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THE MOORE-PENROSE INVERSE IN THE SYMMETRIZED MAX-PLUS ALGEBRAIC MATRIX USING SINGULAR VALUE DECOMPOSITION

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Abstract In this paper, we present the construction of the Moore-Penrose inverse of matrix over the symmetrized max-plus algebra using the singular value decomposition. The Moore-Penrose inverse is defined in a similar manner to linear algebra, by replacing equations into balance. The construction of this Moore-Penrose inverse utilizes the balanced inverse in the context of the symmetrized max-plus algebra. The main result of this paper is the form of the Moore-Penrose inverse of a matrix over the symmetrized max-plus algebra, which is constructed using the singular value decomposition.

1 Introduction

Let $\mathbb{R}$ be the set of all real numbers. The max-plus algebra, denoted by $\mathbb{R}_{\max}$, is a semiring $\mathbb{R} \cup \{-\infty\}$, with addition defined as "maximum" and multiplication defined as "plus". In this algebra, the zero element is $-\infty$ and the unity element is 0 [6, 8]. Unlike in linear algebra, there is no additive inverse for elements in the max-plus algebra, except for the zero element. The max-plus algebra can be extended to a larger set using a process called symmetrization, which is discussed in [2, 5]. The result of symmetrization is called the symmetrized max-plus algebra, denoted by $\mathbb{S}$. Furthermore, $\mathbb{R}_{\max}$ can be viewed as the positive part of $\mathbb{S}$. The construction is similar to the extension of natural numbers to integers in conventional algebra, but complications arise due to the idempotency of the "max" operator. The generalization of the notion of an equation is called a balance, denoted by ∇. Since $\mathbb{R}_{\max}$ can be viewed as the positive part of $\mathbb{S}$, $\mathbb{R}_{\max}$ is a subset of $\mathbb{S}$.

The relationship between symmetrized max-plus algebra and conventional algebra is discussed in [3, 4]. This connection enables problems in symmetrized max-plus algebra to be transformed into the context of conventional algebra. Solving problems in symmetrized max-plus algebra can be approached similarly to conventional algebra, and the solutions obtained in conventional algebra can be transformed back to symmetrized max-plus algebra. In reference [4], the QR and singular value decompositions of the symmetrized max-plus algebraic matrix can be determined using the link between symmetrized max-plus algebra and conventional algebra. Similarly, other matrix decompositions such as the LU decomposition [11] and the Cholesky decomposition [12] can be found using this link. Then, characterization of rank of matrix over the symmetrized max-plus algebra also uses this link [13].

In conventional algebra, the Moore-Penrose inverse of a matrix is the most well-known generalization of the inverse matrix [1, 9]. The term generalized inverse is sometimes used interchangeably with pseudoinverse. The Moore-Penrose inverse can be used to calculate a least squares solution for a system of linear equations that has no solution, find the minimum norm solution for a system of linear equations with multiple solutions, and aid in stating and proving results in linear algebra. The Moore-Penrose inverse can be determined using matrix decomposition methods such as full-rank decomposition, singular value decomposition, and Cholesky decomposition [7]. The discussion of some useful direct (sub)decompositions is explained in [10], decompositions of total graph in [14] and matrix polynomial in [15].

The connection between the symmetrized max-plus algebra and conventional algebra enables the development of the Moore-Penrose inverse in the symmetrized max-plus algebraic matrix. This paper focuses on determining the construction of the Moore-Penrose inverse of a matrix over the symmetrized max-plus algebra using matrix decomposition, specifically the singular value decomposition as discussed in [4]. The results of this paper have the potential to be used in solving linear balance systems problems over the symmetrized max-plus algebra.

Section 1 provides an introduction to motivate this paper. Section 2 discusses the symmetrized max-plus algebra as preliminaries. The main result, which discusses the construction of the Moore-Penrose inverse in the symmetrized max-plus algebraic matrix, is presented in Section 3. Finally, Section 4 concludes the paper.

2 Preliminaries

Let $\mathbb{R}$ be the set of all real numbers. The max-plus algebra is defined as the semiring $\mathbb{R} \cup \{-\infty\}$ with operations $\oplus$ (maximum) and $\otimes$ (plus) defined as follows

$$\begin{aligned} a \oplus b &= \max(a, b) \\ a \otimes b &= a + b \end{aligned}$$

for all $a, b \in \mathbb{R} \cup \{-\infty\}$, where $\max(a, -\infty) = a$ and $a + (-\infty) = -\infty$ [6, 8]. Then, the max-plus algebra is denoted by $\mathbb{R}_{\max}$. The process of max-plus algebraic symmetrization can be used to obtain the negative form instead of the inverse role of addition. This process is similar to expanding natural numbers into integers, and it is carried out to obtain a balanced element. The discussion of max-plus algebra symmetrization is covered in [5].

Definition 2.1. [5] Let $(a, b), (c, d) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}$ The balance relation (denoted by ∇) is defined as follows

$$(a, b)\nabla(c, d) \text{ iff } a \oplus d = b \oplus c.$$

The relation is reflexive and symmetric but not transitive, so it is not an equivalence relation, and the quotient set of $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ by ∇ cannot be defined.

Definition 2.2. [5] Let $(a, b), (c, d) \in \mathbb{R}_{\max} \times \mathbb{R}_{\max}$. The relation $\mathcal{B}$ in $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ is defined as follows

$$(a, b)\mathcal{B}(c, d) = \begin{cases} (a, b)\nabla(c, d) & ; a \neq c \text{ and } b \neq d \\ (a, b) = (c, d) & ; a = c \text{ or } b = d. \end{cases}$$

There are three types of equivalence classes generated by $\mathcal{B}$ i.e $\overline{(w, -\infty)}$ is called max-positive (simply written as w), $\overline{(-\infty), w}$ is called max-negative (simply written as $\ominus w$), and $\overline{(w, w)}$ is called balanced (simply written as $w^\bullet$). The max-zero class is denoted by $\overline{(\varepsilon, \varepsilon)}$ and simply written as ϵ. The quotient set of $\mathbb{R}_{\max} \times \mathbb{R}_{\max}$ by $\mathcal{B}$ is $(\mathbb{R}_{\max} \times \mathbb{R}_{\max})/\mathcal{B}$ and it is denoted by $\mathbb{S}$, where the zero element is $\varepsilon = \overline{(\varepsilon, \varepsilon)}$ and the unity element is $e = \overline{(0, \varepsilon)}$. Furthermore, $\mathbb{S}$ is called the symmetrized max-plus algebra. Then, the set of all max-positive or zero classes is denoted by $\mathbb{S}^\oplus$, the set of all max-negative or zero classes is denoted by $\mathbb{S}^\ominus$, the set of all balanced classes is denoted by $\mathbb{S}^\bullet$, and the set of all signed elements is denoted by $\mathbb{S}^\vee$, where $\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$, $\mathbb{S} = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$, $\{\overline{(\varepsilon, \varepsilon)}\} = \mathbb{S}^\oplus \cap \mathbb{S}^\ominus \cap \mathbb{S}^\bullet$. The set of all elements that have multiplication inverse is denoted by $(\mathbb{S}^\vee)_*$, where $(\mathbb{S}^\vee)_* = \mathbb{S}^\vee - \mathbb{S}^\bullet$.

Theorem 2.3. *[5] If $x, y \in \mathbb{R}_{max}$, then*

$$x \oplus (\ominus y) = \begin{cases} x & ; x > y \\ \ominus y & ; x < y \\ x^\bullet & ; x = y. \end{cases}$$

In the symmetrized max-plus algebra, similar to conventional algebra, for all $a, b, c \in \mathbb{S}$, $a \ominus c \nabla b$ if and only if $a \nabla b \oplus c$. However, there is a difference for the symmetrized max-plus algebraic substitution. For all $a, b, c \in \mathbb{S}$ and $x \in \mathbb{S}^{\vee}$, if $x \nabla a$ and $c \otimes x \nabla b$ then $c \otimes a \nabla b$. This substitution property in the symmetrized max-plus algebra is called the weak substitution. Consequently, if $a, b \in \mathbb{S}^{\vee}$ where $a \nabla b$, then $a = b$, which is known as reduction of balance to equation.

The connection between the symmetrized max-plus algebra and conventional algebra is studied in [4]. This link is used to solve problems in the symmetrized max-plus algebra using conventional algebraic methods, and the mapping will be explained in the following definition.

Definition 2.4. [4] A mapping $\mathcal{F}$ with domain of definition $\mathbb{S} \times \mathbb{R}_0 \times {\mathbb{R}_0}^+$ is defined as follows

$$\mathcal{F}(a, \mu, s) = \begin{cases} |\mu|\, e^{as} & ; a \in \mathbb{S}^{\oplus} \\ -\, |\mu|\, e^{|a|_{\oplus} s} & ; a \in \mathbb{S}^{\ominus} \\ |\mu|\, e^{|a|_{\oplus} s} & ; a \in \mathbb{S}^{\bullet}. \end{cases}$$

where $a \in \mathbb{S}$, $\mu \in \mathbb{R}_0$ and $s \in \mathbb{R}_0^+$.

Let f and g is a function, respectively. The function f is asymtotically equivalent to g in the neighborhood of ∞ is denoted by $f \sim g$ for $x \to \infty$.

Definition 2.5. [4] Let $f(s) \sim v e^{|a|_{\oplus} s}$ in the neighbourhood of ∞. The reverse function $\mathcal{R}$ is defined as

$$\mathcal{R}(f) = \begin{cases} |a|_{\oplus} & ; v \text{ positive} \\ \ominus\, |a|_{\oplus} & ; v \text{ negative.} \end{cases}$$

The study on matrix decomposition in the symmetrized max-plus algebra presented in [4] can be determined by establishing a connection between the symmetrized max-plus algebra and the conventional algebra, as defined in Definition (2.4) and (2.5). The following theorem shows the existence of singular value decomposition in matrices over the symmetrized max-plus algebra.

Theorem 2.6. *[4] Let $A \in \mathbb{S}^{m \times n}$ and $r = min(m, n)$. Then there are a diagonal max-plus matrix $S \in (\mathbb{R}_{max})^{m \times n}$ and $U \in (\mathbb{S}^{\vee})^{m \times m}$, $V \in (\mathbb{S}^{\vee})^{n \times n}$ such that*

$$A \nabla U \otimes S \otimes V^T$$

where $U^T \otimes U \nabla I_m$, $V^T \otimes V \nabla I_n$ and $\| \ A \ \|_{\oplus} = \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r$ with $\sigma_r = [S]_{ii}$ for $i = 1, 2, ..., r$.

3 Main Result

This section is main result of this paper. The main result of this paper is the construction of the Moore-Penrose inverse in the symmetrized max-plus algebraic matrix using singular value decomposition. First, we discuss the balanced inverse and the Moore-Penrose inverse. Then, we demonstrate the existence of the rank decomposition of the symmetrized max-plus algebraic matrix. Finally, we construct the Moore-Penrose inverse in the symmetrized max-plus algebraic matrix using singular value decomposition, by utilizing balanced inverse and rank decomposition. The following definition and theorem explain the balanced inverse and its existence in the symmetrized max-plus algebra. The definition of the balanced inverse is analogous to linear algebra, but with balance replacing the equation.

Definition 3.1. Let $A \in \mathbb{S}^{n \times n}$.If there exists $B \in \mathbb{S}^{n \times n}$ such that $A \otimes B \nabla I_n$ and $B \otimes A \nabla I_n$ then A is said to be balanced invertible and B is the balanced inverse of A. The balanced inverse of A is denoted by ${A_{\nabla}}^{-1}$.

The following theorem explains the existence of a balanced inverse in the symmetrized max-plus algebraic matrix.

Theorem 3.2. *Let $A \in \mathbb{S}^{n \times n}$. If $det(A)$ is not balance ε, then there exists ${A_{\nabla}}^{-1} \in (\mathbb{S}^{\vee})^{n \times n}$ such that*

$$A \otimes A_{\nabla}^{-1} \nabla I_n \text{ and } A_{\nabla}^{-1} \otimes A \nabla I_n.$$

Proof. Let a_{ij} is an entry of $A \in \mathbb{S}^{n \times n}$ for $i, j = 1, 2, \ldots, n$. If there exists balanced entry in $A \in \mathbb{S}^{n \times n}$, then it is defined a signed matrix $\hat{A} = [\hat{a}_{ij}] \in (\mathbb{S}^{\vee})^{n \times n}$ such that

$$\hat{a}_{ij} = \begin{cases} a_{ij} & ; a_{ij} \text{ signed element} \\ |a_{ij}|_{\oplus} & ; a_{ij} \text{ balanced element} \end{cases}$$

for all i, j. Since $\hat{a}_{ij} \nabla a_{ij}$ for all i, j then $\hat{A} \nabla A$. If $\hat{A} \nabla A$, $\hat{A} \otimes A_{\nabla}^{-1} \nabla I_n$ and $A_{\nabla}^{-1} \otimes \hat{A} \nabla I_n$ then $A \otimes A_{\nabla}^{-1} \nabla I_n$ and $A_{\nabla}^{-1} \otimes A \nabla I_n$, respectively. It is sufficient to prove for a signed matrix A.

Let $\tilde{A}(s) = [\tilde{a}_{ij}(s)]$ is matrix which corresponds to $A = [a_{ij}] \in \mathbb{S}^{n \times n}$. Since $det(A)$ is not balance with ε then $det(\tilde{A}(s)) \neq 0$. Let $cof(\tilde{A}(s))^T$ is transpose of cofactor matrix in $\tilde{A}(s)$, then we have

$$\frac{\tilde{A}(s).cof(\tilde{A}(s))^T}{det(\tilde{A}(s))} \text{ and } \frac{cof(\tilde{A}(s))^T.\tilde{A}(s)}{det(\tilde{A}(s))}$$

for $s \to -\infty$. Let

$$\tilde{A}'(s) = \frac{cof(\tilde{A}(s))^T}{det(\tilde{A}(s))}$$

then

$$\tilde{A}(s).\tilde{A}'(s) \sim \tilde{I}_n \text{ and } \tilde{A}'(s).\tilde{A}(s) \sim \tilde{I}_n.$$

Consequently, it corresponds to

$$A \otimes A_{\nabla}^{-1} \nabla I_n \text{ and } A_{\nabla}^{-1} \otimes A \nabla I_n$$

respectively. It is obtained the balanced inverse $A_{\nabla}^{-1} \in (\mathbb{S}^{\vee})^{n \times n}$ of a signed matrix A. By the weak substitution properties, the existence of $A_{\nabla}^{-1} \in (\mathbb{S}^{\vee})^{n \times n}$ also satisfies for a non-signed matrix. □

The following definition explains about the Moore-Penrose inverse of matrices over the symmetrized max-plus algebra. The Moore-Penrose inverse of matrices over the symmetrized max-plus algebra is defined analogously to linear algebra.

Definition 3.3. Suppose $M \in \mathbb{S}^{m \times n}$. The Moore-Penrose inverse of M is an $n \times m$ matrix denoted as M^+ that satisfies the following properties:

(i) $M \otimes M^+ \otimes M \nabla M$

(ii) $\otimes M^+ \otimes M \otimes M^+ \nabla M^+$

(iii) $(M \otimes M^+)^T \nabla M \otimes M^+$

(iv) $(M^+ \otimes M)^T \nabla M^+ \otimes M$

Let $A \in \mathbb{S}^{n \times n}$ and A_{∇}^{-1} be the balanced inverse of A. It's important to note that A_{∇}^{-1} also fulfills conditions 1, 2, 3, and 4 in Definition (3.3). Therefore, the balanced inverse of A, denoted as A_{∇}^{-1}, is also the Moore-Penrose inverse of A in the symmetrized max-plus algebra.

The discussion about the rank in a matrix over the symmetrized max-plus algebra is based on the definition of the minor rank as in [2]. Let A be a matrix over $\mathbb{S}$. The rank of A is defined using minors of A and is called the max-algebraic minor rank of A. The term max-algebraic minor rank will be simply referred to as the minor rank. The characterization of rank in a matrix over $\mathbb{S}$ using the linear independence approach, as in linear algebra, has been discussed in [13]. The rank of a matrix over $\mathbb{S}$ can be determined by calculating the maximum number of rows or columns that are linearly independent in a balanced sense. The following theorem explains the existence of rank decomposition in a matrix over the symmetrized max-plus algebra.

Theorem 3.4. *If $A \in \mathbb{S}^{m \times n}$ and minor rank of A is t, then there exist $C \in (\mathbb{S}^{\vee})^{m \times t}$ and $F \in (\mathbb{S}^{\vee})^{t \times n}$ such that $A \nabla C \otimes F$.*

Proof. In this proof, we will prove only for a signed matrix A. If $A \in \mathbb{S}^{m \times n}$ has balanced entries, then it is defined $\hat{A} = [\hat{a}_{ij}] \in (\mathbb{S}^{\vee})^{m \times n}$ such that

$$\hat{a}_{ij} = \begin{cases} a_{ij} & ; a_{ij} \text{ signed element} \\ |a_{ij}|_{\oplus} & ; a_{ij} \text{ balanced element} \end{cases}$$

for all i, j. For each $a, b \in \mathbb{S}$, if $a \nabla b$ then $a^{\bullet} \nabla b$. Therefore, if we show $A \nabla C \otimes F$, it is enough to show $\hat{A} \nabla C \otimes F$.

Let $\tilde{A}(s) = [\tilde{a}_{ij}(s)]$ is conventional matrix which corresponds to $A = [a_{ij} \in \mathbb{S}^{m \times n}]$. Thus, elementary row operations in $\tilde{A}(s)$ can be performed. The first step is to perform elementary row operations in $\tilde{A}(s)$ until the reduced row echelon form is obtained. Let $\tilde{P}(s)$ be a permutation matrix such that can be partitioned in the form $[\tilde{C}(s)\ \tilde{D}(s)]$, where $\tilde{C}(s)$ is a matrix whose columns are t pivot columns of $\tilde{A}(s)$. Furthermore, the columns of $\tilde{D}(s)$ can be expressed in the form

$$\tilde{d}_i(s) = \tilde{k}_1(s).\tilde{c}_1(s) + \tilde{k}_2(s).\tilde{c}_2(s) + \cdots + \tilde{k}_t(s).\tilde{c}_t(s)$$

for $i = 1, 2, \cdots, t-1$.Thus it is obtained the form $\tilde{D}(s) = \tilde{C}(s).\tilde{G}(s)$ where $\tilde{G}(s)$ is a matrix corresponding to the coefficients in the equation $\tilde{d}_i(s)$. Furthermore,

$$\tilde{A}(s).\tilde{P}(s) = [\tilde{C}(s) \quad \tilde{D}(s)] = [\tilde{C}(s) \quad \tilde{C}(s).\tilde{G}(s)]$$

By using the elementary matrix $\tilde{E}(s)$ the process of changing $\tilde{A}(s)$ is carried out until a reduced echelon matrix form $\tilde{B}(s)$ is obtained. So,

$$\tilde{E}(s)(\tilde{A}(s).\tilde{P}(s)) = \tilde{B}(s).\tilde{P}(s) = \tilde{E}(s)[\tilde{C}(s) \quad \tilde{D}(s)] = \tilde{E}(s).\tilde{C}(s)[\tilde{I}_t(s) \quad \tilde{G}(s)]$$

with $\tilde{E}(s).\tilde{C}(s)$ of the form $\begin{bmatrix} \tilde{I}_t(s) \\ 0 \end{bmatrix}$. Therefore,

$$\tilde{B}(s)(\tilde{P}(s) = \begin{bmatrix} \tilde{I}_t(s) & \tilde{G}(s) \\ 0 & 0 \end{bmatrix}$$

where $\begin{bmatrix} \tilde{I}_t(s) & \tilde{G}(s) \end{bmatrix}$ is a nonzero row of reduced row echelon form. By supposing $\begin{bmatrix} \tilde{I}_t(s) & \tilde{G}(s) \end{bmatrix} = \tilde{F}(s).\tilde{P}(s)$ then we have

$$\tilde{A}(s).\tilde{P}(s) = \tilde{C}(s) \begin{bmatrix} \tilde{I}_t(s) & \tilde{G}(s) \end{bmatrix} = \tilde{C}(s).\tilde{F}(s).\tilde{P}(s).$$

Since $\tilde{P}(s)$ is a permutation matrix, the asymptotically equivalent form $\tilde{A}(s) \sim \tilde{C}(s).\tilde{F}(s)$ for $s \to \infty$ is obtained, which is the rank decomposition form of $\tilde{A}(s)$. By implementing the reverse function $\mathcal{R}$, and suppose

$$A = \mathcal{R}(\tilde{A}(s)), C = \mathcal{R}(\tilde{C}(s)), F = \mathcal{R}(\tilde{F}(s))$$

then we get $A \nabla C \otimes F$ where $C \in (\mathbb{S}^{\vee})^{m \times t}$ and $F \in (\mathbb{S}^{\vee})^{t \times n}$.

□

The matrix S in Theorem (2.6) is a diagonal max-plus matrix in $(\mathbb{R}_{max})^{m \times n}$. When $m \leq n$, S can be represented as:

$$\begin{bmatrix} \sigma_1 & \varepsilon & \dots & \varepsilon & \dots & \varepsilon \\ \varepsilon & \sigma_2 & \dots & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ \varepsilon & \dots & \dots & \sigma_m & \dots & \varepsilon \end{bmatrix}.$$

If the value of $\sigma_1 = \varepsilon$ is E, then $\| A \|_{\oplus} = \varepsilon$, and consequently $\sigma_2 = \sigma_3 = \cdots = \sigma_m = \varepsilon$. Additionally, A is a matrix where each entry is ε. Therefore, in this paper, we assume that $\sigma_1 = \| A \|_{\oplus} \neq \varepsilon$. Since $\| A \|_{\oplus} = \sigma_1 \geq \sigma_2 \cdots \geq \sigma_m$, there is a $t \times t$ submatrix of S whose determinant is a non-zero element of $\mathbb{R}_{max}$, for $1 \leq t \leq m$. Consequently, the minor rank of S is t. For $m \geq n$, this can be done analogously by taking the transpose.

Corollary 3.5. *Let $A \in \mathbb{S}^{m \times n}$ and $A \nabla U \otimes S \otimes V^T$ is singular value decomposition of A. If the minor rank of S is t, then there are $C \in (\mathbb{R}_{max})^{m \times t}$ and $F \in (\mathbb{R}_{max})^{t \times n}$ such that $S = C \otimes F$.*

Proof. According to Theorem (3.4), there exist matrices $C \in (\mathbb{S}^{\vee})^{m\times t}$ and $F \in (\mathbb{S}^{\vee})^{t\times n}$ such that $S = C \otimes F$. Since S is a max-plus algebraic matrix, then both C and F are also max-plus algebraic matrices. By using the reduction of balance in the symmetrized max-plus algebra, we have $S = C \otimes F$. □

For $t = m \leq n$, it can be taken

$$\begin{bmatrix} \sigma_1 & \varepsilon & \dots & \varepsilon \\ \varepsilon & \sigma_2 & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \sigma_m \end{bmatrix}_{m\times m}.$$

and

$$\begin{bmatrix} 0 & \varepsilon & \dots & \varepsilon & \dots & \varepsilon \\ \varepsilon & 0 & \dots & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \varepsilon & \varepsilon & \dots & 0 & \dots & \varepsilon \end{bmatrix}_{m\times n}.$$

Then, it is obtained

$$C^T \otimes T = \begin{bmatrix} \sigma_1^2 & \varepsilon & \dots & \varepsilon \\ \varepsilon & \sigma_2^2 & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & \sigma_m^2 \end{bmatrix}_{m\times m}$$

and

$$C^T \otimes T = \begin{bmatrix} 0 & \varepsilon & \dots & \varepsilon \\ \varepsilon & 0 & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & 0 \end{bmatrix}_{m\times m}.$$

The minor ranks of $(C^T \otimes C)$ and $(F \otimes F^T)$ are m, respectively. Balanced inverses of $(C^T \otimes C)$ and $(F \otimes F^T)$ are guaranteed to exist. For $m \geq m$, this can be achieved by taking the transpose in a similar manner. The following theorem explains how to construct the Moore-Penrose inverse of S using the rank decomposition of S.

Theorem 3.6. *Suppose $A \in \mathbb{S}^{m\times n}$ and $A \nabla U \otimes S \otimes V^T$ is singular value decomposition of A. If the minor rank of S is $t = min(m, n)$ and $S = C \otimes F$ is the rank decomposition of S, then*

$$S^+ = F^T \otimes (F \otimes F^T)^{-1}_{\nabla} \otimes (C^T \otimes C)^{-1}_{\nabla} \otimes C^T$$

is the Moore-Penrose inverse of S.

Proof. Note that

(i) $S \otimes S^+ \otimes S$
$= C \otimes F \otimes F^T \otimes (F \otimes F^T)^{-1}_{\nabla} \otimes (C^T \otimes C)^{-1}_{\nabla} \otimes C^T \otimes C \otimes F$
$\nabla C \otimes I_t \otimes I_t \otimes S$
$= C \otimes F$
$= S$

(ii) $S^+ \otimes S \otimes S^+$
$= F^T \otimes (F \otimes F^T)^{-1}_{\nabla} \otimes (C^T \otimes C)^{-1}_{\nabla} \otimes C^T \otimes S \otimes F^T \otimes (F \otimes F^T)^{-1}_{\nabla} \otimes (C^T \otimes C)^{-1}_{\nabla} \otimes C^T$
$\nabla F^T \otimes (F \otimes F^T)^{-1}_{\nabla} \otimes I_t \otimes I_t \otimes (C^T \otimes C)^{-1}_{\nabla} \otimes C^T$
$= S^+$

(iii) $(S \otimes S^{+})^T$
$= [C \otimes F \otimes F^T \otimes (F \otimes F^T)_{\nabla}^{-1} \otimes (C^T \otimes C)_{\nabla}^{-1} \otimes C^T]^T$
$\nabla [C \otimes I^T \otimes (C^T \otimes C)_{\nabla}^{-1} \otimes C^T]^T$
$= [C \otimes (C^T \otimes C)_{\nabla}^{-1} \otimes C^T]^T$
$\nabla C \otimes (C^T \otimes C)_{\nabla}^{-1} \otimes C^T$
$\nabla S \otimes S^{+}$

(iv) $(S^{+} \otimes S)^T$
$= [F^T \otimes (F \otimes F^T)_{\nabla}^{-1} \otimes (C^T \otimes C)_{\nabla}^{-1} \otimes C^T \otimes C \otimes F]^T$
$\nabla [F^T \otimes (F \otimes F^T)_{\nabla}^{-1} \otimes I_t \otimes F]^T$
$= [F^T \otimes (F \otimes F^T)_{\nabla}^{-1} \otimes F]^T$
$\nabla F^T \otimes (F \otimes F^T)_{\nabla}^{-1} \otimes F$
$\nabla S^{+} \otimes S$

Since S^{+} satisfies condition 1, 2, 3 and 4 in Definition (3.3), then S^{+} is the Moore-Penrose inverse of S

□

Example 3.7. Suppose that

$$A = \begin{bmatrix} \ominus 5 & 1 & \ominus 0 \\ -3 & \varepsilon & (-2)^{\bullet} \end{bmatrix}$$

and the singular value decomposition of A is $U \otimes S \otimes V^T$ is, where

$$U = \begin{bmatrix} \ominus 0 & \ominus(-8) \\ 8 & \ominus 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & -2 & \varepsilon \end{bmatrix}$$

and

$$V = \begin{bmatrix} 0 & -5 & \ominus(-4) \\ \ominus(-4) & \ominus(-5) & \ominus 0 \\ -5 & \ominus 0 & -5 \end{bmatrix}.$$

Since $S^{+} = \begin{bmatrix} -5 & \varepsilon \\ \varepsilon & 2 \\ \varepsilon & \varepsilon \end{bmatrix}$ fullfils condition 1, 2, 3 and 4 in Definition (3.3), then S^{+} is the Moore-Penrose of S.

The main theorem discusses the construction of the Moore-Penrose inverse of A using the singular value decomposition of A.

Theorem 3.8. *If $A \in \mathbb{S}^{m \times n}$ and $A \nabla U \otimes S \otimes V^T$ is singular value decomposition of A, then $A^{+} = V \otimes S^{+} \otimes U^T$ is the Moore-Penrose inverse of A.*

Proof. In [4], the discussion of singular value decomposition of A is obtained using an assumption for a signed matrix A. We use the singular value decomposition of the signed matrix A i.e $A \nabla U \otimes S \otimes V^T$. Note that $A^{+} = V \otimes S^{+} \otimes U^T$ satisfies

(i) $A \otimes A^{+} \otimes A$
$\nabla U \otimes S \otimes V^T \otimes V \otimes S^{+} \otimes U^T \otimes U \otimes S \otimes V^T$
$\nabla U \otimes S \otimes V^T$
∇A

(ii) $A^{+} \otimes A \otimes A^{+}$
$\nabla V \otimes S^{+} \otimes U^T \otimes U \otimes S \otimes V^T \otimes V \otimes S^{+} \otimes U^T$
$\nabla V \otimes S^{+} \otimes U^T$
$= A^{+}$

(iii) $(A \otimes A^+)^T$
$\nabla [U \otimes S \otimes V^T \otimes V \otimes S^+ \otimes U^T]^T$
$= U \otimes S \otimes V^T \otimes V \otimes S^+ \otimes U^T$
$= A \otimes A^T$

(iv) $(A^+ \otimes A)^T$
$\nabla [V \otimes S^+ \otimes U^T \otimes U \otimes S \otimes V^T]^T$
$= V \otimes S^+ \otimes U^T \otimes U \otimes S \otimes V^T$
$= A^+ \otimes A$

Thus, A^+ is a Moore-Penrose inverse for A.

□

Example 3.9. Suppose

$$A = \begin{bmatrix} \ominus 5 & 1 & \ominus 0 \\ -3 & \varepsilon & (-2)^{\bullet} \end{bmatrix}$$

in Example (3.7), where

$$S^+ = \begin{bmatrix} -5 & \varepsilon \\ \varepsilon & 2 \\ \varepsilon & \varepsilon \end{bmatrix}$$

is the Moore-Penrose invere of S. Note that

$$A^+ = V \otimes S^+ \otimes U^T = \begin{bmatrix} \ominus(-5) & \ominus(-3) \\ -9 & -3 \\ -6 & 2 \end{bmatrix}$$

satisfies conditions 1, 2, 3 and 4 in Definition (3.3). Then, A^+ is a Moore-Penrose inverse of A.

4 Conclusion

The Moore-Penrose inverse in the symmetrized max-plus algebraic matrix is defined similarly to linear algebra and can be determined using singular value decomposition. This inverse is constructed using the balanced inverse and the full-rank decomposition of the matrix in the symmetrized max-plus algebra. Future research could focus on constructing the Moore-Penrose inverse in the symmetrized max-plus algebraic matrix using QR decomposition and Cholesky decomposition.

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