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DORROH EXTENSION OF NIL, NILPOTENT, AND K-BOOLEAN RINGS

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Abstract We investigate the transfer of the notions of the nil rings, the nilpotent rings and the K-Boolean rings in different ring extensions such as the trivial ring extension and Dorroh extension. We use these results to provide examples of the mentioned notions and also to test the transfer of these notions in other ring constructions such as the amalgamated duplication of a ring.

1 Introduction

Let R be a ring (not necessarily with a unity) and $\mathbb{Z}$ denote the ring of integers under the usual addition and multiplication. The set $\mathbb{Z} \times R$ under the component-wise addition and the multiplication given by the equation: $(m, x)(n, y) = (mn, my + nx + xy)$ for any $m, n \in \mathbb{Z}$ and $x, y \in R$ is a ring with identity $(1, 0)$. This ring is denoted by $\mathbb{Z} * R$ and is called the Dorroh extension of R. Notice that if R is identified by $\{0\} \times R$, then $\mathbb{Z} * R$ contains R as a subring. Hence, by Dorroh extension, one can embed a ring without a unity (R) in a ring contains a unity ($\mathbb{Z} * R$) [1].

In 2006, Anderson mentioned that the Dorroh extension can be lifted to any ring R (instead of $\mathbb{Z}$) and any R-algebra M (instead of R) in the following manner:

Let R be a ring (not necessarily with a unity) and M be an R-algebra. Then, $R \times M$ under the component-wise addition and the multiplication: $(a, m)(b, n) = (ab, an + bm + mn)$ for all $a, b \in R$ and $m, n \in M$ is a ring denoted by $R \dot{\oplus} M$ [2]. In this article we consider $R \dot{\oplus} M$ to be Dorroh extension of the ring R by its R-algebra M.

In 2007, Marco D'Anna and Marco Fontana used Dorroh extension to introduce a new construction called the amalgamated duplication of a ring R along an $R - module\ M$ (usually, M is a submodule of an over ring of R) and is denoted by $R \bowtie M$ [3]. They defined $R \bowtie M$ by the equation:

$$R \bowtie M = \psi(R \dot{\oplus} M) = \{(r, r + m) \mid r \in R, m \in M\}$$

where ψ is a homomorphism map from $R \dot{\oplus} M$ to $R \times (R + M)$ given by $\psi(r, m) = (r, r + m)$ for any $r \in R$ and $m \in M$.

It deserves to be mentioned that ψ is an injective ring homomorphism that enables us to see $R \bowtie M$ as a subring of $R \times (R + M)$ (being a homomorphic image of $R \dot{\oplus} M$). Actually it is a subdirect product of the rings R and $(R + M)$. For more about this construction, we refer the reader to [3, 4].

Let R be a commutative ring with unity and M be an $R-module$. In 1955, Nagata introduced the idealization of M in R (also called the trivial extension of R by M), denoted by $R \ltimes M$, which is endowed with the component-wise addition and the multiplication given by: $(r,m)(r',m') := (rr', rm' + r'm)$ *where* $r, r' \in R$ *and* $m, m' \in M$. For more about Nagata idealization, we refer the reader to [5, 6, 7, 8, 9].

Recall that a ring R is called a Boolean ring if $x^2 = x$ for all $x \in R$. In 1992, Kandasamy introduced the notion of the K-Boolean rings. He defined a ring R to be K-Boolean if $x^{2k} = x$ for any $x \in R$, where k is a positive integer [10]. It is very clear that Boolean rings are 1-Boolean, and in general, n-Boolean implies m-Boolean $\forall n \leq m$. Moreover, In the same paper, Kandasamy proved that; if R is K-Boolean, then $char(R) = 2$, where $char(R)$ denotes the characteristic of a ring R and is defined to be the smallest positive integer n such that $nr = 0$ for any $r \in R$, or 0 if there is no integer n such that $nr = 0$.

A ring R is called a nil ring if every element in R is nilpotent. (i.e., for any $x \in R$, there is a positive integer n such that $x^n = 0$), and is called a nilpotent ring if there is a positive integer m such that $R^m = \{0\}$ ($R^m := \{r^m \mid r \in R\}$). If R is nilpotent then the smallest positive integer k such that $R^k = \{0\}$ is called the nilpotency degree of R. It is very clear that nilpotent rings are nil, while the converse is not true in general; one may consider $\frac{R[X_1,X_2,...]}{(X_1{}^2,X_2{}^3,...)}$ as a counter-example of a nilpotent ring which is not nil, where R is a nilpotent ring. Moreover, they coincide when R is a finitely generated commutative ring without a unity. A good reference for these notions and facts is [11].

The first section of this article is devoted to establishing the transfer of the nil rings, nilpotent rings, and K-Boolean rings in different settings of Dorroh extension as well as deducing the transfer of these notions in the amalgamated duplication of a ring R along an R-module M. In the second section, we investigate the transfer of the nil rings and the nilpotent rings in the general context of the trivial ring extension. In both of the sections, illustrative and counter-examples are provided.

Throughout, R denotes a commutative ring; $Nil(R)$ denotes the set of all nilpotents of R; $char(R)$ denotes the characteristic of R; $Q(R)$ denotes the total ring of quotient of R; $R[X]$ denotes the ring of polynomials with one indeterminate X and coefficients from R; $R[X_1, X_2, ...]$ denotes the ring of polynomials with infinite number of indeterminates $X_1, X_2, ...$ and coefficients from R.

2 Dorroh extension of nil, nilpotent and K-Boolean rings

Let R be a ring and M be an R-algebra, we start this section with the following theorem which describes the nilpotent elements of $R\dot{\oplus}M$.

Theorem 2.1. *Let R be a ring and S be an $R-algebra$. If $M \subseteq S$, then $Nil(R\dot{\oplus}M) = Nil(R) \times (Nil(S) \cap M)$.*

Proof. Assume that $(x,y) \in Nil(R) \times Nil(S) \cap M$. Then there are positive integers m, n with $x^m = 0$ and $y^n = 0$. But then $(x,y)^{mn} = (0,0)$ and consequently, $(x,y) \in Nil(R\dot{\oplus}M)$. Conversely, assume that $(x,y) \in Nil(R\dot{\oplus}M)$. Then, there is a positive integer m such that $(x,y)^m = (0,0)$. So $x^m = 0$, and hence $x \in Nil(R)$ as well as $(x,0) \in Nil(R\dot{\oplus}M)$. Now, $Nil(R\dot{\oplus}M)$ being an ideal of $R\dot{\oplus}M$ (since R is commutative) implies that $(x,y) - (x,0) = (0,y) \in Nil(R\dot{\oplus}M)$. So that there is a positive integer k such that $(0,y)^k = (0,y^k) = (0,0)$, and consequently, $y \in Nil(S)$. Therefore, $(x,y) \in Nil(R) \times Nil(S) \cap M$. □

Next, we use Theorem 2.1 to establish the transfer of the nil ring notion in the Dorroh extension.

Corollary 2.2. *Let R be a ring and S be an $R-algebra$. If $M \subseteq S$ then $R\dot{\oplus}M$ is a nil ring if and only if R is a nil ring and $M \subseteq Nil(S)$.*

Proof. $R\dot{\oplus}M$ is a nil ring if and only if $Nil(R\dot{\oplus}M) = R\dot{\oplus}M$ if and only if $Nil(R) \times (M \cap Nil(S)) = R \times M$ if and only if $Nil(R) = R$ and $M \cap Nil(S) = M$ if and only if R is nil and $M \subseteq Nil(S)$. □

A special important case of Corollary 2.2 can be obtained by assuming that M is a submodule of $Q(R)$ with $M^2 \subseteq M$. It deserves to notify that all nilpotent elements are zero divisors and hence, in the nilpotent rings and also in the nil rings, there is no regular element which is not a unit, and hence $Q(R) = R$.

Corollary 2.3. *Let R be a ring and M a submodule of $Q(R)$ with $M^2 \subseteq M$. Then $R\dot{\oplus}M$ is a nil ring if and only if R is.*

Proof. Notice that R nil implies that $Q(R) = R$ and hence, $M \subseteq R$. Now by Corollary 2.2, $R\dot{\oplus}M$ is nil if and only if R is nil and $M \subseteq Nil(R) = R$ if and only if R is nil. □

Lemma 2.4. *Let R be a nilpotent ring with a nilpotency degree m. Then for any $x, y \in R$,*

$$\binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + ... + \binom{m}{m-1}xy^{m-1} = 0$$

Proof. Since R is a nilpotent ring, $r^m = 0$ for any $r \in R$. In particular, $(x+y)^m = 0$ for any $x, y \in R$. Hence, by Binomial theorem:

$$x^m + \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + ... + \binom{m}{m-1}xy^{m-1} + y^m = 0$$

But again, sincr R is nilpotent with nilpotency degree m, $x^m = y^m = 0$. Whence

$$\binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + ... + \binom{m}{m-1}xy^{m-1} = 0$$

□

As a consequence of the previous lemma, we conclude the following main result, which provides the transfer of the nilpotent notion in the special setting of Dorroh extension considered in Corollary 2.3.

Theorem 2.5. *Let R be a ring and M be a submodule of $Q(R)$ with $M^2 \subseteq M$. Then R is a nilpotent ring with nilpotency degree m if and only if $R\dot{\oplus}M$ is nilpotent with nilpotency degree m.*

Proof. Assume that $R\dot{\oplus}M$ is nilpotent with nilpotency degree m. Then $(R\dot{\oplus}M)^m = \{(0,0)\}$. Hence, $(r,0)^m = (r^m, 0) = (0,0)$ and thus $R^m = \{0\}$. So we conclude that R is nilpotent with nilpotency degree less than or equal to m which implies that $Q(R) = R$. The last fact turns to M is an ideal of R. Now, if the nilpotency degree of R is n, then using the fact that $M \subseteq R$ and Lemma 2.4, we deduce that for any $(x,y) \in R\dot{\oplus}M$,

$$(x,y)^n = (x^n, \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + ... + \binom{n}{n-1}xy^{n-1} + y^n) = (0,0)$$

and hence, $(R\dot{\oplus}M)^n = \{(0,0)\}$. But the nilpotency degree of $R\dot{\oplus}M$ is m implies that $n \geq m$. Consequently, $m = n$.
Conversely, assume that R is nilpotent with nilpotency degree m. Then, again $M \subseteq R$ and we have, by Lemma 2.4

$$(x,y)^m = (x^m, \binom{m}{1}x^{m-1}y + \binom{m}{2}x^{m-2}y^2 + ... + \binom{m}{m-1}xy^{m-1} + y^m) = (0,0)$$

. Thus, $(R\dot{\oplus}M)^n = \{(0,0)\}$. Now, if $(R\dot{\oplus}M)^k = \{(0,0)\}$ with $k \leq m$, then $(r,0)^k = (0,0)$ for any $r \in R$ which implies that $R^k = \{0\}$. Lastly, the nilpotency degree of R is m implies that $k = m$. □

Theorem 2.5 provides examples of nilpotent rings, as shown below.

Example 2.6. Let R be a nilpotent ring with nilpotency degree 2. For example, take $R = \{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{R}\}$. Then, for any I ideal of R, $R\dot{\oplus}I$ is another nilpotent ring with nilpotency degree 2.

Further, one may use Theorem 2.5 to provide new examples of nil rings that are not nilpotent.

Example 2.7. Let R be a nil ring which is not nilpotent.
(1) Consider the R-algebra $\frac{R[X]}{(X^2)}$. If $M = \frac{(x)}{(x^2)}$. Then, $M^2 = (\frac{(x)}{(x^2)})^2 = 0$, and hence $M \subseteq Nil(\frac{R[X]}{(X^2)})$. Now using Corollary 2.2, $R\dot{\oplus}\frac{(x)}{(x^2)}$ is a nil ring.
(2) $R\dot{\oplus}R$ is a nil ring (Corollary 2.3) which is not nilpotent (Theorem 2.5).

Next, we test the transfer of the K-Boolean rings in Dorroh extension. The following theorem and proposition are needed to prove Theorem 2.11.

Theorem 2.8 (Lucas's theorem). *For non-negative integers m and n and a prime p, the following congruence relation holds:*

$$\binom{m}{n} = \prod_{i=0}^{k} \binom{m_i}{n_i} (mod\ p)$$

where $m = m_k p^k + m_{k-1}p^{k-1} + ... + m_1 p + m_0$ and $n = n_k p^k + n_{k-1}p^{k-1} + ... + n_1 p + n_0$ are the base p expansions of m and n, respectively. This uses the convention that $\binom{m}{n} = 0$ if $m < n$.

Proof. See [12, 13]. □

Corollary 2.9. *Let n be an even positive integer and r be an odd positive integer. Then, $\binom{n}{r}$ is even.*

Proof. Assume that $n = n_k 2^k + n_{k-1}2^{k-1} + ... + n_1 2 + n_0$ and $r = r_k 2^k + r_{k-1}2^{k-1} + ... + r_1 2 + r_0$. Since n is even and r is odd, we must have $n_0 = 0$ and $r_0 = 1$. Thus, by Lucas's theorem, $\binom{n}{r} = \binom{n_k}{r_k}\binom{n_{k-1}}{r_{k-1}}...\binom{n_1}{r_1}\binom{0}{1}(mod 2)$. Now, using the convention $\binom{m}{n} = 0$ if $m < n$, we conclude that $\binom{0}{1} = 0$, and consequently, $\binom{n}{r} = 0(mod 2)$. Equivalently, $\binom{n}{r}$ is even. □

Proposition 2.10. *Let R be a K-Boolean ring. Then for any $x, y \in R$ we have;*

$$\binom{2k}{2} x^{2k-2}y^2 + ... + \binom{2k}{2k-2} x^2 y^{2k-2} = 0$$

Proof. Assume that R is a K-Boolean ring and $x, y \in R$. Then trivially, we have $x^{2k} = x$, $y^{2k} = y$, and $(x + y)^{2k} = x + y$ (all of them are elements of R). But according to Birnoulli's equation,

$$(x+y)^{2k} = x^{2k} + \binom{2k}{1} x^{2k-1}y + \binom{2k}{2} x^{2k-2}y^2 + ... + \binom{2k}{2k-2} x^2 y^{2k-2} + \binom{2k}{2k-1} xy^{2k-1} + y^{2k}$$

Consequently,

$$x + y = x + y + \binom{2k}{1} x^{2k-1}y + \binom{2k}{2} x^{2k-2}y^2 + ... + \binom{2k}{2k-2} x^2 y^{2k-2} + \binom{2k}{2k-1} xy^{2k-1}$$

Or equivalently,

$$\binom{2k}{1} x^{2k-1}y + \binom{2k}{2} x^{2k-2}y^2 + ... + \binom{2k}{2k-2} x^2 y^{2k-2} + \binom{2k}{2k-1} xy^{2k-1} = 0 \qquad ... (1)$$

Now, Corollary 2.9 implies that all integers $\binom{2k}{1}, \binom{2k}{3} ..., \binom{2k}{2k-1}$ are positive even integers. Thus, $\binom{2k}{1} = 2t$ for some integer t, and consequently, $\binom{2k}{1}x^{2k-1}y = 2tx^{2k-1}y = 2(tx^{2k-1}y)$. But $char(R) = 2$ (R is K-Boolean) implies that $\binom{2k}{1}x^{2k-1}y = 2(tx^{2k-1}y) = 0$.
Similar arguments lead to $\binom{2k}{3}x^{2k-3}y^3 = ... = \binom{2k}{2k-1}xy^{2k-1} = 0$. So that (1) becomes $\binom{2k}{2}x^{2k-2}y^2 + ... + \binom{2k}{2k-2}x^2y^{2k-2} = 0$. □

The following theorem establishes the transfer of the K-Boolean rings in the Dorroh extension. Notice that, if R is a K-Boolean ring with unity 1, then each element of R is either 1 or a zero divisor, and hence $Q(R) = R$.

Theorem 2.11. *Let R be a ring with unity, and M be a sub-module of $Q(R)$ with $M^2 \subseteq M$. Then, $R\dot{\oplus}M$ is K-Boolean if and only if R is.*

Proof. Assume that $R\dot{\oplus}M$ is K-Boolean. Let x be any element of R. Then $(x,0)^{2k} = (x^{2k},0) = (x,0)$. Hence, $x^{2k} = x$ for any $x \in R$, and consequently, R is K-Boolean.
Conversely, Assume that R is K-Boolean. Then, $Q(R) = R$, and hence $M \subseteq R$. Now, Let $(x,y) \in R\dot{\oplus}M$. Then $(x,y)^{2k} = (x^{2k}, \binom{2k}{1}x^{2k-1}y + ... + y^{2k})$. Using Corollary 2.9 and Proposition 2.10, we deduce that $(x,y)^{2k} = (x,y)$ for any $(x,y) \in R\dot{\oplus}M$. Therefore, $R\dot{\oplus}M$ is K-Boolean. □

Example 2.12. Let $R = \prod_{i=1}^{\infty} R_i$ and $I = \oplus_{i=1}^{\infty} R_i$ where R_i is $\mathbb{Z}_2$. Then $R\dot{\oplus}I$ is a Boolean ring. Indeed, trivially R is Boolean (being a product of Boolean rings) and I is an ideal of R. Thus $R\dot{\oplus}I$ is a Boolean ring by Theorem 2.11.

Example 2.13. Let R be a 2-Boolean ring which is not Boolean (Proposition 3 in [10]). Then, by Theorem 2.11, $R\dot{\oplus}R$ is another example of a 2-Boolean ring which is not Boolean.

3 The amalgamated duplication and the trivial ring extension of nil, nilpotent and K-Boolean rings

In this section, we will test the transfer of the nil, nilpotent and K-Boolean rings in the amalgamated duplication and the trivial ring extension "idealization". We start with the nilpotent rings. The following is a corollary of Corollary 2.3 and Theorem 2.5.

Corollary 3.1. *Let R be a ring and M is an R-submodule of $Q(R)$ with $M^2 \subseteq M$. Then: (1) $R \bowtie M$ is a nilpotent ring if and only if R is.*
(2) $R \bowtie M$ is a nil ring if and only if R is.

Proof. (1) Assume that R is nilpotent. By Theorem 2.5, $R\dot{\oplus}M$ is a nilpotent ring. But a homomorphic image of a nilpotent element is nilpotent implies that $R \bowtie M$ is nilpotent. Conversely, assume that $R \bowtie M$ is nilpotent. Then, $(x, x + 0)$ is a nilpotent element for any $x \in R$. Hence, there exists an integer k with $(x,x)^k = (x^k, x^k) = (0,0)$. Thus, $x^k = 0$. Similar proof for (2). □

The next is an example of nilpotent rings.

Example 3.2. Let $R = M = \{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}\}$. Then trivially R is a nilpotent ring. Hence, by Corollary 3.1 $R \bowtie R$ is a nilpotent ring.

Now, we will test the transfer of the nil and nilpotent rings in the trivial ring extension "idealization".

Theorem 3.3. *Let R be a ring and M an R-module. Then,*
(1) $R \ltimes M$ is nilpotent if and only if R is nilpotent. Moreover, if the nilpotency degree of R is n, then the nilpotency degree of $R \ltimes M$ is n or $n+1$.
(2) $R \ltimes M$ is nil if and only if R is.

Proof. (1) First, assume that $R \ltimes M$ is nilpotent. Since the quotient of a nilpotent ring is nilpotent, and given the isomorphism $R \cong \frac{R \ltimes M}{0 \ltimes M}$, we deduce that R is nilpotent. Conversely, assume that R is nilpotent with nilpotency degree n. If $(x,m) \in R \ltimes M$ where $x \in R$ and $m \in M$, then trivially, $(x,m)^k = (x^k, kx^{k-1}m)$, for any positive integer k. So that, $(x,m)^{n+1} = (x^{n+1}, (n+1)x^n m) = (0,0)$. Now, (x,m) being arbitrary in $R \ltimes M$ implies that $(R \ltimes M)^{n+1} = 0$, so the nilpotency degree of $R \ltimes M$ is less than or equal to $n+1$. Now if the nilpotency degree of $R \ltimes M$ is l , then $(x,m)^l = (0,0)$ for any $(x,m) \in R \ltimes M$. Particularly, $(x,0)^l = (0,0)$ for any $x \in R$. So that,

$R^l = 0$, and hence, $l \geq n$ which proves the moreover statement.
(2) By the idealization of R-module in a ring R and Theorem 2.1, we have $Nil(R \ltimes M) = Nil(R) \ltimes M$. But $R \ltimes M$ is nil ring if and only if $Nil(R \ltimes M) = R \ltimes M$ if and only if $Nil(R) \ltimes M = R \ltimes M$ if and only if $Nil(R) = R$ if and only if R is nil ring. □

The following corollary establishes the transfer of the K-Boolean notion in the amalgamated duplication.

Corollary 3.4. *Let R be a ring and M be an R-submodule of $Q(R)$ with $M^2 \subseteq M$. Then, $R \bowtie M$ is K-Boolean if and only if R is.*

Proof. Assume that $R \bowtie M$ is K-Boolean. Then $(x, x+0)^{2k} = (x, x)$ for all $x \in R$, and consequently, R is K-Boolean. Conversely, Assume that R is K-Boolean. Then by Theorem 2.11 $R = Q(R)$ and $R \bigoplus M$ is K-Boolean, consequently, any of its homomorphic image is K-Boolean. Particularly, $R \bowtie M$ is K-Boolean. □

The next is an example of K-Boolean rings.

Example 3.5. Let R be a Boolean ring (e.g., $\mathbb{Z}_2 \times \mathbb{Z}_2$). Then $R \bowtie R$ is also a Boolean ring. This is a direct consequence of Corollary 3.4.

Lastly, we will test the transfer of the K-Boolean rings in the trivial ring extension "idealization". The main theorem ensures that the idealization of a non-zero module over a ring R is never K-Boolean even if the ring R is K-Boolean.

Theorem 3.6. *Let R be a ring and M be an R-module. $R \ltimes M$ is K-Boolean if and only if R is K-Boolean and $M = 0$.*

Proof. Assume that $R \ltimes M$ is K-Boolean. Then, $(x, m)^{2k} = (x, m)$ for any $(x, m) \in R \ltimes M$. But it is easy to see that $(x, m)^{2k} = (x^{2k}, 2kx^{2k-1}m)$. So we have the equality:

$$(x, m) = (x, m)^{2k} = (x^{2k}, 2kx^{2k-1}m) \qquad ...(2)$$

Hence, $x^{2k} = x$ for any $x \in R$, and consequently, R is K-Boolean. Now, R being K-Boolean implies that $char(R) = 2$ and hence, $2kx^{2k-1}m = 0$. Thus we obtain from (2), $(x, m) = (x^{2k}, 0)$ which implies that $m = 0$ for any $m \in M$. Therefor, $M = 0$. The converse is very trivial as $R \ltimes 0 \cong R$. □

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