M & Y \end{bmatrix}$ is an r-clean ring. Hence, we have that T is an r-clean $\mathbb{Z}$-algebra.

Example 2.3. Let e be a central idempotent of the ring R. According to [2], if R is an r-clean ring, then eRe is also an r-clean ring. Thus, we have that eRe is an r-clean R-algebra.

Let R and S be two rings, J an ideal of S, and $f : R \to S$ be a ring homomorphism. D'Anna, et al. in [12] and [13] introduced and studied the subring

$$R \bowtie^f J = \{(a, f(a) + j) \mid a \in R, j \in J\}$$

of $R \times S$, called the amalgamation of R and S along J with respect to f. Moreover, $R \bowtie^f J$ forms an algebra over the ring R. We obtain the following example by generalizing the clean properties of the amalgamation ring in [14].

Example 2.4. If the ring $R \bowtie^f J$ is an r-clean R-algebra, then both the ring R and $f(R) + J$ are r-clean R-algebras. Furthermore, if the ideal $J = S$, then $R \bowtie^f S$ is an r-clean R-algebra if and only if both R and S are r-clean R-algebras.

According to [11], if A is an R-algebra and M is an A-module, then the endomorphism ring $End_A(M)$ is an R-algebra. Moreover, we have the relationship between the R-algebra A and the endomorphism ring $End_R(M)$ as follows.

Lemma 2.5. *Let M be an R-module and A an R-algebra. Then, M is a module over A if and only if there exists an R-algebra morphism $\psi : A \to End_R(M)$.*

We present the definition of the r-endoclean module as a generalization of the clean module.

Definition 2.6. Let R be a non-commutative ring and M an R-module. Then, M is called an r-endoclean R-module if the endomorphism ring $End_R(M)$ is r-clean.

Referring to [6], we have the following example of an r-endoclean module.

Example 2.7. Let D be a division ring and V a vector space over D. Then, V is an r-endoclean D-module.

Example 2.8. Let $R = M_n(D)$ be the matrix ring over a division ring D, where $n > 0$ is an integer. Then,

(i) The free module F is an r-endoclean R-module.

(ii) The module M is an r-endoclean R-module.

Example 2.9. Let R be a semisimple Artinian ring. The module M is an r-endoclean R-module.

Let R be a non-commutative ring, M an R-module, $S = End_R(M)$ the endomorphism ring, and element $\alpha \in S$. We recall that both $Ker(\alpha)$ and $Im(\alpha)$ are direct summands of M if and only if there exist idempotents $\pi_1, \pi_2 \in S$ such that $Ker(\pi_1) = Ker(\alpha)$ and $Im(\pi_2) = Im(\alpha)$. We use this property to prove the next proposition.

In the following, we present the necessary and sufficient conditions for an element in the endomorphism ring to be regular.

Proposition 2.10. *Let R be a non-commutative ring, M an R-module, $S = End_R(M)$ the endomorphism ring, and element $\alpha \in S$. The element α is a regular element, i.e. $\alpha = \alpha\gamma\alpha$ for some element $\gamma \in S$, if and only if M can be decomposed as*

$$M = Im(\alpha) \oplus Ker(\alpha\gamma) = Ker(\alpha) \oplus Im(\gamma\alpha).$$

Proof. Given the element $\alpha = \alpha\gamma\alpha$ for some element $\gamma \in S$. Let any element $m \in Im(\alpha) \cap Ker(\alpha\gamma)$, there exists an element $x \in M$ such that $\alpha(x) = m$ and $\alpha\gamma(m) = 0$. Consequently, we have $m = \alpha(x) = \alpha\gamma\alpha(x) = \alpha\gamma(m) = 0$. Thus, $Im(\alpha) \cap Ker(\alpha\gamma) = 0$. Now, let $m \in M$. We obtain $\alpha\gamma(m - \alpha\gamma(m)) = \alpha\gamma(m) - \alpha\gamma\alpha\gamma(m) = \alpha\gamma(m) - \alpha\gamma(m) = 0$, which shows that $(m - \alpha\gamma(m)) \in Ker(\alpha\gamma)$. Since $\alpha\gamma(m) \in Im(\alpha)$, we get $m = \alpha\gamma(m) + (m - \alpha\gamma(m)) \in Im(\alpha) + Ker(\alpha\gamma)$. Hence, $M = Im(\alpha) + Ker(\alpha\gamma)$. Thus, $M = Im(\alpha) \oplus Ker(\alpha\gamma)$. Analogously, we can show that $M = Ker(\alpha) \oplus Im(\gamma\alpha)$. Conversely, let the R-module M be decomposed as $M = Im(\alpha) \oplus Ker(\alpha\gamma) = Ker(\alpha) \oplus Im(\gamma\alpha)$. Thus, $Im(\alpha)$ and $Ker(\alpha)$ are direct summands of M. Consequently, there exist idempotents $\pi_1, \pi_2 \in S$ such that $Ker(\alpha) = Ker(\pi_1)$ and $Im(\alpha) = Im(\pi_2)$. Since $Im(\pi_2) \subseteq Im(\alpha)$, then we can take the corestriction

$$\begin{aligned} \alpha' : M &\to Im(\alpha) \\ x &\mapsto \alpha'(x) = \alpha(x), \qquad \text{for all } x \in M. \end{aligned}$$

and

$$\begin{aligned} \pi_2' : M &\to Im(\alpha) \\ x &\mapsto \pi_2'(x) = \pi_2(x), \qquad \text{for all } x \in M. \end{aligned}$$

According to the Fundamental Homomorphism Theorem, there exists an R-module isomorphism $i : Im(\alpha) \to M/Ker(\alpha)$ such that $(i\alpha')(x) = i(\alpha'(x)) = x + Ker(\alpha)$, where $i\alpha' : M \to M/Ker(\alpha)$ is the canonical surjection. Since $Ker(\alpha) \subseteq Ker(\pi_1)$, there exists a mapping $\overline{\pi_1} : M/Ker(\alpha) \to M$ such that $\overline{\pi_1} i\alpha' = \pi_1$. Next, we choose $\gamma = \overline{\pi_1} i\pi_2'$. Since $Im(\alpha) \subseteq Im(\pi_2)$, for each $x \in M$ there exists $y \in M$ such that $\alpha(x) = \pi_2(y)$. As a result, we get

$$\begin{aligned} (\pi_2'\alpha)(x) &= \pi_2'(\alpha(x)) = \pi_2'(\pi_2(y)) = \pi_2(\pi_2(y)) \\ &= \pi_2^2(y) = \pi_2(y) = \alpha(x) = \alpha'(x). \end{aligned}$$

From here, we obtain

$$\begin{aligned} \alpha\gamma\alpha(x) &= \alpha\overline{\pi_1} i\pi_2'\alpha(x) = (\alpha\overline{\pi_1} i)(\pi_2'\alpha)(x) \\ &= (\alpha\overline{\pi_1} i)\alpha'(x) = \alpha(\overline{\pi_1} i\alpha')(x) = \alpha\pi_1(x). \end{aligned}$$

Moreover, since $\pi_1^2 = \pi_1$ and $\pi_1(\pi_1 - id_S) = 0_S$, it follows that $Ker(\pi_1) \subseteq Ker(\alpha)$ implies $\alpha(\pi_1 - id_S) = 0_S$. Consequently, we obtain $\alpha = \alpha\pi_1$. Thus, we have proved that $\alpha = \alpha\gamma\alpha$. □

From Proposition 2.10, it is clear that $Im(\alpha) \cong Im(\gamma\alpha)$ and $Ker(\alpha\gamma) \cong Ker(\alpha)$. Moreover, using Propositon 2.10, we provide the necessary and sufficient conditions for a module to be r-endoclean as follows.

Proposition 2.11. *Let R be a non-commutative ring, M an R-module, and $S = End_R(M)$ the endomorphism ring. Then, M is an r-endoclean R-module if and only if for each $\alpha \in S$, there exists an idempotent $\pi \in S$ such that M can be decomposed as*

$$M = Im(\alpha - \pi) \oplus Ker((\alpha - \pi)\gamma) = Ker(\alpha - \pi) \oplus Im(\gamma(\alpha - \pi)),$$

for some element $\gamma \in S$.

Proof. Let M be an r-endoclean R-module, which implies that S is an r-clean ring. For any element $\alpha \in S$, there exists an idempotent $\pi \in S$ such that $\alpha - \pi$ is a regular element of S. Referring to the Proposition 2.10, there exists an element $\gamma \in S$ such that $M = Im(\alpha - \pi) \oplus Ker((\alpha - \pi)\gamma) = Ker(\alpha - \pi) \oplus Im(\gamma(\alpha - \pi))$. Conversely, for each element $\alpha \in S$ there exists an idempotent $\pi \in S$ such that the R-module M can be decomposed as $M = Im(\alpha - \pi) \oplus Ker((\alpha - \pi)\gamma) = Ker(\alpha - \pi) \oplus Im(\gamma(\alpha - \pi))$. Based on Proposition 2.10, $\alpha - \pi$ is a regular element. Consequently, α is an r-clean element of S. Thus, S is an r-clean ring, and therefore, M is an r-endoclean R-module. □

Considering the fact that ring $R \cong End_R(R)$, we obtain the following properties.

Proposition 2.12. *Let R be a non-commutative ring and $S = End_R(R)$ the endomorphism ring. The following statements are equivalent:*

(i) *R is an r-clean ring.*

(ii) *R is an r-endoclean R-module.*

(iii) *For each element $\alpha \in S$, there exists an idempotent $\pi \in S$ such that R can be decomposed as $R = Im(\alpha - \pi) \oplus Ker((\alpha - \pi)\gamma) = Ker(\alpha - \pi) \oplus Im(\gamma(\alpha - \pi))$, for some element $\gamma \in S$.*

(iv) *For each element $x \in R$, there exists an idempotent $e \in R$ such that R can be decomposed as $R = ((x - e))R \oplus (1_R - (x - e)b)R = (1_R - (x - e))R \oplus (b(x - e))R$, for some element $b \in R$.*

Proof. $(i) \Leftrightarrow (ii)$, $(ii) \Leftrightarrow (iii)$, and the implication $(iii) \Rightarrow (iv)$ are evident from the fact that $R \cong S$. To Prove $(iv) \Rightarrow (iii)$, let any element $\alpha \in S$. Since $R \cong S$, for every α, there exists an element $r \in R$ such that $\alpha(a) = ra$ for every $a \in R$. Thus, the statement (iii) follows. □

3 Some Properties of r-Clean Algebras

In this section, we present some properties of r-clean algebras. The first property provides a sufficient condition for an R-algebra to contain an r-clean subalgebra.

Proposition 3.1. *Let A be an R-algebra and R an r-clean ring. Then, A contains an r-clean subalgebra.*

Proof. First, define the mapping

$$\begin{aligned} \theta : R &\rightarrow A \\ r &\mapsto \theta(r) = r1_A, \qquad \text{for all } r \in R. \end{aligned}$$

We find that θ is a ring homomorphism. Let $a \in R$. Since R is an r-clean ring, there exist $e \in Id(R)$ and $r \in Reg(R)$ such that $a = e + r$. As θ is a ring homomorphism, we have $\theta(a) = \theta(e + r) = \theta(e) + \theta(r)$. Note that $\theta(e) = \theta(ee) = \theta(e)\theta(e)$, so we conclude that $\theta(e) \in Id(A)$. Since $r \in Reg(R)$, there exists $s \in R$ such that $r = rsr$. Furthermore, since θ is a ring homomorphism, we have $\theta(r) = \theta(rsr) = \theta(r)\theta(s)\theta(r)$. Thus, we conclude that $\theta(r) \in Reg(A)$. Therefore, $\theta(a)$ is an r-clean element of A. Hence, $Im(\theta)$ is an r-clean subalgebra of A. □

The following property concerns the necessary conditions for an R-algebra to be r-clean.

Proposition 3.2. *Let A be an R-algebra and M an A-module. If M is a faithful R-module and A is an r-clean algebra, then $End_R(M)$ contains an r-clean subring.*

Proof. Let M be an A-module, which means that there is a ring homomorphism

$$\begin{aligned} f : A &\rightarrow End_R(M) \\ a &\mapsto f(a) = f_a, \qquad \text{for all } a \in A \end{aligned}$$

where

$$\begin{aligned} f_a : M &\rightarrow M \\ m &\mapsto = am, \qquad \text{for all } m \in M. \end{aligned}$$

Let $a \in A$. Since A is an r-clean algebra, there exist $e \in Id(A)$ and $r \in Reg(A)$ such that $a = e + r$. As f is a ring homomorphism, we have

$$f(a) = f(e + r) = f(e) + f(r).$$

Clearly, $f(e) \in Id(End_R(M))$ and $f(r) \in Reg(End_R(M))$. Hence, $f(a)$ is an r-clean element of $End_R(M)$. Therefore, $Im(f)$ is an r-clean subring of $End_R(M)$. □

Furthermore, we provide sufficient conditions for an R-module to decompose as a direct sum of two submodules.

Proposition 3.3. *Let R be a non-commutative ring and M a faithful left R-module. If R is an r-clean ring and contains no zero-divisor elements, then there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$.*

Proof. Since M is a left R-module, there is the scalar multiplication defined by:

$$\begin{aligned} \cdot : R \times M &\rightarrow M \\ (r, m) &\mapsto rm, \qquad \text{for all } (r, m) \in R \times M. \end{aligned}$$

Let R be an r-clean ring, meaning that every element of R is r-clean. Suppose we take an element $x \in R$, there exists an idempotent $e \in R$ such that the ring R can be decomposed as $R = (1_R - (x-e))R \oplus (b(x-e))R$, for some element $b \in R$. This means that for every $m \in M$ there exist $s \in (1_R - (x-e))R$ and $t \in (b(x-e))R$ such that $m = 1_R m = (s+t)m = sm + tm$. Thus, we obtain $M = sM + tM$. Moreover, we will show that $sM \cap tM = \{0_M\}$. Let $x \in sM \cap tM$. There exist $m_1, m_2 \in M$ such that $x = sm_1 = tm_2$. Consequently, we have $sm_1 - tm_2 = 0_M$. If $m_1 = m_2$, then $s = t$. Since $(1_R - (x-e))R \cap (b(x-e))R = \{0_R\}$, we get $s = t = 0_R$. If $m_1 \neq m_2$ and they are linearly independent, it follows that $s = t = 0_R$. Moreover, if $m_1 \neq m_2$ and they are not linearly independent, there exists $0_R \neq k \in R$ such that $m_1 = km_2$. Consequently, we have $skm_2 = tm_2$, which implies $(sk - t)m_2 = 0_M$. Since M is a faithful R-module, we conclude that $sk - t = 0_R$. Thus, we have $sk = t$. This means $t \in (1_R - (x-e))R \cap (b(x-e))R$, whereas $(1_R - (x-e))R \cap (b(x-e))R = \{0_R\}$. Hence, $sk = 0_R$. Furthermore, since $k \neq 0_R$ and R contain no zero-divisor elements, we obtain $s = 0_R$. Therefore, $s = t = 0_R$. Thus, we conclude that $sM \cap tM = \{0_M\}$. By taking $M_1 = sM$ and $M_2 = tM$, we have proved that $M = M_1 \oplus M_2$. □

Next, based on [11], we have $End_R(M_1 \oplus M_2) = End_R(M_1) \oplus End_R(M_2)$. Referring to Proposition 3.3, we obtain the following property.

Proposition 3.4. *Let R be a non-commutative ring and M a faithful left R-module. If R is an r-clean ring and contains no zero-divisor elements, then there exists a decomposition $End_R(M) = End_R(M_1) \oplus End_R(M_2)$, where M_1 and M_2 are submodules of M.*

Proof. Since R is an r-clean ring and contains no zero-divisor elements, based on Proposition 3.3, there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$. Therefore, we have

$$End_R(M) = End_R(M_1 \oplus M_2).$$

Since $End_R(M_1 \oplus M_2) = End_R(M_1) \oplus End_R(M_2)$, we obtain

$$End_R(M) = End_R(M_1) \oplus End_R(M_2).$$

□

The following presents the consequences of Proposition 3.4.

Corollary 3.5. *Let R be an integral domain and A a faithful R-algebra. If R is an r-clean ring, then there exists a decomposition $End_R(A) = End_R(A_1) \oplus End_R(A_2)$, where A_1 and A_2 are subalgebras of A.*

Referring to Proposition 2.11, we present the necessary and sufficient conditions for an R-algebra to be r-endoclean.

Proposition 3.6. *Let A be an R-algebra and $S = End_R(A)$ the endomorphism ring. Then, A is an r-endoclean R-module if and only if for every element $\alpha \in S$, there exists an idempotent $\pi \in S$ such that A can be decomposed as*

$$A = Im(\alpha - \pi) \oplus Ker((\alpha - \pi)\gamma) = Ker(\alpha - \pi) \oplus Im(\gamma(\alpha - \pi)),$$

for some element $\gamma \in S$.

At the end of this paper, we present the necessary and sufficient conditions for an R-algebra to be r-clean.

Proposition 3.7. *Let A be an R-algebra and $S = End_A(A)$ the endomorphism ring. Then, A is an r-clean R-algebra if and only if for every element $\alpha \in S$, there exists an idempotent $\pi \in S$ such that A can be decomposed as*

$$A = Im(\alpha - \pi) \oplus Ker((\alpha - \pi)\gamma) = Ker(\alpha - \pi) \oplus Im(\gamma(\alpha - \pi)),$$

for some element $\gamma \in S$.

Proof. Since $A \cong S$ and A is an r-clean R-algebra, we conclude that S is an r-clean ring. In other words, the A-module A is r-endoclean. Referring to Proposition 2.11, for each $\alpha \in S$, there exists an idempotent $\pi \in S$ such that the A-module A can be decomposed as $A = Im(\alpha - \pi) \oplus Ker((\alpha - \pi)\gamma) = Ker(\alpha - \pi) \oplus Im(\gamma(\alpha - \pi))$, for some element $\gamma \in S$. Conversely, for each element $\alpha \in S$, there exists an idempotent $\pi \in S$ such that A can be decomposed as $A = Im(\alpha - \pi) \oplus Ker((\alpha - \pi)\gamma) = Ker(\alpha - \pi) \oplus Im(\gamma(\alpha - \pi))$, for some element $\gamma \in S$. Based on Proposition 2.10, $\alpha - \pi$ is a regular element. Consequently, α is an r-clean element of S. Thus, S is an r-clean ring. Since $A \cong S$, we conclude that A is an r-clean R-algebra. □

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