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Uniqueness Results of Entire Functions Concerning Their Shifts and Derivatives

Pulak Sahoo and Suman Pal

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Corresponding Author: Pulak Sahoo

Abstract: In the paper, we study the uniqueness of entire functions when their shifts and kth derivative of the function share two values ignoring multiplicities. The result of the paper generalizes a recent result due to Huang and Fang [Computational Methods and Function Theory, 21 (2021), 523-532].

1 Introduction, Definitions and Results

In what follows, we assume that the reader is familiar with standard notations and main results of Nevanlinna value distribution theory as explained in [7, 20, 21]. Let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic function of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty,\ r \notin E$. Let $f,\ g$ be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$ and let a be a complex value. We say that f and g share a CM(IM), provided that $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities (ignoring multiplicities). Define

$$\rho(f) = \overline{lim}_{r\to\infty}\frac{log^{+}T(r, f)}{log\ r},$$

$$\rho_2(f) = \overline{lim}_{r\to\infty}\frac{log^{+}log^{+}T(r, f)}{log\ r}$$

as the order and the hyper-order of f, respectively. For a meromorphic function $f(z)$, we define its shift by $f(z + c)$. Moreover, we introduce the following notation: $S_{(m,n)}(a) = \{z|\ z$ is a common zero of $f(z+c)-a$ and $f^{(k)}(z)-a$ with multiplicities m and n respectively$\}$. $\overline{N}_{(m,n)}(r, 1/(\text{f}-a))$ denotes the counting function of f with respect to the set $S_{(m,n)}(a)$. Here $\overline{N}_{n)}(r, 1/(f - a))$ denotes the counting function of zeros of $f - a$ with multiplicities at most n. $\overline{N}_{(n}(r, 1/(f - a))$ denotes the counting function of all zeros of $f - a$ with multiplicities at least n. $\overline{N}_n(r, 1/(f - a))$ denotes the counting function of all zeros of $f - a$ with multiplicity exactly n. For some related studies we refer the reader to see [17, 18, 19].

In 1977, Rubel and Yang [16] first investigated the uniqueness of an entire function concerning its first derivatives and proved the following result.

Theorem A. Let f be a nonconstant entire function and $a,\ b$ be two distinct finite complex values. If $f(z)$ and $f'(z)$ share $a,\ b$ CM, then $f(z) \equiv f'(z)$.

In 1979, Mues and Steinmetz [12] replaced CM sharing values by IM sharing values and proved the following result which improves Theorem A.

Theorem B. Let f be a nonconstant entire function and $a,\ b$ be two distinct finite complex values. If $f(z)$ and $f'(z)$ share $a,\ b$ IM, then $f(z) \equiv f'(z)$.

In the last decade, the value distribution of entire and meromorphic functions with respect to the difference analogue has become a subject of great interest for researchers, see[1-6, 8, 10,11, 13-15, 22]. In 2011, Heittokanges et al. [8] proved a similar analogue of Theorem A concerning their shifts.

Theorem C. Let f be a nonconstant entire function of finite order, let c be a nonzero finite complex value, and let $a,\ b$ be two distinct finite complex values. If $f(z)$ and $f(z+c)$ share $a,\ b$ CM, then $f(z) \equiv f(z+c)$.

In the same year, Qi [13] proved the following theorem for IM shared values.

Theorem D. Let f be a nonconstant entire function of finite order, let c be a nonzero finite complex value, and let $a,\ b$ be two distinct finite complex values. If $f(z)$ and $f(z+c)$ share $a,\ b$ IM, then $f(z) \equiv f(z+c)$.

In 2018, Qi, Li and Yang [14] investigated the value sharing problem related to $f'(z)$ and $f(z+c)$ and proved the following result.

Theorem E. Let f be a nonconstant entire function of finite order, and let $a,\ c$ be two finite nonzero complex values. If $f'(z)$ and $f(z+c)$ share $0,\ a$ CM, then $f'(z) \equiv f(z+c)$.

In 2020, Qi and Yang [15] improved Theorem E and proved the following results.

Theorem F. Let f be a nonconstant entire function of finite order, and let $a,\ c$ be two nonzero finite complex values. If $f'(z)$ and $f(z+c)$ share 0 CM and a IM, then $f'(z) \equiv f(z+c)$.

Theorem G. Let f be a transcendental entire function of finite order, let c be a nonzero finite complex value, and let $a,\ b$ be two distinct finite complex values. If $f'(z)$ and $f(z+c)$ share $a,\ b$ IM, then $T(r, f(z+c)) = O(T(r, f')),\ \ T(r, f'(z)) = O(T(r, f(z+c)))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure.

Theorem H. Let f be a transcendental entire function of finite order, let c be a nonzero finite complex value, and let $a,\ b$ be two distinct finite complex values. If $f'(z)$ and $f(z+c)$ share $a,\ b$ IM, and $\overline{N}(r, \frac{1}{f'-a}) = S(r, f)$, then $f'(z) \equiv f(z+c)$.

Regarding Theorem H, the following question is inevitable.

Question 1.1. Is the condition $\overline{N}(r, \frac{1}{f'-a}) = S(r, f)$ in Theorem H necessary or not?

Recently, Huang and Fang [9] answered the above question in a positive sense and proved the following result.

Theorem I. Let f be a transcendental entire function with $\rho_2(f) < 1$, let c be a nonzero finite complex value, and let $a,\ b$ be two distinct finite complex values. If $f'(z)$ and $f(z+c)$ share $a,\ b$ IM, then $f'(z) \equiv f(z+c)$.

From Theorem I, it is natural to ask the following question which motivate us to write up this paper.

Question 1.2. Is the same type of conclusion can be drawn if we replace the first derivative of f by kth derivatives of f, where k is some positive integer?

In this paper we investigate to find out a possible answer of the above question and obtain the following theorem.

Theorem 1.1. *Let f be a transcendental entire function of hyper order $\rho_2(f) < 1$, let c be a nonzero finite complex value, and let $a,\ b$ be two distinct finite complex values. If $f^{(k)}(z)$ and $f(z+c)$ share $a,\ b$ IM, then $f^{(k)}(z) \equiv f(z+c)$, where k is some positive integer.*

2 Lemmas

In this section we present some known results which will be needed in the sequel.

Lemma 2.1. *[6] Let f be a nonconstant meromorphic function of hyper order $\rho_2(f) < 1$. Then for any $c \in \mathbb{C}\backslash\{0\}$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f).$$

Lemma 2.2. *[21] Let $f_1,\ f_2$ be two nonconstant meromorphic functions defined in $|z| < \infty$. Then*

$$N\left(r, f_1 f_2\right) - N\left(r, \frac{1}{f_1 f_2}\right) = N\left(r, f_1\right) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right),$$

where $0 < r < \infty$.

Lemma 2.3. *Let f be a nonconstant meromorphic function, and let $P(f) = a_0 f^p + a_1 f^{p-1} + ... + a_p$, $a_0 \neq 0$, be a polynomial of degree p with constant coefficients a_j, $j = 0, 1, 2, ..., p$. Suppose that $b_j, j = 0, 1, 2, ..., q$, $q > p$, are distrinct finite complex numbers. Then for some positive integer k, we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

$$\textit{and} \qquad m\left(r, \frac{P(f) f^{(k)}}{(f-b_1)(f-b_2)....(f-b_q)}\right) = S(r,f).$$

Proof. The 1st one follows from Milloux Theorem [7] and the 2nd one follows from 1st by factorising it as,

$$\frac{P(f) f^{(k)}}{(f-b_1)(f-b_2)...(f-b_q)} = \sum_{i=1}^{q} \frac{c_i f^{(k)}}{f-b_i},$$

where $c_1\ c_2,\ .\ .\ .,\ c_q$ are nonzero complex numbers. □

3 PROOF OF THE THEOREMS

Proof of Theorem 1.1. If possible, we assume that $f(z+c) \not\equiv f^{(k)}(z)$. Since $f^{(k)}(z)$ and $f(z+c)$ share $a,\ b$ IM and f is transcendental entire function with $\rho_2(f) < 1$, using Nevanlinna's second fundamental theorem, Lemmas 2.1 and 2.3 we obtain

$$\begin{aligned} T(r, f(z+c)) &\leq \overline{N}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b}\right) + \overline{N}(r, f(z+c)) \\ &\quad + S(r,f) \\ &= \overline{N}\left(r, \frac{1}{f^{(k)}(z)-a}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}(z)-b}\right) + S(r,f) \\ &\leq N\left(r, \frac{1}{f(z+c)-f^{(k)}(z)}\right) + S(r,f) \\ &\leq T(r, f(z+c)-f^{(k)}(z)) + S(r,f) \\ &\leq m(r, f(z+c)-f^{(k)}(z)) + S(r,f) \\ &\leq m(r, f(z+c)) + m\left(r, 1-\frac{f^{(k)}(z)}{f(z+c)}\right) + S(r,f) \\ &\leq T(r, f(z+c)) + S(r,f). \end{aligned}$$

This shows that

$$T(r, f(z+c)) = \overline{N}\left(r, \frac{1}{f^{(k)}(z)-a}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}(z)-b}\right) + S(r, f). \tag{3.1}$$

Put

$$\phi(z) = \frac{f'(z+c)(f^{(k)}(z) - f(z+c))}{(f(z+c)-a)(f(z+c)-b)}, \tag{3.2}$$

$$\psi(z) = \frac{f^{(k+1)}(z)(f^{(k)}(z) - f(z+c))}{(f^{(k)}(z)-a)(f^{(k)}(z)-b)}. \tag{3.3}$$

Let z_0 be a common zero of $f(z+c)-a$ (resp. $f(z+c)-b$) and $f^{(k)}(z)-a$ (resp. $f^{(k)}(z)-b$) with multiplicities m and n respectively. Assume that $m \geq n$. Then z_0 is a zero of $f(z+c) - f^{(k)}(z)$ with multiplicity at least n. Obviously, z_0 is a zero of $f'(z+c)$ with mulplicity $(m-1)$. It, therefore, follows that $\phi(z_0) \neq \infty$. Since $f(z+c)$ and $f^{(k)}(z)$ share the values $a,\ b$ IM, from (3.2), we see that $\phi(z)$ has no poles and hence $\phi(z)$ is an entire function. Therefore, by Lemmas 2.1 and 2.3 we have

$$\begin{aligned} T(r, \phi(z)) &= m(r, \phi(z)) = m\left(r, \frac{f'(z+c)(f^{(k)}(z) - f(z+c))}{(f(z+c)-a)(f(z+c)-b)}\right) \\ &\leq m\left(r, \frac{f'(z+c)f(z+c)}{(f(z+c)-a)(f(z+c)-b)}\right) + m\left(r, \frac{f^{(k)}(z)}{f(z+c)} - 1\right) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{3.4}$$

Let $d = a + l(a-b),\ l \neq 0, -1$ Then using (3.4), Lemmas 2.1 and 2.3 we obtain

$$\begin{aligned} m\left(r, \frac{1}{f(z+c)-d}\right) &= m\left(r, \frac{f'(z+c)(f^{(k)}(z) - f(z+c))}{\phi(z)(f(z+c)-a)(f(z+c)-b)(f(z+c)-d)}\right) \\ &\leq m\left(r, \frac{f'(z+c)f(z+c)}{\phi(z)(f(z+c)-a)(f(z+c)-b)(f(z+c)-d)}\right) \\ &\quad + m\left(r, \frac{f^{(k)}(z)}{f(z+c)} - 1\right) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{3.5}$$

Rewrite (3.2) as

$$\phi(z)(f(z+c))^2 = f'(z+c)(f^{(k)}(z) - f(z+c)) + ((a+b)f(z+c) - ab)\,\phi(z).$$

Since, by assumption, $f(z+c) \not\equiv f^{(k)}(z)$, we have $\phi(z) \not\equiv 0$. Therefore, by Nevanlinna's first fundamental theorem, Lemmas 2.1 and 2.3 we obtain

$$\begin{aligned}
2T(r,f(z+c)) &= T(r,(f(z+c))^2)\\
&= T\left(r, \frac{f'(z+c)f^{(k)}(z) - f'(z+c)f(z+c) + (a+b)f(z+c)\phi(z) - ab\phi(z)}{\phi(z)}\right)\\
&\leq T(r, f'(z+c)f^{(k)}(z) - f'(z+c)f(z+c) + (a+b)f(z+c)\phi(z) - ab\phi(z))\\
&\quad +T\left(r,\frac{1}{\phi(z)}\right)\\
&\leq m\left(r, \frac{f'(z+c)f^{(k)}(z) - f'(z+c)f(z+c) + (a+b)f(z+c)\phi(z)}{f(z+c)}\right)\\
&\quad +m(r,f(z+c)) + S(r,f)\\
&\leq m(r,f^{(k)}(z)) + m\left(r, \frac{f'(z+c)f^{(k)}(z) - f'(z+c)f(z+c)}{f(z+c)f^{(k)}(z)}\right)\\
&\quad +m(r,f(z+c)) + S(r,f)\\
&\leq T(r,f(z+c)) + T(r,f^{(k)}(z)) + S(r,f).
\end{aligned}$$

This gives

$$T(r,f(z+c)) \leq T(r,f^{(k)}(z)) + S(r,f).$$

Also, using Lemmas 2.1 and 2.3 we obtain

$$\begin{aligned}
T(r,f^{(k)}(z)) &= m(r,f^{(k)}(z)) + S(r,f)\\
&\leq m\left(r,\frac{f^{(k)}(z)}{f(z+c)}\right) + m(r,f(z+c)) + S(r,f)\\
&\leq m\left(r,\frac{f^{(k)}(z)}{f(z)}\right) + m\left(r,\frac{f(z)}{f(z+c)}\right) + m(r,f(z+c)) + S(r,f)\\
&\leq T(r,f(z+c)) + S(r,f).
\end{aligned}$$

Thus, we have

$$T(r,f(z+c)) = T(r,f^{(k)}(z)) + S(r,f). \tag{3.6}$$

Now, using Nevanlinna's second fundamental theorem, (3.1) and (3.6), we get

$$\begin{aligned}
2T(r,f(z+c)) &= 2T(r,f^{(k)}(z)) + S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{f^{(k)}(z)-a}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}(z)-b}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}(z)-d}\right) + S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + T\left(r,\frac{1}{f^{(k)}(z)-d}\right)\\
&\quad -m\left(r,\frac{1}{f^{(k)}(z)-d}\right) + S(r,f)\\
&\leq T(r,f(z+c)) + T(r,f^{(k)}(z)) - m\left(r,\frac{1}{f^{(k)}(z)-d}\right) + S(r,f)\\
&\leq 2T(r,f(z+c)) - m\left(r,\frac{1}{f^{(k)}(z)-d}\right) + S(r,f).
\end{aligned}$$

Thus,

$$m\left(r,\frac{1}{f^{(k)}(z)-d}\right) = S(r,f). \tag{3.7}$$

From the first fundamental theorem of Nevanlinna, Lemmas 2.1, 2.2, (3.5), (3.6), (3.7) and the condition that f is an entire function with hyper order $\rho_2(f) < 1$, we obtain

$$\begin{aligned} m\left(r,\frac{f(z+c)-d}{f^{(k)}(z)-d}\right) - m\left(r,\frac{f^{(k)}(z)-d}{f(z+c)-d}\right) &= T\left(r,\frac{f(z+c)-d}{f^{(k)}(z)-d}\right) \\ -N\left(r,\frac{f(z+c)-d}{f^{(k)}(z)-d}\right) - T\left(r,\frac{f^{(k)}(z)-d}{f(z+c)-d}\right) &+ N\left(r,\frac{f^{(k)}(z)-d}{f(z+c)-d}\right) \\ &= N\left(r,\frac{1}{f(z+c)-d}\right) - N\left(r,\frac{1}{f^{(k)}(z)-d}\right) + S(r,f) \\ = T\left(r,\frac{1}{f(z+c)-d}\right) - m\left(r,\frac{1}{f(z+c)-d}\right) &- T\left(r,\frac{1}{f^{(k)}(z)-d}\right) \\ &+m\left(r,\frac{1}{f^{(k)}(z)-d}\right) + S(r,f) \\ &= T(r,f(z+c)) - T(r,f^{(k)}(z)) + S(r,f) = S(r,f). \end{aligned}$$

Thus, we have

$$m\left(r,\frac{f(z+c)-d}{f^{(k)}(z)-d}\right) - m\left(r,\frac{f^{(k)}(z)-d}{f(z+c)-d}\right) = S(r,f). \tag{3.8}$$

Using Lemma 2.3, (3.5) and (3.8) we obtain

$$\begin{aligned} m\left(r,\frac{f(z+c)-d}{f^{(k)}(z)-d}\right) &= m\left(r,\frac{f^{(k)}(z)-d}{f(z+c)-d}\right) + S(r,f) \\ \le m\left(r,\frac{f^{(k)}(z)}{f(z+c)-d}\right) &+ m\left(r,\frac{d}{f(z+c)-d}\right) + S(r,f) = S(r,f). \end{aligned} \tag{3.9}$$

Rewrite (3.3) as

$$\psi(z) = \left[\frac{a-d}{a-b}\frac{f^{(k+1)}(z)}{f^{(k)}(z)-a} - \frac{b-d}{a-b}\frac{f^{(k+1)}(z)}{f^{(k)}(z)-b}\right]\left[1-\frac{f(z+c)-d}{f^{(k)}(z)-d}\right].$$

Noting that $\psi(z)$ is an entire function, using Lemma 2.3 and (3.9) we obtain

$$T(r,\psi(z)) = m(r,\psi(z)) = S(r,f). \tag{3.10}$$

Let m and n be two positive integers and let $z_1 \in S_{(m,n)}(a) \cup S_{(m,n)}(b)$. That means z_1 is a common zero of $f(z+c)-a$ (resp. $f(z+c)-b$) and $f^{(k)}(z)-a$ (resp. $f^{(k)}(z)-b$) with multiplicity m and n respectively. Then from (3.2) and (3.3) we get

$$n\phi(z_1) - m\psi(z_1) = 0.$$

We now consider the following two cases separately.

Case 1. $n\phi(z) - m\psi(z) \equiv 0$ for some positive integers $m,\ n$. Therefore,

$$n\phi(z) \equiv m\psi(z).$$

By simple calculation we obtain

$$n\left[\frac{f'(z+c)}{f(z+c)-a} - \frac{f'(z+c)}{f(z+c)-b}\right] \equiv m\left[\frac{f^{(k+1)}(z)}{f^{(k)}(z)-a} - \frac{f^{(k+1)}(z)}{f^{(k)}(z)-b}\right].$$

Integrating we obtain

$$\left(\frac{f(z+c)-a}{f(z+c)-b}\right)^n \equiv A\left(\frac{f^{(k)}(z)-a}{f^{(k)}(z)-b}\right)^m, \tag{3.11}$$

where A is nonzero constant. Hence $m=n$, otherwise we would have a contradiction to (3.6). Now, it follows from (3.11) that

$$B\left(\frac{f(z+c)-a}{f(z+c)-b}\right) \equiv \left(\frac{f^{(k)}(z)-a}{f^{(k)}(z)-b}\right),$$

where $B \neq 1$ is a nonzero constant. This gives,

$$\frac{b-a}{f^{(k)}(z)-b} = \frac{(B-1)f(z+c)+(b-aB)}{f(z+c)-b}.$$

Since $f(z)$ is entire with $\rho(f)<1$, it follows that

$$f(z+c) \neq \frac{b-aB}{1-B}.$$

Obviously,

$$\frac{b-aB}{1-B} \neq a, b.$$

Thus, we have

$$\begin{aligned} 2T(r,f(z+c)) &\leq \overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-\frac{b-aB}{1-B}}\right) \\ &\quad +S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b}\right)+S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{f^{(k)}(z)-a}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}(z)-b}\right)+S(r,f), \end{aligned}$$

a contradiction with (3.1).

Case 2. Let $n\phi(z) \not\equiv m\psi(z)$, for any positive integers m and n. Then using (3.4) and (3.10) we have

$$\begin{aligned} \overline{N}_{(m,n)}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}_{(m,n)}\left(r,\frac{1}{f(z+c)-b}\right) &\leq \overline{N}\left(r,\frac{1}{n\phi(z)-m\psi(z)}\right) \\ &\leq T(r,n\phi(z)-m\psi(z))+S(r,f) \\ &\leq T(r,\phi(z))+T(r,\psi(z))+S(r,f) \\ &= S(r,f), \end{aligned} \tag{3.12}$$

for all positive integers m and n. Therefore by (3.6) and (3.12), we get

$$
\begin{aligned}
T(r,f(z+c)) &\leq \overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b}\right)+S(r,f)\\
&\leq \overline{N}_{1)}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}_{2}\left(r,\frac{1}{f(z+c)-a}\right)\\
&\quad+\overline{N}_{3}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}_{4}\left(r,\frac{1}{f(z+c)-a}\right)\\
&\quad+\overline{N}_{(5}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}_{1)}\left(r,\frac{1}{f(z+c)-b}\right)\\
&\quad+\overline{N}_{2}\left(r,\frac{1}{f(z+c)-b}\right)+\overline{N}_{3}\left(r,\frac{1}{f(z+c)-b}\right)\\
&\quad+\overline{N}_{4}\left(r,\frac{1}{f(z+c)-b}\right)+\overline{N}_{(5}\left(r,\frac{1}{f(z+c)-b}\right)+S(r,f)\\
&\leq \sum_{n=1}^{4}\sum_{m=1}^{4}\overline{N}_{(m,n)}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}_{(5}\left(r,\frac{1}{f^{(k)}(z)-a}\right)\\
&\quad+\overline{N}_{(5}\left(r,\frac{1}{f(z+c)-a}\right)\\
&\quad+\sum_{n=1}^{4}\sum_{m=1}^{4}\overline{N}_{(m,n)}\left(r,\frac{1}{f(z+c)-b}\right)+\overline{N}_{(5}\left(r,\frac{1}{f^{(k)}(z)-b}\right)\\
&\quad+\overline{N}_{(5}\left(r,\frac{1}{f(z+c)-b}\right)+S(r,f)\\
&\leq \frac{1}{5}\left[N\left(r,\frac{1}{f(z+c)-a}\right)+N\left(r,\frac{1}{f(z+c)-b}\right)\right]\\
&\quad+\frac{1}{5}\left[N\left(r,\frac{1}{f^{(k)}(z)-a}\right)+N\left(r,\frac{1}{f^{(k)}(z)-b}\right)\right]+S(r,f)\\
&\leq \frac{2}{5}T(r,f(z+c))+\frac{2}{5}T(r,f^{(k)}(z))+S(r,f)\\
&= \frac{4}{5}T(r,f(z+c))+S(r,f),
\end{aligned}
$$

a contradiction. This completes the proof of Theorem 1.1. □

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Author information

Pulak Sahoo, Department of Mathematics, University of Kalyani, West Bengal-741235, India.
E-mail: `sahoopulak1@gmail.com`

Suman Pal, Department of Mathematics, University of Kalyani, West Bengal-741235, India.
E-mail: `sumanpal201300@gmail.com`

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