Palestine Journal of Mathematics

Vol 13(4)(2024) , 1366–1375

Applications of Noor type Differential Operator on Multivalent Functions

F. Müge Sakar, Saqib Hussain, Muhammad Naeem, Shahid khan, Gangadharan Murugusundaramoorthy and Zahid Shareef

Communicated by Thabet Abdeljawadi

MSC 2010 Classifications: Primary 30C45; Secondary 26A33, 33D15.

Keywords and phrases: Analytic functions; Multivalent functions; Noor type operator; Hadamard product (or convolution) .

Abstract In current study, we use the definition of convolution (or Hadamard product) and consider the Noor type differential operator to define a new class $\Omega_p(\lambda, \alpha; \Psi)$ of multivalent functions in open unit disk. We also give some interesting applications of this operator for multivalent functions by using the method of convolution and derive some useful results.

Acknowledgement: We record our sincere thanks to the referees for their insightful suggestions to improve the paper in present form

1 Introduction

Let $A_{(p)}$ be the class of analytic and p-valent functions which have the form

$$r(z) = z^p + \sum_{j=2}^{\infty} b_{j+p-1} z^{j+p-1} \quad p \in N = \{1, 2, 3, ...\}, \tag{1.1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For $p = 1$, it is clear that $A = A_{(1)}$.
If $r \in A_{(p)}$ satisfies the following condition

$$\Re\left(\frac{zr'(z)}{r(z)}\right) > \beta \qquad (z \in \Delta; 0 \leq \beta < p; p \in N),$$

the function $r \in A_{(p)}$ is p-valently starlike of order β in Δ.
If $r \in A_{(p)}$ holds the following condition

$$\Re\left(\frac{(zr'(z))'}{r'(z)}\right) > \beta \qquad (z \in \Delta; 0 \leq \beta < p; p \in N),$$

the function $r \in A_{(p)}$ is p-valently convex of order β in Δ.
The convolution of $r_1(z)$ and $r_2(z)$

$$(r_1 * r_2)(z) = z^p + \sum_{j=2}^{\infty} b_{j+p-1,1} b_{j+p-1,2} z^{j+p-1} = (r_2 * r_1)(z),$$

where

$$r_i(z) = z^p + \sum_{j=2}^{\infty} b_{j+p-1,i} z^{j+p-1} \in A_{(p)} \qquad (i = 1, 2).$$

Let P denotes the class of functions Ψ with $\Psi(0) = 1$. If r_1 and r_2 are analytic in Δ, we know that r_1 is subordinate to r_2, written as $r_1 \prec r_2$, if there is a analytic Schwarz function w, in Δ

which holds the conditions of $w(0) = 0$ and $|w(z)| < 1$ such that $r_1 = r_2(w(z))$. Moreover, we get the equivalence given below if r_2 is univalent in Δ, see [8].

$$r_1(z) \prec r_2(z) \Longleftrightarrow r_1(0) = r_2(0) \text{ and } r_1(\Delta) \subset r_2(\Delta). \quad z \in \Delta.$$

Operators theory has an important act in the field of Geometric Function Theory. Generally, operators are used to define new subclasses. The method of convolution has extraordinary role in the evolution of this field. Several integral and differential (linear operators) can be appointed in view of convolution. Alexander [1] presented the first integral operator for the class analytic functions. Further, numerous common integral operators are researched by authors, such as Libera [14], Bernardi [5], El-Ashwah and Aouf [6]. Srivastava et al. [23] geometrically investigated the class of complex fractional operators (differential and integral) and Ibrahim [4] defined differential operator and investigated a new class of analytic functions into two-dimensional fractional parameters in the open unit disk. For more details see [2, 3, 24]For complex or real number $a, b, c \notin \{0, -1, -2, ...\}$, the hypergeometric series is denoted by

$$_2F_1(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + ..., \tag{1.2}$$

It is important that the series (1) converges absolutely for $z \in \Delta$ and hereby it presents a function which is analytic in Δ. Patel and Cho [21] presented the operator $D^{\delta+p-1} : A_{(p)} \to A_{(p)}$ given by

$$D^{\delta+p-1}r(z) = \frac{z^p}{(1-z)^{\delta+p}} * r(z) \quad (\delta > -p), \tag{1.3}$$

on an equality with

$$D^{\delta+p-1}r(z) = \frac{z^p\left(z^{n-1}r(z)\right)^{n+p-1}}{(n+p-1)!},$$

where $n > -p$. From the equality, (1.2) and $r(z)$ is presented by (1.1), following equality occurs

$$D^{n+p-1}r(z) = z^p + \sum_{j=p+1}^{\infty} \binom{n+j+p-2}{j-1} b_{j+p-1}z^{j+p-1} \quad (p \in N\,; n > -p).$$

Upon $p = 1$, D^{n+p-1} was presented by Ruscheweyh [22] and Goel and Sohi [7]. D^{n+p-1} is known as the Ruscheweyh derivative of $(n+p-1)\,th$ order. Lately, similar to D^{n+p-1}, $I^{\lambda+p-1} : A_{(p)} \to A_{(p)}$, which is an integral operator, was introduced by Liu and Noor [15]. Let

$$r^{\lambda+p-1}(z) = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p),$$

and let $\left(r^{\lambda+p-1}(z)\right)^{\dagger}$ be given as

$$r^{\lambda+p-1}(z) * \left(r^{\lambda+p-1}(z)\right)^{\dagger} = \frac{z^p}{(1-z)^{p+1}}. \tag{1.4}$$

Then

$$I^{\lambda+p-1}r(z) = \left(r^{\lambda+p}(z)\right)^{\dagger} * r(z) = \left(\frac{z^p}{(1-z)^{\lambda+p}}\right)^{\dagger} * r(z) \quad r \in A_{(p)} \tag{1.5}$$

If $r(z)$ of (1.1), is used, then from the equations (1.4) and (1.5), we get

$$\begin{aligned} I^{\lambda+p-1}r(z) &= z^p + \sum_{j=p+1}^{\infty} \frac{(p+1)(p+2)...(j+p-1)}{(\lambda+p)(\lambda+p-1)...(\lambda+j+p-2)} b_{j+p-1}z^{j+p-1} \\ &= z^p_2F_1(1, p+1; \lambda+p; z) * r(z) \quad (\lambda > -p). \end{aligned} \tag{1.6}$$

Now, we introduce the function $Q^{\lambda}_{j+p-1} \in A_{(p)}$ by:

$$Q^{\lambda}_{j+p-1} = z^p + \sum_{j=2}^{\infty} \Phi_{j+p-1} z^{j+p-1}, \tag{1.7}$$

where

$$\Phi_{j+p-1} = \frac{(j+p-1)!\Gamma(p+\lambda)}{\Gamma(\lambda+j+p-1)}, \tag{1.8}$$

and

$$\Gamma(\lambda+1) = \lambda!. = \lambda(\lambda-1)(\lambda-2)..., \ \ (\lambda > -p, \ \ p \in N)$$

For $r \in A_{(p)}$, using the knowledgements about Noor type operator given above, we introduce a new operator $I^{\lambda+p-1} : A_{(p)} \to A_{(p)}$ by:

$$I^{\lambda+p-1} r(z) = Q^{\lambda}_{j+p-1} * r(z) = z^p + \sum_{j=2}^{\infty} \Phi_{j+p-1} b_{j+p-1} z^{j+p-1}. \tag{1.9}$$

Note that; upon taking $\lambda = 1$, we get

$$I^p r(z) = Q^1_{j+p-1} * r(z) = z^p + \sum_{j=2}^{\infty} \Gamma(p+1) b_{j+p-1} z^{j+p-1}.$$

upon taking $\lambda = 0$, we obtain

$$I^{p-1} r(z) = Q^0_{j+p-1} * r(z) = z^p + \sum_{j=2}^{\infty} (j+p-1)\Gamma(p) b_{j+p-1} z^{j+p-1}.$$

upon taking $\lambda = 1$ and $p = 1$, we obtain $r(z)$ as follows:

$$Ir(z) = Q_j * r(z) = z + \sum_{j=2}^{\infty} b_j z^j.$$

Furthermore, Noor integral operator of $(\lambda+p-1)\,th$ order of r is indicated by $I^{\lambda+p-1}r$ and given in (1.6), [15]. Several authors have studied on the concept of some analytic classes connected with Noor operator, [17, 18, 19]. Lately, Noor [20] mentioned new analytic subclasses connected with Noor integral operator and their geometric perspective. Now, from the equality (1.9) , we have easily the following identity:

$$(\lambda+p) I^{\lambda+p-1} r(z) = z\left(I^{\lambda+p} r(z)\right)' + \lambda I^{\lambda+p} r.(z) \tag{1.10}$$

Definition 1.1. The function $r \in A_{(p)}$ is belong to $\Omega_p(\lambda, \alpha; \Psi)$, if the first order differential subordinate condition, given below, holds:

$$(1-\alpha) z^{-p} I^{\lambda+p-1} r(z) + \frac{\alpha}{p} z^{-p+1}\left(I^{\lambda+p-1} r(z)\right)' \prec \Psi(z),$$

where $\alpha \in \mathbb{C}$, $\alpha > -p$, $p \in N$ and $\Psi \in P$.

Remark 1.2. For special cases of $\lambda = 1$, $\alpha = 1$ and $p = 1$, we can obtain following classes respectively

(i) $\Omega_p(1, \alpha; \Psi) = \Omega_p(\alpha, \Psi) = \left\{ r(z) \in A_{(p)} : (1-\alpha) z^{-p} r(z) + \frac{\alpha}{p} z^{-p+1} (r(z))' \prec \Psi(z) \right\}$

(ii) $\Omega_p(\lambda, 1; \Psi) = \Omega_p(\lambda, \Psi) = \left\{ r(z) \in A_{(p)} : \frac{1}{p} z^{-p+1} \left(I^{\lambda+p-1} r(z)\right)' \prec \Psi(z) \right\}$

(iii) $\Omega_1(\lambda, \alpha; \Psi) = \Omega(\lambda, \alpha, \Psi) = \left\{ r(z) \in A_{(p)} : (1-\alpha) \frac{1}{z} I^{\lambda} r(z) + \alpha \left(I^{\lambda} r(z)\right)' \prec \Psi(z) \right\}$

Lemma 1.3. *[16] Let* $s(z) = 1 + \sum_{n=m}^{\infty} b_n z^n$ $(m \in \mathbb{N})$ *be analytic in* Δ. *If* $Re(s(z)) > 0$ $(z \in \Delta)$, *then*

$$Re(s(z)) \geq \frac{1 - |z|^m}{1 + |z|^m} \quad (z \in \Delta).$$

In this study, we continue our investigations in the some properties of the class $\Omega_p(\lambda, \alpha; \Psi)$ connected with Noor type operator.

2 Main Results

In present section, we will derive our main theorems as characters of the class $\Omega_p(\lambda, \alpha; \Psi)$.

Theorem 2.1. *Let* $\alpha \geq 0$ *and*

$$r_i(z) = z^p + \sum_{j=2}^{\infty} b_{j+p-1,i} z^{j+p-1} \in \Omega_p(\lambda, \alpha; \Psi) \quad (i = 1, 2), \tag{2.1}$$

where

$$\Psi_i(z) = \frac{1 + C_i z}{1 + D_i z} \text{ and } -1 \leq D_i < C_i \leq 1. \tag{2.2}$$

If $r \in A_{(p)}$ *is given by*

$$I^{\lambda+p-1} r(z) = \left(I^{\lambda+p-1} r_1(z)\right) * \left(I^{\lambda+p-1} r_2(z)\right), \tag{2.3}$$

then $r \in \Omega_p(\lambda, \alpha; \Psi)$, *where*

$$\Psi(z) = \tau + (1 - \tau)\frac{1 + z}{1 - z}, \tag{2.4}$$

and τ *is given by*

$$\tau = \begin{cases} 1 - \frac{4(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)}\left(1 - \frac{p}{\alpha}\right)\int_0^1 \frac{t^{\frac{p}{\alpha}-1}}{1+t} dt, & \alpha > 0 \\ 1 - \frac{2(C_1 - D_1)(C_2 - D_2)}{(1 - D_1)(1 - D_2)}, & \alpha = 0. \end{cases} \tag{2.5}$$

the bound τ *is sharp when* $D_1 = D_2 = -1$.

Proof. For $\alpha > 0$. Since $r_i(z) \in \Omega_p(\lambda, \alpha; \Psi)$, it follows that

$$\begin{aligned} s_i(z) &= (1 - \alpha) z^{-p} I^{\lambda+p-1} r_i(z) + \frac{\alpha}{p} z^{-p+1}\left(I^{\lambda+p-1} r_i(z)\right)' \\ &\prec \frac{1 + C_i z}{1 + D_i z} \qquad (i = 1, 2), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} I^{\lambda+p-1} r_i(z) &= \frac{p}{\alpha} z^{-\frac{p(1-\alpha)}{\alpha}} \int_0^z t^{\frac{p}{\alpha}-1} s_i(t)\, dt \\ &= \frac{p}{\alpha} z^p \int_0^1 t^{\frac{p}{\alpha}-1} s_i(tz)\, dt \qquad (i = 1, 2). \end{aligned} \tag{2.7}$$

Now, if $r \in A_{(P)}$ is defined by (2.3), using the condition (2.6), we derive that

$$\begin{aligned} I^{\lambda+p-1}r(z) &= \left(I^{\lambda+p-1}r_1(z)\right) * \left(I^{\lambda+p-1}r_2(z)\right) \\ &= \left(\frac{p}{\alpha}z^p \int_0^1 t^{\frac{p}{\alpha}-1}s_1(tz)\,dt\right) * \left(\frac{p}{\alpha}z^p \int_0^1 t^{\frac{p}{\alpha}-1}s_2(tz)\,dt\right) \\ &= \frac{p}{\alpha}z^p \int_0^1 t^{\frac{p}{\alpha}-1}s_0(tz)\,dt, \end{aligned} \tag{2.8}$$

where

$$s_0(z) = \frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}(s_1 * s_2)(tz)\,dt. \tag{2.9}$$

Further, by using (2.6) and Herglotz theorem [8], we have

$$\Re\left\{\left(\frac{s_1(z)-\tau_1}{1-\tau_1}\right) * \left(\frac{1}{2}+\frac{s_2(z)-\tau_2}{2(1-\tau_2)}\right)\right\} > 0 \quad (z \in \Delta),$$

which leads to

$$\Re\{(s_1 * s_2)(z)\} > \tau_0 = 1 - 2(1-\tau_1)(1-\tau_2) \qquad (z \in \Delta),$$

where

$$0 \le \tau_i = \frac{1-C_i}{1-D_i} < 1 \quad (i = 1, 2).$$

Moreover, according to Lemma 1.3, we have

$$\Re\{(s_1 * s_2)(z)\} \ge \tau_0 + (1-\tau_0)\frac{1-|z|}{1+|z|} \quad (z \in \Delta). \tag{2.10}$$

Thus, it concludes from (2.8) to (2.10) that

$$\begin{aligned} \Re\{s_0(z)\} &= \Re\left\{(1-\alpha)z^{-p}I^{\lambda+p-1}r(z) + \frac{\alpha}{p}z^{-p+1}\left(I^{\lambda+p-1}r(z)\right)'\right\} \\ &= \frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}\Re\{(s_1 * s_2)\}(tz)\,dt \\ &\ge \frac{p}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}\left(\tau_0 + (1-\tau_0)\frac{1-|z|t}{1+|z|t}\right)dt \\ &> \tau_0 + \frac{p(1-\tau_0)}{\alpha}\int_0^1 t^{\frac{p}{\alpha}-1}\left(\frac{1-t}{1+t}\right)dt \\ &= 1 - 4(1-\tau_1)(1-\tau_2)\left(1 - \frac{p}{\alpha}\int_0^1 \frac{t^{\frac{p}{\alpha}-1}}{1+t}dt\right) \\ &= \tau, \end{aligned}$$

which proves that $r \in \mathbf{\Omega}_p(\lambda, \alpha; \mathbf{\Psi})$ for $\mathbf{\Psi}$ presented by (2.4).

To justify the sharpness of the bound τ, we get $r_i \in A_{(p)}$ $(i = 1, 2)$ defined by

$$I^{\lambda+p-1}r_i(z) = \frac{p}{\alpha}z^{-\frac{p(1-\alpha)}{\alpha}}\int_0^z t^{\frac{p}{\alpha}-1}\left(\frac{1+C_i t}{1-t}\right)dt \quad (i = 1, 2), \tag{2.11}$$

for which we have

$$s_i(z) = (1-\alpha) z^{-p} I^{\lambda+p-1} r_i(z) + \frac{\alpha}{p} z^{-p+1} \left(I^{\lambda+p-1} r_i(z)\right)'$$
$$\prec \frac{1+C_i z}{1-z} \qquad (i=1,2),$$

and

$$(s_1 * s_2)(z) = \left(\frac{1+C_1 z}{1-z}\right) * \left(\frac{1+C_2 z}{1-z}\right)$$
$$= 1-(1+C_1)(1+C_2) + \frac{(1+C_1)(1+C_2)}{1-z}.$$

Hence, for the function r in (2.3), we get

$$(1-\alpha) z^{-p} I^{\lambda+p-1} r(z) + \frac{\alpha}{p} z^{-p+1} \left(I^{\lambda+p-1} r(z)\right)'$$
$$= \frac{p}{\alpha} \int_0^1 t^{\frac{p}{\alpha}-1} \left(1-(1+C_1)(1+C_2) + \frac{(1+C_1)(1+C_2)}{1-z}\right) dt$$
$$\to \tau \quad (z \to -1),$$

which shows that the number τ is the best possible when $D_1 = D_2 = -1$.

For $\alpha = 0$, the proof of Theorem 2.1 is obvious, and we do not need to show details. □

Corollary 2.2. *Let* $\alpha \geq 0$ *and*

$$r_i(z) = z + \sum_{j=2}^{\infty} b_{j,i} z^j \in \mathbf{\Omega}_p(\lambda, \alpha; \mathbf{\Psi}) \quad (i=1,2),$$

where

$$\mathbf{\Psi}_i(z) = \frac{1+C_i z}{1+D_i z} \;\; \text{and} \;\; -1 \leq D_i < C_i \leq 1.$$

If $r \in A$ *is defined by*

$$I^{\lambda} r(z) = \left(I^{\lambda} r_1(z)\right) * \left(I^{\lambda} r_2(z)\right),$$

then $r \in \mathbf{\Omega}_p(\lambda, \alpha; \mathbf{\Psi})$, *where*

$$\mathbf{\Psi}(z) = \tau + (1-\tau)\frac{1+z}{1-z},$$

and τ *is given by*

$$\tau = \begin{cases} 1 - \frac{4(C_1-D_1)(C_2-D_2)}{(1-D_1)(1-D_2)}\left(1-\frac{1}{\alpha}\right)\int_0^1 \frac{t^{\frac{1}{\alpha}-1}}{1+t} dt, & \alpha > 0 \\ 1 - \frac{2(C_1-D_1)(C_2-D_2)}{(1-D_1)(1-D_2)}, & \alpha = 0. \end{cases} \tag{2.12}$$

the bound τ *is sharp when* $D_1 = D_2 = -1$.

Theorem 2.3. *Let* α *be a positive real number. Let* $r(z) = z^p + \sum_{j=2}^{\infty} b_{j+p-1} z^{j+p-1} \in A_{(p)}$, $h_1(z) = z^p$ *and* $h_m = z^p + \sum_{j=2}^{m} b_{j+p-1} z^{j+p-1}$ $(m \geq 2)$. *Suppose that*

$$\sum_{j=2}^{\infty} d_j \left|b_{j+p-1}\right| \leq 1, \tag{2.13}$$

where

$$d_j = \frac{1-D}{C-D}\Phi_{j+p-1}\left(1+\frac{\alpha}{p}(j-1)\right), \tag{2.14}$$

$$\Phi_{j+p-1} = \frac{(j+p-1)!\Gamma(p+\lambda)}{\Gamma(\lambda+j+p-1)} \text{ and } -1 \leq D < C \leq 1.$$

(i) *If* $-1 \leq D \leq 0$*, then* $r \in \Omega_p(\lambda, \alpha; \Psi)$*, where*

$$\Psi(z) = \frac{1+Cz}{1+Dz}.$$

(ii) *If* $\{d_j\}_1^\infty$ *is nondecreasing, then*

$$\Re\left\{\frac{r(z)}{h_m(z)}\right\} > 1 - \frac{1}{d_{m+1}}, \tag{2.15}$$

and

$$\Re\left\{\frac{h_m(z)}{r(z)}\right\} > \frac{d_{m+1}}{1+d_{m+1}}, \tag{2.16}$$

for $z \in \Delta$*. the estimates in* (2.15) *and* (2.16) *are sharp for each* $m \in \mathbb{N}$*.*

Proof. From the assumptions of Theorem 2.3, we get $d_j > 0\ (j \in \mathbb{N})$. Let

$$\begin{aligned} S(z) &= (1-\alpha) z^{-p} I^{\lambda+p-1} r(z) + \frac{\alpha}{p} z^{-p+1}\left(I^{\lambda+p-1} r(z)\right)' \\ &= 1 + \sum_{j=2}^{\infty} \frac{(j+p-1)!\Gamma(p+\lambda)}{\Gamma(\lambda+j+p-1)}\left(1+\frac{\alpha}{p}(j-1)\right) b_{j+p-1} z^{j+p-1}. \end{aligned} \tag{2.17}$$

(i) For $-1 \leq D \leq 0$ and $z \in \Delta$, it continues from (2.13), (2.14) and , (2.17) that

$$\begin{aligned} &\left|\frac{S(z)-1}{C-DS(z)}\right| \\ &= \left|\frac{\sum_{j=2}^{\infty} \frac{(j+p-1)!\Gamma(p+\lambda)}{\Gamma(\lambda+j+p-1)}\left(1+\frac{\alpha}{p}(j-1)\right) b_{j+p-1} z^{j+p-1}}{(C-D) - D\sum_{j=2}^{\infty} \frac{(j+p-1)!\Gamma(p+\lambda)}{\Gamma(\lambda+j+p-1)}\left(1+\frac{\alpha}{p}(j-1)\right) b_{j+p-1} z^{j+p-1}}\right| \\ &\leq \frac{\sum_{j=2}^{\infty} d_j |b_{j+p-1}|}{(1-D) + D\sum_{j=2}^{\infty} d_j |b_{j+p-1}|} \\ &\leq 1, \end{aligned}$$

which implies that

$$(1-\alpha) z^{-p} I^{\lambda+p-1} r(z) + \frac{\alpha}{p} z^{-p+1}\left(I^{\lambda+p-1} r(z)\right)' \prec \frac{1+Cz}{1+Dz} = \Psi,$$

hence, $r \in \Omega_p(\lambda, \alpha; \Psi)$.

(ii) Under the hypothesis in part (ii) of Theorem 2.3, we see from (2.14) that

$$d_{j+1} > d_j > 1 \quad (j \in \mathbb{N}).$$

Therefore, we have

$$\sum_{j=2}^{m} |b_{j+p-1}| + d_{m+1} \sum_{j=m+1}^{\infty} |b_{j+p-1}| \leq \sum_{j=2}^{\infty} d_j |b_{j+p-1}| \leq 1. \tag{2.18}$$

By setting

$$s_1(z) = d_{m+1}\left\{\frac{r(z)}{h_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right\}$$
$$= 1 + \frac{d_{m+1}\sum\limits_{j=m+1}^{\infty} b_{j+p-1}z^{j-1}}{1 + \sum\limits_{j=2}^{\infty} b_{j+p-1}z^{j-1}},$$

and applying (2.18) , we deduce that

$$\left|\frac{s_1(z)-1}{s_1(z)+1}\right| \leq \frac{d_{m+1}\sum\limits_{j=m+1}^{\infty}|b_{j+p-1}|}{2 - 2\sum\limits_{j=2}^{m}|b_{j+p-1}| - d_{m+1}\sum\limits_{j=m+1}^{\infty}|b_{j+p-1}|}$$
$$\leq 1 \quad (z \in \Delta),$$

which readily yields (2.15) .

If we take

$$r(z) = z^p - \frac{z^{m+p}}{d_{m+1}}, \tag{2.19}$$

then

$$\frac{r(z)}{h_m(z)} = 1 - \frac{z^m}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}} \quad \text{and} \quad z \to 1^-,$$

which shows that the inequality in (2.15) is the best likelihood for every $m \in \mathbb{N}$.

Similarly, if we take

$$s_2(z) = (1 + d_{m+1})\left(\frac{h_m(z)}{r(z)} - \frac{d_{m+1}}{1 + d_{m+1}}\right),$$

then we can obtain that

$$\left|\frac{s_2(z)-1}{s_2(z)+1}\right| \leq \frac{(1 + d_{m+1})\sum\limits_{j=m+1}^{\infty}|b_{j+p-1}|}{2 - 2\sum\limits_{j=2}^{m}|b_{j+p-1}| - (d_{m+1} - 1)\sum\limits_{j=m+1}^{\infty}|b_{j+p-1}|}$$
$$\leq 1 \quad (z \in \Delta),$$

which yields (2.16) . The inequality in (2.15) is the best likelihood for every $m \in \mathbb{N}$, with the extreme a function r given by (2.19) . □

Corollary 2.4. *Let α be a positive real number. Let $r(z) = z + \sum\limits_{j=2}^{\infty} b_j z^j \in A$, $h_1(z) = z$ and $h_m = z + \sum\limits_{j=2}^{m} b_j z^j$ $(m \geq 2)$. Suppose that*

$$\sum_{j=2}^{\infty} d_j |b_j| \leq 1,$$

where

$$d_j = \frac{1 - D}{C - D}\Phi_j(1 + \alpha(j-1)),$$

$$\Phi_j = \frac{j!\Gamma(1+\lambda)}{\Gamma(\lambda + j)} \text{ and } -1 \leq D < C \leq 1.$$

(i) If $-1 \leq D \leq 0$, then $r \in \Omega\left(\lambda, \alpha; \Psi\right)$, where

$$\Psi\left(z\right) = \frac{1 + Cz}{1 + Dz}.$$

(ii) If $\{d_j\}_1^\infty$ is non decreasing, then

$$\Re\left\{\frac{r\left(z\right)}{h_m\left(z\right)}\right\} > 1 - \frac{1}{d_{m+1}},$$

and

$$\Re\left\{\frac{h_m\left(z\right)}{r\left(z\right)}\right\} > \frac{d_{m+1}}{1 + d_{m+1}},$$

for $z \in \Delta$. the estimates in (ii) are sharp for every $m \in \mathbb{N}$.

Conclusion: Many differential and integral operator have been defined for analytic and multivalent functions. In this paper we used the definition of convolution (or Hadamard product) defined the Noor type differential operator. We consider this operator and investigated a new class $\Omega_p\left(\lambda, \alpha; \Psi\right)$ of multivalent functions in open unit disk disk. We also gave some interesting applications of this operator for multivalent functions by using the method of convolution and derive some useful results. For the future work on the subject of our study, we cite a number of articles [9, 10, 11, 12, 13] for the developments of geometric function theory related with q-calculus operator theory.

Acknowledgement: We record our sincere thanks to the referees for their insightful suggestions to improve the paper in present form

References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. Math*. **17**(1) (1915), 12–22.

[2] R. W. Ibrahim, Ulam stability for fractional differential equation in complex domain, *Abstract & Applied Analysis*, V(2012), Article ID 649517, 8pages, 2012.

[3] R.W, Ibrahim, Generalized Ulam-Hyers stability for fractional differential equations, *Inter. J.Math.*, **23**(5), article 1250056, 2012.

[4] R. W. Ibrahim, On generalized Srivastava-Owa fractional operators in the unit disk, *Advances in difference equations*, (2011), no 1, Article ID 55, 2011.

[5] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc*. **135** (1969), 429–446.

[6] R. M. El-Ashwah and M. K. Aouf, New classes of p-valent harmonic functions, *Bull. Math. Anal. Appl*. **2**(3) (2010), 53–64.

[7] R.M. Goel, N.S. Sohi, A new criterion for p-valent functions, *Proc. Amer. Math. Soc*. **78** (1980) 353–357.

[8] A. W. Goodman, Univalent Functions, vols. I, II. Polygonal Publishing House, New Jersey, (1983).

[9] Q. Hu, H. M. Srivastava, B. Ahmad, N. Khan, M. G. Khan, W. K. Mashwani and B. Khan, A subclass of multivalent Janowski type q-Starlike functions and its consequences, *Symmetry*, (13), (2021), 1275

[10] B. Khan, Z. G. Liu, T. G. Shaba, S. Araci, N. Khan, M. G. Khan, Applications of q-derivative operator to the subclass of bi-univalent functions involving q-Chebyshev polynomials. *J. Math*, (2022), 2022, ID8162182.

[11] M. F. Khan, A. Goswami and S. Khan, Certain new subclass of multivalent q-starlike functions associated with qsymmetric calculus, *Fractal Fract*. 2022, 6, 367. https://doi.org/10.3390. 39.

[12] B. Khan, H.M. Srivastava, S. Arjika, S. Khan, N. Khan and Q. Z. Ahmad, A certain q-Ruscheweyh type derivative operator and its applications involving multivalent functions, *Advances in Difference Equations* (2021) 2021:279 https://doi.org/10.1186/s13662-021-03441-6.

[13] S. Khan , S. Hussain and M. Darus, Inclusion relations of q-Bessel functions associated with generalized conic domain, *AIMS Mathematics*, **6**(4): 3624–3640.

[14] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc*. **16** (1965), 755–758.

[15] J.-L. Liu, K.I. Noor, Some properties of Noor integral operator, *J. Natur. Geom.* **21** (2002) 81–90.

[16] T. H. Macgregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc*. (1962).

[17] K.I. Noor, M.A. Noor, On integral operators, *J. Math. Anal. Appl*. **238** (1999), 341–352.

[18] K.I. Noor, On new classes of integral operators, *J. Natur. Geom.* **16** (1999), 71–80.

[19] K.I. Noor, M.A. Noor, On certain classes of analytic functions defined by Noor integral operator, *J. Math. Anal. Appl*. **281** (2003), 244–252.

[20] K.I. Noor, Some classes of p-valent analytic functions defined by certain integral operator, *Appl. Math.Comput.* **157** (2004) 835–840.

[21] J. Patel and N. E. Cho, Some classes of analytic functions involving Noor integral operator. *J. Math. Anal. Appl*. **312**(2) (2005), 564-75.

[22] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc*. **49** (1975), 109–115.

[23] H. M. Srivastava and S. Owa, Eds, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in: H.M. Srivastava, S. Owa, Univalent functions, fractional Calculus, and their applications, John Wiley & Sons, New York, etc. 1989.

[24] H. M. Srivastava, M. Darus and R. W. Ibrahim, Classes of analytic functions with fractional powers defined by means of a certain linear operators, *Integral Transforms and Special Functions* **22**(1), (2011), 17-28.

Author information

F. Müge Sakar, Saqib Hussain, Muhammad Naeem, Shahid khan, Gangadharan Murugusundaramoorthy and Zahid Shareef, Department of Business Administration, Dicle University, Diyarbakır, Turkey. Department of Mathematics, COMSATS University Islamabad, Abbottabad campus, Pakistan. Department of Mathematics and Statistic, International Islamic University, Islamabad, Pakistan. Department of Mathematics ,Riphah International University, Islamabad, Pakistan. chool of Advanced Sciences, Vellore Institute of Technology, Vellore - 632014, India. Division of Mathematics and Natural Science, Higher Colleges of Technology, Fujairah, United Arab Emirates.
E-mail: mugesakar@hotmail.com, saqibhussain@gmail.com, naeem.phdma75@iiu.edu.pk, shahidmath761@gmail.com, gmsmoorthy@yahoo.com, zshareef@hct.ac.ae

Received: 2022-07-20
Accepted: 2024-09-10