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On $(2,$radical)-ideals of noncommutative rings

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Abstract Let R be a commutative ring and let $J(R)$ denote the Jacobson radical and $\mathcal{P}(R)$ the prime radical of the ring R. Recently the notions of $(2;J)$ and $(2;\mathcal{P})$-ideals as generalizations of J-ideals and n-ideals were introduced and studied in commutative rings. Let ρ be a special radical and R a noncommutative ring. In this paper we introduce the concept of $(2;\rho)$-ideals as a generalization of radical-ideals. A proper ideal P is a $(2;\rho)$-ideal if whenever $aRbRc \subseteq P$ for $a;b;c \in R$, then $ab \in P$ or $ac \in \rho(R)$ or $bc \in \rho(R)$. We show that $(2;J)$ and $(2;\mathcal{P})$-ideals are special cases of $(2;\rho)$-ideals and that many of the results for $(2;J)$ and $(2;\mathcal{P})$-ideals are also satisfied for noncommutative rings in general.

1 Introduction

For some related study see [6] and [10]. In [9] the notion of an n-ideal was introduced. Later, following this, in [5] the notion of a $\mathcal{J}$-ideal was introduced in [3]. The $\mathcal{J}$-ideal is connected to the Jacobson radical and the n-radical is connected to the prime radical. Lately in [3] we extended these notions to noncommutative rings and show that these notions are special cases of a general type of ideal connected to a special radical. In [8] the notion of a $(2,n)$-ideal was introduced for a commutative ring as new generalization of n-ideals and in [9] the notion of a $(2,\mathcal{J})$-ideal was introduced for a commutative ring as new generalization of $\mathcal{J}$-ideals. In this note we extend the notion of $(2,n)$ and $(2,\mathcal{J})$-ideals to noncommutative rings and we also show that many of these ideals are special cases of a more general type of ideal connected to a special radical. For the following definitions of radicals and related results we refer the reader to [13].

A class ρ of rings forms a radical class in the sense of Amitsur-Kurosh if ρ has the following three properties

(i) The class ρ is closed under homomorphism, that is, if $R \in \rho$, then $R/I \in \rho$ for every $I \lhd R$.

(ii) Let R be any ring. If we define $\rho(R) = \sum\{I \lhd R : I \in \rho\}$, then $\rho(R) \in \rho$.

(iii) For any ring R the factor ring $R/\rho(R)$ has no nonzero ideal in ρ i.e. $\rho(R/\rho(R)) = 0$.

A class $\mathcal{M}$ of rings is a **special class** if it is hereditary, consists of prime rings and satisfies the following condition $(*)$ if $0 \neq I \lhd R$, $I \in \mathcal{M}$ and R a prime ring, then $R \in \mathcal{M}$.

Let $\mathcal{M}$ be any special class of rings. The class $\mathcal{U}(\mathcal{M}) = \{R : R$ has no nonzero homomorphic image in $\mathcal{M}\}$ of rings forms a radical class of rings and the upper radical class $\mathcal{U}(\mathcal{M})$ is called a special radical class.

Let ρ be a special radical with special class $\mathcal{M}$ i.e. $\rho = \mathcal{U}(\mathcal{M})$. Now let $\mathcal{S}_\rho = \{R : \rho(R) = 0\}$. If $\mathcal{P}$ denotes the class of prime rings, then for the special radical ρ it follows from [13] that $\rho = \mathcal{U}(\mathcal{P} \cap \mathcal{S}_\rho)$. For a ring R we have $\rho(R) = \cap\{I \lhd R : R/I \in \mathcal{P} \cap \mathcal{S}_\rho\}$ i.e. ρ has the intersection property relative to the class $\mathcal{P} \cap \mathcal{S}_\rho$.

Let $I \lhd R$, then $\rho(R/I) = \rho^*(I)/I$ for some uniquely determined ideal $\rho^*(I)$ of R with $\rho(I) \subseteq I \subseteq \rho^*(I)$ and $\rho^*(I)$ is called the radical of the ideal I while $\rho(I)$ is the radical of the ring I.

We also have $\rho^*(I) = \rho(R)$ if and only if $I \subseteq \rho(R)$.

In what follows let ρ be a special radical with special class $\mathcal{M}$. Hence $\rho = \mathcal{U}(\mathcal{P} \cap \mathcal{S}_\rho)$.

The following are some of the well known special radicals which are defined in [13], prime radical $\mathcal{P}$, Levitski radical $\mathcal{L}$, Kőthe's nil radical $\mathcal{N}$, Jacobson radical $\mathcal{J}$ and the Brown McCoy radical $\mathcal{G}$.

2 Definitions and general results

Throughout this paper, all rings are associative, noncommutative, and without identity unless stated otherwise; by ideal, we mean two-sided ideal.

Definition 2.1. Let ρ be a special radical. A proper ideal I of the ring R is called a ρ-ideal if whenever $a, b \in R$ and $aRb \subseteq I$ and $a \notin \rho(R)$, then $b \in I$.

In [9] and [5] the notions of n-ideals and J-ideals were introduced for commutative rings.

Definition 2.2. [9, Definition 2.1] and [5, Definition 2.1] If ρ is the prime radical or the Jacobson radical of a commutative ring, then a proper ideal I of R is a ρ-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \rho(R)$, then $b \in I$.

Definition 2.3. Let ρ be a special radical. A proper ideal I of the ring R is called a $(2, \rho)$-ideal if whenever $a, b, c \in R$ and $aRbRc \subseteq I$, then $ab \in I$ or $ac \in \rho(R)$ or $bc \in \rho(R)$.

If ρ is equal to the prime radical $\mathcal{P}$ or the Jacobson radical $\mathcal{J}$ in the above definition, then we have the notions of $(2, \mathcal{P})$ and $(2, \mathcal{J})$-ideals.

Remark 2.4. Let R be a commutative ring with identity and I a proper ideal of R. I is a $(2, \rho)$-ideal if and only if $a, b, c \in R$ with $abc \in I$ then $ab \in I$ or $ac \in \rho(R)$ or $bc \in \rho(R)$. Let I be a $(2, \rho)$-ideal and $a, b, c \in R$ with $abc \in I$. Now $aRbRc \subseteq I$ and hence $ab \in I$ or $ac \in \rho(R)$ or $bc \in \rho(R)$. Now, suppose that if $a, b, c \in R$ with $abc \in I$ then $ab \in I$ or $ac \in \rho(R)$ or $bc \in \rho(R)$. Let $aRbRc \subseteq I$. Since R is a ring with identity, $abc \in aRbRb \subseteq I$. Hence $ab \in I$ or $ac \in \rho(R)$ or $bc \in \rho(R)$ and we are done.

Proposition 2.5. *Let ρ be a special radical and let R be a ring with identity and I a proper ideal of R. We have the following:*

(i) Every ρ-ideal of R is a $(2, \rho)$-ideal.

(ii) If I is a $(2, \rho)$-ideal of R then $I \subseteq \rho(R)$.

(iii) If ρ_1 is a special radical such that $\rho_1 \leqslant \rho$, then every $(2, \rho_1)$-ideal is a $(2, \rho)$-ideal of R.

Proof. (i) Let I be a ρ-ideal of R and $a, b, c \in R$ such that $aRbRc \subseteq I$. If $ac \in \rho(R)$, then we done. So suppose $ac \notin \rho(R)$. Hence we have $a \notin \rho(R)$. Since $aRbc \subseteq I$ and I a ρ-ideal of R, we have $bc \in I$. Now from [3, Proposition 1.5] we have $I \subseteq \rho(R)$ and therefore $bc \in \rho(R)$ and hence I is a $(2, \rho)$-ideal.

(ii) Suppose I is a $(2, \rho)$-ideal of R and $I \nsubseteq \rho(R)$. Hence there exists $a \in I - \rho(R)$. Now $1R1Ra \subseteq I$ and $1a \notin \rho(R)$ and therefore $1 \in I$ since I is a $(2, \rho)$-ideal of R. This is not possible since I is a proper ideal of R. Hence $I \subseteq \rho(R)$.

(iii) Let I be a $(2, \rho_1)$-ideal of R and $a, b, c \in R$ such that $aRbRc \subseteq I$ and $ac \notin \rho(R)$ and $bc \notin \rho(R)$. Since $\rho_1(R) \subseteq \rho(R)$, we have $ac \notin \rho_1(R)$ and $bc \notin \rho_1(R)$. Since I is a $(2, \rho_1)$-ideal of R, we have $ab \in I$ and we are done.

□

Example 2.6. A $(2, \rho)$-ideal of R need not be a ρ-ideal. Let $R = M_2(\mathbb{Z}_{15})$ and $\rho = \mathcal{J}$ the Jacobson radical $I = M_2(\langle\overline{0}\rangle)$ is a $(2, \mathcal{J})$-ideal but not a $\mathcal{J}$-ideal since $\begin{bmatrix} \overline{3} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} M_2(\mathbb{Z}_{15}) \begin{bmatrix} \overline{5} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \subseteq I$ but $\begin{bmatrix} \overline{3} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \notin \mathcal{J}(R)$ and $\begin{bmatrix} \overline{5} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \notin I$.

Example 2.7. Consider $R = M_2(\mathbb{Z}_{120})$ and $I = M_2(\langle\overline{60}\rangle)$. For the Jacobson radical $\mathcal{J}$ we have $\mathcal{J}(R) = M_2(\langle\overline{30}\rangle)$. Clearly $I \subseteq \rho(R)$. I is not a $(2,\mathcal{J})$-ideal of R since $\begin{bmatrix}\overline{3} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix} R \begin{bmatrix}\overline{4} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix} R \begin{bmatrix}\overline{5} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix} \subseteq I$ but $\begin{bmatrix}\overline{3} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix}\begin{bmatrix}\overline{4} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix} \notin I$ and $\begin{bmatrix}\overline{3} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix}\begin{bmatrix}\overline{5} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix} \notin \mathcal{J}(R)$ and $\begin{bmatrix}\overline{4} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix}\begin{bmatrix}\overline{5} & \overline{0}\\ \overline{0} & \overline{0}\end{bmatrix} \notin \mathcal{J}(R)$.

Example 2.8. The converse of Proposition 2.5 3 is not true in general as can be seen from the following example. Let R be a commutative local domain which is not a field. From [7, (3) page 56] $\mathcal{J}(R) \neq 0$ and $\mathcal{N}(R) = 0$. From Theorem 5.9 $\mathcal{J}(R)$ is a $(2,\mathcal{J})$-ideal. Let $0 \neq a \in \mathcal{J}(R) - \mathcal{N}(R)$. Now $1R1Ra \subseteq \mathcal{J}(R)$ where $1 \notin \mathcal{J}(R)$ and $a \notin \mathcal{N}(R)$. Therefore $\mathcal{J}(R)$ is not an $(2,\mathcal{N})$-ideal of R.

Proposition 2.9. *Let ρ be a special radical and R a ring with $I \subseteq K$ be proper ideals. If K is a ρ-ideal of R, then I is a $(2,\rho)$-ideal.*

Proof. Assume that $aRbRc \subseteq I$ for some $a,b,c \in R$ and $ab \notin I$. If $a \in \rho(R)$, we are done. Suppose that $a \notin \rho(R)$. Since K is a ρ-ideal and $aRbc \subseteq K$, we have $bc \in K$, Since K is a ρ-ideal of R, it follows from [3, Proposition 1.5] that $K \subseteq \rho(R)$ and hence $bc \in \rho(R)$ as desired. □

Corollary 2.10. *Let ρ be a special radical and R a ring with I_1, I_2 two proper ideals of R. If I_1 or I_2 is a ρ-ideal of R, then I_1I_2 and $I_1 \cap I_2$ are $(2,\rho)$-ideals.*

Proposition 2.11. *Let ρ be a special radical and R a ring with proper ideal I. If I is such that $R/I \in \mathcal{S}_\rho \cap \mathcal{P}$, then the following are equivalent:*

(i) *I is a $(2,\rho)$-ideal of R.*

(ii) *$I = \rho(R)$.*

(iii) *I is a ρ-ideal of R.*

Proof. $1 \Leftrightarrow 2$ Let I be a $(2,\rho)$-ideal of R. Since ρ is a special radical, we have $\rho(R) = \cap\{A \lhd R : R/A \in \mathcal{S}_\rho \cap \mathcal{P}\}$. Now, since $R/I \in \mathcal{S}_\rho \cap \mathcal{P}$, we have $\rho(R) \subseteq I$. From Proposition 2.5 we have $I \subseteq \rho(R)$. Hence $I = \rho(R)$. For the converse, let $I = \rho(R)$ with $R/I \in \mathcal{S}_\rho \cap \mathcal{P}$. Let $a,b,c \in R$ such that $aRbRc \subseteq I$. If $bc \in \rho(R)$, then we done. Suppose $bc \notin \rho(R) = I$. Since I is a prime ideal and $aRbc \subseteq I$, we have $a \in I$ and therefore $ab \in I$. Hence I is a $(2,\rho)$-ideal of R.

$2 \Leftrightarrow 3$ This follows from [3, Proposition1.13].

□

Recall from [4, Definition 2.1] A proper ideal I of R is called a principally right 2-absorbing primary ideal of R if whenever $a,b,c \in R$ and $aRbRc \subseteq I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ where $\sqrt{I} = \{V \lhd R | V^n \subseteq I$ for some positive integer $n\}$. Recall also that $\sqrt{I} = \mathcal{P}^*(I) = \rho^*(I)$ for some special radical ρ and where $\mathcal{P}^*(I)$ is equal to the intersection of all the prime ideals of the ring R containing the ideal I.

Proposition 2.12. *Let ρ be a special radical and R a ring with a proper ideal I. Suppose I is a principally right 2-absorbing primary ideal of R. I is a $(2,\rho)$-ideal of R if and only if $I \subseteq \rho(R)$.*

Proof. $\Rightarrow$This follows from Proposition 2.5.

$\Leftarrow$Let I be a principally right 2-absorbing primary ideal of R with $I \subseteq \rho(R)$. Let $a,b,c \in R$ such that $aRbRc \subseteq I$. Since $I \subseteq \rho(R)$ and since ρ is a special radical, we have $\sqrt{I} \subseteq \mathcal{P}^*(I) = \rho^*(I) = \rho(R)$. Suppose $ac \notin \rho(R)$ and $bc \notin \rho(R)$ hence we have $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Now, since I is a principally right 2-absorbing primary ideal of R, we have $ab \in I$ and we are done. □

Example 2.13. If ρ is equal to the prime radical, then a $(2,\rho)$-ideal is a principally right 2-absorbing primary ideal. However, these are different concepts. For instance, consider the ideal $I = \langle 12\rangle$ of $\mathbb{Z}$. The ideal $M_2(I)$ of the matrix ring $M_2(\mathbb{Z})$ is not a $(2,\rho)$- ideal, since

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} M_2(\mathbb{Z}) \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} M_2(\mathbb{Z}) \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \subseteq M_2(I)$, but $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin M_2(I)$ and $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \notin \sqrt{M_2(\mathbb{Z})}$. However, $M_2(I)$ is a 2-absorbing principally right primary ideal of $M_2(\mathbb{Z})$ since $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \in \sqrt{M_2(I)}$. It is also clear that $M_2(I) \nsubseteq \sqrt{M_2(\mathbb{Z})}$.

Theorem 2.14. *Let ρ be a special radical and R a ring with a proper ideal P. The following statements are equivalent:*

(i) P is a $(2, \rho)$-ideal of R.

(ii) $(P : xRyR) \subseteq (\rho(R) : xR) \cup (\rho(R) : yR)$ for all $x, y \in R$ with $xy \notin P$.

(iii) $(P : xRyR) \subseteq (\rho(R) : xR)$ or $(P : xRyR) \subseteq (\rho(R) : yR)$ for all $x, y \in R$ with $xy \notin P$.

(iv) For all $x, y \in R$ and each ideal J of R, $xRyJ \subseteq P$ implies either $xy \in P$ or $xJ \subseteq \rho(R)$ or $yJ \subseteq \rho(R)$.

(v) For all $x \in R$ and ideals J and K of R, $xJK \subseteq P$ implies either $xJ \subseteq P$ or $xK \subseteq \rho(R)$ or $JK \subseteq \rho(R)$.

(vi) For all ideals I, J, K of R such that $IJK \subseteq P$ either $IJ \subseteq P$ or $IK \subseteq \rho(R)$ or $JK \subseteq \rho(R)$.

Proof. $1 \Rightarrow 2$ Suppose P is a $(2, \rho)$-ideal of R and choose $x, y \in R$ with $xy \notin P$. Take $z \in (P : xRyR)$. Hence $xRyRz \subseteq P$. Since P is a $(2, \rho)$-ideal of R, we have $xz \in \rho(R)$ or $yz \in \rho(R)$. Hence $Rxz \subseteq \rho(R)$ or $Ryz \subseteq \rho(R)$ and therefore $(P : xRyR) \subseteq (\rho(R) : xR) \cup (\rho(R) : yR)$.

$2 \Rightarrow 3$ Follows from the fact that an ideal that is contained in the union of two ideals must be contained in one of them.

$3 \Rightarrow 4$ Let $xRyJ \subseteq P$ for some $x, y \in R$ and an ideal J of R. Hence $xRyRJ \subseteq xRyJ \subseteq P$. Suppose $xy \notin P$. From 3 we have $J \subseteq (P : xRyR) \subseteq (\rho(R) : xR)$ or $J \subseteq (P : xRyR) \subseteq (\rho(R) : yR)$. Hence $xRJ \subseteq \rho(R)$ or $yRJ \subseteq \rho(R)$. Hence $xJ \subseteq \rho(R)$ or $yJ \subseteq \rho(R)$.

$4 \Rightarrow 5$ Suppose $xJK \subseteq P$ for some $x \in R$ and ideals J and K of R. Suppose $xJ \nsubseteq P$ or $xK \nsubseteq \rho(R)$. Choose $j_0 \in J$ and $k_0 \in K$ such that $xj_0 \notin P$ and $xk_0 \notin \rho(R)$. Since $xRj_0K \subseteq P$, it follows from 4 that $j_0K \subseteq \rho(R)$. Now we will show that $JK \subseteq \rho(R)$. Take any $j \in J$. If $jK \subseteq \rho(R)$, then we done. So suppose $jK \nsubseteq \rho(R)$. Since $xRjK \subseteq P$, from 4 we have $xj \in P$. This implies $x(j + j_0) \notin P$. As $xR(j + j_0)K \subseteq P$, by 4 we have $(j + j_0)K \subseteq \rho(R)$. Since $j_0K \subseteq \rho(R)$, we conclude that $jK \subseteq \rho(R)$ and so $JK \subseteq \rho(R)$.

$5 \Rightarrow 6$ Suppose $IJK \subseteq P$ for some ideals I, J, K of R. Assume $IJ \nsubseteq P$ and $JK \nsubseteq \rho(R)$. Then there exists $x \in I$ such that $xJ \nsubseteq P$. Since $xJK \subseteq P$, by 5 we have $xK \subseteq \rho(R)$. Let $a \in I$, then since $aJK \subseteq P$, by 5 we have $aJ \subseteq P$ or $aK \subseteq \rho(R)$.

Case 1: Let $aJ \subseteq P$. Then since $(x + a)JK \subseteq P$, by 5 we have $(x + a)K \subseteq \rho(R)$. As $xK \subseteq \rho(R)$, we have $aK \subseteq \rho(R)$.

Case 2: Let $aJ \nsubseteq P$. Since $aJK \subseteq P$, by 5 we get $aK \subseteq \rho(R)$. Thus we have $IK \subseteq \rho(R)$ which completes the proof.

$6 \Rightarrow 1$ Suppose $aRbRc \subseteq P$ for $a, b, c \in R$ with $ac \notin \rho(R)$ and $bc \notin \rho(R)$. Put $I = RaR$, $J = RbR$ and $K = RcR$. Now $IJK = RaRRbRRcR \subseteq RaRbRcR = RPR \subseteq P$. Since $IK \nsubseteq \rho(R)$ and $JK \nsubseteq \rho(R)$, it follows from 6 that $ab \in IJ \subseteq P$. Therefore P is a $(2, \rho)$-ideal of R.

□

Proposition 2.15. *Let ρ be a special radical.*

(i) The intersection of any set of $(2, \rho)$-ideals of the ring R is a $(2, \rho)$-ideal.

(ii) Let R be a ring. $\rho(R)$ is a $(2, \rho)$-ideal of R if and only if it is a 2-absorbing ideal of R.

Proof. (i) Let $\{P_i : i \in \Delta\}$ be a set of $(2,\rho)$-ideals. Let $aRbRc \subseteq \cap\{P_i : i \in \Delta\}$ and suppose $ac \notin \rho(R)$ and $bc \notin \rho(R)$. Since $aRbRc \subseteq P_i$ for every $i \in \Delta$, we have $ab \in P_i$ for every $i \in \Delta$. Hence $ab \in \cap\{P_i : i \in \Delta\}$ and therefore $\cap\{P_i : i \in \Delta\}$ is a $(2,\rho)$-ideal.

(ii) This follows from the definition of a $(2,\rho)$-ideal and a 2-absorbing ideal.

□

Definition 2.16. A proper ideal P of a ring R is said to be a $(2,\rho)$-primary ideal if $aRbRc \subseteq P$ for $a,b,c \in R$, then $ab \in P$ or $ac \in \rho^*(P)$ or $bc \in \rho^*(P)$.

Proposition 2.17. *Let ρ be a special radical and P a proper ideal of the ring R. The following statements are equivalent:*

(i) P is a $(2,\rho)$-primary ideal of the ring R and $\rho^*(P) = \rho(R)$.

(ii) P is a $(2,\rho)$-ideal of R.

Proof. $1 \Rightarrow 2$ Suppose P is a $(2,\rho)$-primary ideal of the ring R and $\rho^*(P) = \rho(R)$. Let $aRbRc \subseteq P$ for $a,b,c \in R$ and suppose $ac \notin \rho(R) = \rho^*(P)$ and $bc \notin \rho(R) = \rho^*(P)$. Since P is a $(2,\rho)$-primary ideal, we have $ab \in P$ and we are done.

$2 \Rightarrow 1$ Suppose P is a $(2,\rho)$-ideal of R. From Proposition 2.5 we have $P \subseteq \rho(R)$. Now, since ρ is a special radical and $P \subseteq \rho(R)$, we have $\rho(R) = \rho^*(P)$. Let $aRbRc \subseteq P$ for $a,b,c \in R$. Since P is a $(2,\rho)$-ideal of R, we have $ab \in P$ or $ac \in \rho(R) = \rho^*(P)$ or $bc \in \rho(R) = \rho^*(P)$. Hence P is a $(2,\rho)$-primary ideal.

□

Theorem 2.18. *Let R and S be rings and $f : R \to S$ be a surjective ring-homomorphism. If ρ is a special radical, then the following statements hold:*

(i) If I is a $(2,\rho)$-ideal of R and $\ker(f) \subseteq I$, then $f(I)$ is a $(2,\rho)$-ideal of S.

(ii) If J is a $(2,\rho)$-ideal of S and $\ker(f) \subseteq \rho(R)$, then $f^{-1}(J)$ is a $(2,\rho)$-ideal of R.

Proof. (i) Let $d,e,h \in S$ such that $dSeSh \subseteq f(I)$ and $dh \notin \rho(S)$ and $eh \notin \rho(S)$. Since f is surjective, we can choose $a,b,c \in R$ such that $f(a) = d$ and $f(b) = e$ and $f(c) = h$. Now, $dSeSh = f(a)f(R)f(b)f(R)f(c) = f(aRbRc) \subseteq f(I)$ and since $\ker(f) \subseteq I$, we have $aRbRc \subseteq I$. Then $ab \in I$ or $ac \in \rho(R)$ or $bc \in \rho(R)$ since I is a $(2,\rho)$-ideal of R. If $ac \in \rho(R)$ then $f(ac) = f(a)f(c) = dh \in f(\rho(R)) \subseteq \rho(S)$ since ρ is a special radical which is a contradiction. Similarly, if $bc \in \rho(R)$, then $eh \in \rho(S)$. Hence $ab \in I$. This gives $f(ab) = f(a)f(b) = de \in f(I)$, as needed

(ii) Let $a,b, \in R$ such that $aRbRc \subseteq f^{-1}(J)$ and $ac \notin \rho(R)$ and $bc \notin \rho(R)$. Now, $f(a)Sf(b)Sf(c) = f(aRbRc) \subseteq J$ with $f(ac) \notin f(\rho(R))$ and $f(bc) \notin f(\rho(R))$. From [3, Theorem 1.10 (2)] we have $f(a)f(c) \notin \rho(S)$ and $f(b)f(c) \notin \rho(S)$. Since J is a $(2,\rho)$-ideal of S, we have $f(a)f(b) \in J$. Hence $ab \in f^{-1}(J)$. It follows that $f^{-1}(J)$ is a $(2,\rho)$-ideal of R.

□

Corollary 2.19. *Let ρ be a special radical. Let R be a ring and let I, K be two ideals of R with $K \subseteq I$. Then the following hold.*

(i) If I is a $(2,\rho)$-ideal of R, then I/K is a $(2,\rho)$-ideal of R/K.

(ii) If I/K is a $(2,\rho)$-ideal of R/K and $K \subseteq \rho(R)$, then I is a $(2,\rho)$-ideal of R.

(iii) If I/K is a $(2,\rho)$-ideal of R/K and K is a $(2,\rho)$-ideal of R, then I is a $(2,\rho)$-ideal of R.

Proof. (i) Assume that I is a $(2,\rho)$-ideal of R with $K \subseteq I$. Let $\pi : R \to R/K$ be the natural epimorphism defined by $\pi(R) = r + K$. Note that $\ker(\pi) = K \subseteq I$. Thus, by Theorem 2.18 1., it follows that $\pi(I) = I/K$ is a $(2,\rho)$-ideal of R/K.

(ii) Again, consider the natural epimorphism $\pi : R \to R/K$. Since $K \subseteq \rho(R)$, by Theorem 2.18 2., $I = \pi^{-1}(I/K)$ is a $(2,\rho)$-ideal of R.

(iii) This is clear by 2. and Theorem 2.18.

□

Proposition 2.20. *Let R be a ring, I and J be two $(2,\rho)$-ideals of R. Then $I+J$ is a $(2,\rho)$-ideal of R.*

Proof. Since $I \subseteq \rho(R)$ and J is a proper ideal, we then have $I+J$ is proper ideal of R. On the other hand, $I \cap J$ and $I/I \cap J$ are $(2,\rho)$-ideals of R and $R/I \cap J$, respectively. From the isomorphism $(I+J)/J \simeq I/I \cap J$ and Corollary 2.19, we conclude that $I+J$ is a $(2,\rho)$-ideal, as needed. □

Lemma 2.21. *Let ρ be a special radical and let R be a ring and I an ideal of R. If $\rho(R)$ is a prime ideal of R and $I \subseteq \rho(R)$ then I is a $(2,\rho)$-ideal.*

Proof. Let $aRbRc \subseteq I$ for $a,b,c \in R$. If $bc \in \rho(R)$, then we done so suppose $bc \notin \rho(R)$. Now $aRbc \subseteq \rho(R)$ and since $\rho(R)$ is a prime ideal, we get $a \in \rho(R)$. Hence $ac \in \rho(R)$ and therefore I is a $(2,\rho)$-ideal. □

3 Idealization

We now show how to construct ρ-ideals using the Method of Idealization. In what follows, R is a ring (associative, not necessarily commutative and not necessarily with identity) and M is an $R-R$-bimodule. The idealization of M is the ring $R \boxplus M$ with $(R \boxplus M,+) = (R,+) \oplus (M,+)$ and the multiplication is given by $(r,m)(s,n) = (rs, rn+ms)$. $R \boxplus M$ itself is, in a canonical way, an $R-R$-bimodule and $M \simeq 0 \boxplus M$ is a nilpotent ideal of $R \boxplus M$ of index 2. We also have $R \simeq R \boxplus 0$ and the latter is a subring of $R \boxplus M$. Note also that $R \boxplus M$ is a subring of the Morita ring $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ via the mapping $(r,m) \mapsto \begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$. We will require some knowledge about the ideal structure of $R \boxplus M$. If I is an ideal of R and N is an $R-R$-bi-submodule of M, then $I \boxplus N$ is an ideal of $R \boxplus M$ if and only if $IM + MI \subseteq N$.

If ρ is a special radical, it follows from [11] that if R is any ring, then $\rho(R \boxplus M) = \rho(R) \boxplus M$ for all $R-R$-bimodules M.

Proposition 3.1. *Let ρ is a special radical. Let I be a proper ideal of R. Let M be a $R-R$-bi-module. Now I is a $(2,\rho)$-ideal of R if and only if $I \boxplus M$ is a $(2,\rho)$-ideal of $R \boxplus M$.*

Proof. ⇒Let $x_i = (r_i, m_i) \in R \boxplus M$ for $1 \leqslant i \leqslant 3$. Suppose $x_1 R \boxplus M x_2 R \boxplus M x_3 \subseteq I \boxplus M$ with $x_1x_3 \notin \rho(R \boxplus M) = \rho(R) \boxplus M$ and $x_2x_3 \notin \rho(R \boxplus M) = \rho(R) \boxplus M$. Now we have $r_1Rr_2Rr_3 \subseteq I$ and $r_2r_3 \notin \rho(R)$ and $r_1r_3 \notin \rho(R)$. Since I is a $(2,\rho)$-ideal of R, we have $r_1r_2 \in I$. Hence $x_1x_2 = (r_1r_2, r_1m_2 + m_1r_2) \in I \boxplus M$ and $I \boxplus M$ is a $(2,\rho)$-ideal.

⇐Let $aRbRc \subseteq I$ for $a,b,c \in R$ and suppose $ac \notin \rho(R)$ and $bc \notin \rho(R)$. Now $(a,0)R \boxplus M(b,0)R \boxplus M(c,0) \subseteq I \boxplus M$ with $(a,0)(c,0) = (ac,0) \notin \rho(R) \boxplus M = \rho(R \boxplus M)$ and $(b,0)(c,0) = (bc,0) \notin \rho(R) \boxplus M = \rho(R \boxplus M)$. Since $I \boxplus M$ is a $(2,\rho)$-ideal of $R \boxplus M$, we get $(a,0)(b,0) = (ab,0) \in I \boxplus M$. Hence $ab \in I$ and therefore I is a $(2,\rho)$-ideal of R. □

Definition 3.2. [3, Definition 2.4] Let ρ be a special radical and let M be an $R-R$-bi module. The proper $R-R$ bi-submodule N of M is a $\rho-$submodule if for $a \in R$ and $m \in M$, whenever $mRa \subseteq N$ and $a \notin (\rho(R)M : M)$, then $m \in N$.

Theorem 3.3. *Let ρ is a special radical. Let I be a $(2,\rho)$-ideal of R and N an $R-R$-bi-submodule of the $R-R$-bi-module M. Then if $(\rho(R)M : M) = \rho(R)$ and N is a ρ-submodule of M with $IM + MI \subseteq N$, then $I \boxplus N$ is a $(2,\rho)$-ideal of $R \boxplus M$.*

Proof. Let $x_i = (r_i, m_i) \in R \boxplus M$ for $1 \leqslant i \leqslant 3$. such that $x_1 R \boxplus M x_2 R \boxplus M x_3 \subseteq I \boxplus N$. Suppose $x_2x_3 \notin \rho(R) \boxplus M = \rho(R \boxplus M)$ and $x_1x_3 \notin \rho(R \boxplus M) = \rho(R) \boxplus M$. We have $r_1Rr_2Rr_3 \subseteq I$ and $r_1r_3 \notin \rho(R)$ and $r_2r_3 \notin \rho(R)$. Since I is a $(2,\rho)$-ideal of R and $r_1r_3 \notin \rho(R)$ and $r_2r_3 \notin \rho(R)$, we have $r_1r_2 \in I$. Now, $(r_1,m_1)(1,0)(r_2,m_2)(1,0)(r_3,m_3) \in x_1 R \boxplus M x_2 R \boxplus M x_3 \subseteq I \boxplus N$. Hence $(r_1r_1r_3, r_1r_2m_3 + (r_1m_2 + m_1r_2)r_3) \in I \boxplus N$ and we have $r_1r_2m_3 + (r_1m_2 + m_1r_2)r_3 \in N$. Since $r_1r_2m_3 \in N$, we have $(r_1m_2 + m_1r_2)r_3 \in N$. Since

$r_2r_3 \notin \rho(R)$, we have $r_3 \notin \rho(R)$. Now since $(r_1m_2 + m_1r_2)r_3 \in N$ and $r_3 \notin \rho(R)$ and N is a ρ-submodule of M, we have $(r_1m_2+m_1r_2) \in N$. Hence $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2+m_1r_2) \in I \boxplus N$ and $I \boxplus N$ is a $(2, \rho)$-ideal of $R \boxplus M$. □

Proposition 3.4. *Let ρ is a special radical. Let I be an ideal of R and N a proper $R-R$-bi-submodule of the $R-R$-bi-module M. Suppose $\rho(R)$ is a prime ideal. If $I \subseteq \rho(R)$ then $I \boxplus N$ is a $(2, \rho)$-ideal of $R \boxplus M$.*

Proof. Since $\rho(R)$ is a prime ideal, it follows from [11, Proposition 8] that $\rho(R) \boxplus M$ is a prime ideal of $R \boxplus M$. Also $I \boxplus N \subseteq \rho(R) \boxplus N \subseteq \rho(R) \boxplus M = \rho(R \boxplus M)$. Hence from Lemma 2.21 $I \boxplus N$ is a $(2, \rho)$-ideal. □

4 Product of rings

Next, we characterize $(2, \rho)$-ideals of a Cartesian product of two rings.

Lemma 4.1. *[1, Lemma] Suppose that $\mathcal{R}$ is a Amitsur-Kurosh radical class of rings. If $R_1, R_2, \cdots, R_n$ are rings, then $\mathcal{R}(R_1 \times R_2 \times \cdots \times R_n) = \mathcal{R}(R_1) \times \mathcal{R}(R_2) \times \cdots \times \mathcal{R}(R_n)$. Hence, $\mathcal{R}$ is always closed under finite products.*

Let in what follows R_1 and R_2 be rings, not necessarily commutative and let ρ be a special radical.

Remark 4.2. Let R_1 and R_2 be noncommutative rings and let I be a $(2, \rho)$-ideal of R_1. Then we do not necessarily have that $I \times R_2$ is a $(2, \rho)$-ideal of $R = R_1 \times R_2$. For example if ρ is the prime radical $M_2(4\mathbb{Z}_8) \subseteq M_2(\mathbb{Z}_8)$ is a $(2, \rho)$-ideal, but $I = M_2(4\mathbb{Z}_8) \times M_2(\mathbb{Z}_8) \subseteq M_2(\mathbb{Z}_8) \times M_2(\mathbb{Z}_8) = R$ is not $(2, \rho)$-ideal. Indeed,

$$\left(\begin{bmatrix}2&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right) R \left(\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right) R \left(\begin{bmatrix}2&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right) \subseteq I,$$

$$\left(\begin{bmatrix}2&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right)\left(\begin{bmatrix}2&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right) \notin \rho(R) = M_2(\langle\overline{2}\rangle) \times M_2(\langle\overline{2}\rangle)$$

and $$\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right)\left(\begin{bmatrix}2&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right) \notin \rho(R)$$

and $$\left(\begin{bmatrix}2&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right)\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right) \notin I.$$

Theorem 4.3. *Let ρ be a special radical and R_1 and R_2 be noncommutative rings with identities. Consider the ideal $I \subset R$. The following statements are equivalent.*

(i) $I \times R_2$ is a (2,ρ)-ideal of $R = R_1 \times R_2$.

(ii) I is a prime ideal of R_1.

(iii) $I \times R_2$ is a prime-ideal of $R_1 \times R_2$.

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R_1$ such that $aR_1b \subseteq I$. Now $(a, 1)R(1, 1)R(b, 1) \subseteq I \times R_2$. Since $(a, 1)(b, 1) \notin \rho(R_1) \times \rho(R_2) = \rho(R)$ and $(1, 1)(b, 1) \notin \rho(R_1) \times \rho(R_2) = \rho(R)$, we have $(a, 1)(1, 1) = (a, 1) \in I \times R_2$. Hence $a \in I$ and therefore I is a prime ideal.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) is clear. □

Theorem 4.4. *Let ρ be a special radical and R_1 and R_2 be noncommutative rings with identities. Consider ideals $I_1 \subset R_1$ and $I_2 \subset R_2$. The following statements are equivalent.*

(i) $I = I_1 \times I_2$ is a (2,ρ)-ideal of $R = R_1 \times R_2$.

(ii) $I_1 = \rho(R_1) \subset R_1$ and $I_2 = \rho(R_2) \subset R_2$ are prime ideals.

(iii) $I = I_1 \times I_2$ is a 2-absorbing ideal of R and $I \subseteq \rho(R)$.

Proof. (1) $\Rightarrow$ (2) Assume that $I_1 \neq \rho(R_1)$, and take $a \in I_1 \backslash \rho(R_1)$. Then $(a,1)R_1 \times R_2(1,1)R_1 \times R_2(1,0) \subseteq I$. Now $(a,1)(1,0) \notin \rho(R) = \rho(R_1) \times \rho(R_2)$. Since I is a $(2,\rho)$-ideal of R, we conclude that $(a,1)(1,1) \in I_1 \times I_2$ or
$(1,1)(1,0) \in \rho(R) = \rho(R_1) \times \rho(R_2)$, a contradiction. Thus $I_1 = \rho(R_1)$. If $I_1 = \rho(R_1)$ is not prime, then there are elements $a, b \notin I_1 = \rho(R_1)$ such that $aR_1b \subseteq I_1$. Then $(a,1)R(1,0)R(b,1) \subseteq I$, $(a,1)(1,0) \notin I_1 \times I_2$,
$(a,1)(b,1) \notin \rho(R) = \rho(R_1) \times \rho(R_2)$ and $(1,0)(b,1) \notin \rho(R_1) \times \rho(R_2)$, a contradiction. Thus I_1 is prime in R_1. The same arguments show that $I_2 = \rho(R_2)$ is a prime ideal of R_2.

(2) $\Rightarrow$ (3) Suppose that $I_1 = \rho(R_1) \subseteq R_1$ and $I_2 = \rho(R_2) \subseteq R_2$ are prime ideals. Hence, $(I_1 \times R_2)$ and $(R_1 \times I_2)$ are prime ideals of $R = R_1 \times R_2$. Since the intersection of two prime ideals is always 2-absorbing by [2, Proposition 1.9], we conclude that $I = (I_1 \times R_2) \cap (R_1 \times I_2)$ is a 2-absorbing ideal of R.

(3) $\Rightarrow$ (1) Let $(x_i, y_i) \in R$ for $1 \leqslant i \leqslant 3$ such that $(x_1,y_1)R(x_2,y_2)R(x_3,y_3) \subseteq I$. Since I is 2-absorbing and $I \subseteq \rho(R)$, we have $(x_1,y_1)(x_2,y_2) \in I$ or $(x_1,y_1)(x_3,y_3) \in I \subseteq \rho(R)$ or $(x_2,y_2)(x_3,y_3) \in I \subseteq \rho(R)$. Hence I is a $(2,\rho)$-ideal of R. ◻

Corollary 4.5. *Let ρ be a special radical and R_1 and R_2 be noncommutative rings with identities. Consider ideals $I_1 \subset R_1$ and $I_2 \subset R_2$. If $I = I_1 \times I_2$ is a $(2,\rho)$-ideal of $R = R_1 \times R_2$, then I_1 is a ρ-ideal of R_1 and I_2 is a ρ-ideal of R_2*

Proof. Suppose $I = I_1 \times I_2$ is a $(2,\rho)$-ideal of $R = R_1 \times R_2$. From 4.4 it follows that $I_1 = \rho(R_1) \subset R_1$ and $I_2 = \rho(R_2) \subset R_2$ are prime ideals. It now follows from [3, Proposition Proposition 1.13] that I_1 is a ρ-ideal of R_1 and I_2 is a ρ-ideal of R_2. ◻

5 (2,$\mathcal{J}$)-ideals

In this section the special radical will be the Jacobson radical. In [5] Khashan et al. introduced the notion of $J-$ideals for commutative rings with identity element. In [12] Yildiz et al. introduced the concept of $(2,J)$-ideal as a generalization of a J-ideal. They investigate many properties of $(2,J)$-ideals. We show that for the Jacobson radical many of the results proved by Yildiz et al. are also true for noncommutative rings.

In what follows for the noncommutative ring R, $\mathcal{J}(R)$ will denote the Jacobson radical of the ring R.

Definition 5.1. A proper ideal I of a ring R is a $\mathcal{J}$-ideal if whenever $a, b \in R$ such that $aRb \subseteq I$ and $a \notin \mathcal{J}(R)$, then $b \in I$.

If R is a commutative ring, then the notion of a $\mathcal{J}$-ideal coincides with a J-ideal as been defined by Khashan et al. in [5].

Definition 5.2. A proper ideal I of a ring R is a $(2,\mathcal{J})$-ideal if whenever $a, b, c \in R$ such that $aRbRc \subseteq I$ then $ab \in I$ or $ac \in \mathcal{J}(R)$, or $bc \in \mathcal{J}(R)$. If R is a commutative ring, then the notion of a $(2,\mathcal{J})$-ideal coincides with a $(2,J)$-ideal as been defined by Yildiz et al. in [12].

Proposition 5.3. *(See [12, Proposition 1])Let $\mathcal{J}$ be the Jacobson radical and I a proper ideal of the ring R. We have the following:*

(i) Every $\mathcal{J}$-ideal of R is a $(2,\mathcal{J})$-ideal.

(ii) If I is a $(2,\mathcal{J})$-ideal of R then $I \subseteq \mathcal{J}(R)$.

(iii) If ρ_1 is a special radical such that $\rho_1 \leqslant \mathcal{J}$, then every $(2,\rho_1)$-ideal is a $(2,\mathcal{J})$-ideal of R.

Proof. This follows from Proposition 2.5 by taking ρ equal to $\mathcal{J}$. ◻

Proposition 5.4. *(See [12, Proposition 2])Let $\mathcal{J}$ be the Jacobson radical and I a proper ideal I the ring R. Suppose I is a principally right 2-absorbing primary ideal of R. I is a $(2,\mathcal{J})$-ideal of R if and only if $I \subseteq \mathcal{J}(R)$.*

Proof. This follows from Proposition 2.12 by taking ρ equal to $\mathcal{J}$. ◻

Proposition 5.5. *Let $\mathcal{J}$ be the Jacobson radical and R a ring with $I \subseteq K$ proper ideals. If K is a $\mathcal{J}$-ideal of R, then I is a $(2, \mathcal{J})$-ideal.*

Proof. This follows from Proposition 2.9 by taking ρ equal to $\mathcal{J}$. □

Corollary 5.6. *Let $\mathcal{J}$ be the Jacobson radical and R a ring with I_1, I_2 two proper ideals of R. If I_1 or I_2 is a $\mathcal{J}$-ideal of R, then I_1I_2 and $I_1 \cap I_2$ are $(2, \mathcal{J})$-ideals.*

Theorem 5.7. *(See [12, Theorem 1]) Let $\mathcal{J}$ be the Jacobson radical and P a proper ideal of the ring R.The following are equivalent:*

(i) P *is a* $(2, \mathcal{J})$*-ideal of* R.

(ii) $(P : xRyR) \subseteq (\mathcal{J}(R) : xR) \cup (\mathcal{J}(R) : yR)$ *for all* $x, y \in R$ *with* $xy \notin P$.

(iii) $(P : xRyR) \subseteq (\mathcal{J}(R) : xR)$ *or* $(P : xRyR) \subseteq (\mathcal{J}(R) : yR)$ *for all* $x, y \in R$ *with* $xy \notin P$.

(iv) For all $x, y \in R$ *and each ideal* J *of* R, $xRyJ \subseteq P$ *implies either* $xy \in P$ *or* $xJ \subseteq \mathcal{J}(R)$ *or* $yJ \subseteq \mathcal{J}(R)$.

(v) For all $x \in R$ *and ideals* J *and* K *of* R, $xJK \subseteq P$ *implies either* $xJ \subseteq P$ *or* $xK \subseteq \mathcal{J}(R)$ *or* $JK \subseteq \mathcal{J}(R)$.

(vi) For all ideals I, J, K *of* R *such that* $IJK \subseteq P$ *either* $IJ \subseteq P$ *or* $IK \subseteq \mathcal{J}(R)$ *or* $JK \subseteq \mathcal{J}(R)$.

Proof. This follows from Theorem 2.14 by taking ρ equal to $\mathcal{J}$. □

Proposition 5.8. *(See [12, Proposition 3])Let $\mathcal{J}$ be the Jacobson radical.*

(i) The intersection of any set of $(2, \mathcal{J})$*-ideals of the ring* R *is a* $(2, \mathcal{J})$*-ideal.*

(ii) Let R *be a ring.* $\mathcal{J}(R)$ *is a* $(2, \mathcal{J})$*-ideal of* R *if and only if it is a 2-absorbing ideal of* R.

Proof. This follows from Proposition2.15 by taking ρ equal to $\mathcal{J}$. □

Theorem 5.9. *(See [12, Theorem 2])Let R be a ring and ρ be the Jacobson radical $\mathcal{J}$.The following statements are equivalent:*

(i) R *is a local ring.*

(ii) Every proper ideal of R *is a* $(2, \mathcal{J})$*-ideal.*

(iii) Every proper principal ideal of R *is a* $(2, \mathcal{J})$*-ideal.*

Proof. $1 \Rightarrow 2$ Let R be a local ring and let P be a proper ideal of R such that $xRyRz \subseteq P$ for $x, y, z \in R$. Suppose $xz \notin \mathcal{J}(R)$ and $yz \notin \mathcal{J}(R)$. Since R is a local ring, xz and yz are unit elements. If xz is a unit, then $xy = x1y(xz)^{-1}xz \in xRyRz \subseteq P$. If yz is a unit then $xy = x1y(yz)^{-1}yz \in xRyRz \subseteq P$. Hence P is a $(2, \mathcal{J})$-ideal.

$2 \Rightarrow 3$ This is clear,

$3 \Rightarrow 1$ Suppose every proper principal ideal of R is a $(2, \mathcal{J})$-ideal and take a maximal left ideal M of R. Now $\mathcal{J}(R) \subseteq M$. We show $M \subseteq \mathcal{J}(R)$. Let $a \in M$ and suppose $a \notin \mathcal{J}(R)$. Now $1R1Ra \subseteq \langle a \rangle$. Since $\langle a \rangle$ is a $(2, \mathcal{J})$-ideal, we have $1 \in \langle a \rangle \subseteq M$. A contradiction and hence $\mathcal{J}(R) = M$ is the unique maximal left ideal of R. Therefore R is a local ring from [7, Theorem 19.1].

□

Proposition 5.10. *(See [12, Proposition 4]) Let $\mathcal{J}$ be the Jacobson radical and P a proper ideal of the ring R. The following statements are equivalent:*

(i) P *is a* $(2, \mathcal{J})$*-primary ideal of the ring* R *and* $\mathcal{J}(P) = \mathcal{J}(R)$.

(ii) P *is a* $(2, \mathcal{J})$*-ideal of* R.

Proof. This follows from Proposition2.17 by taking ρ equal to $\mathcal{J}$. □

Theorem 5.11. *(See [12, Proposition 5]) Let R and S be rings and $f : R \to S$ be a surjective ring-homomorphism. For the Jacobson radical radical $\mathcal{J}$, the following statements hold:*

(i) If I is a $(2, \mathcal{J})$-ideal of R and $\ker(f) \subseteq I$, then $f(I)$ is a $(2, \mathcal{J})$-ideal of S.

(ii) If J is a $(2, \mathcal{J})$-ideal of S and $\ker(f) \subseteq \mathcal{J}(R)$, then $f^{-1}(J)$ is a $(2, \mathcal{J})$-ideal of R.

Proof. This follows from Theorem 2.18 by taking ρ equal to $\mathcal{J}$. □

Proposition 5.12. *(See [12, Proposition 6])Let $\mathcal{J}$ be the Jacobson radical and let R be a ring and let I, K be two ideals of R with $K \subseteq I$. Then the following hold:*

(i) If I is a $(2, \mathcal{J})$-ideal of R, then I/K is a $(2, \mathcal{J})$-ideal of R/K.

(ii) If I/K is a $(2, \mathcal{J})$-ideal of R/K and $K \subseteq \mathcal{J}(R)$, then I is a $(2, \mathcal{J})$-ideal of R.

(iii) If I/K is a $(2, \mathcal{J})$-ideal of R/K and K is a $(2, \mathcal{J})$-ideal of R, then I is a $(2, \mathcal{J})$-ideal of R.

Proof. This follows from Corollary2.19 by taking ρ equal to $\mathcal{J}$. □

Proposition 5.13. *Let R be a ring, I and J be two $(2, \mathcal{J})$-ideals of R. Then $I+J$ is a $(2, \mathcal{J})$-ideal of R.*

Proof. This follows from Proposition 2.20 by taking ρ equal to $\mathcal{J}$. □

Proposition 5.14. *Let $\mathcal{J}$ be the Jacobson radical and let R be a noncommutative ring with identity. Let I be a proper ideal of R. Let M be a $R-R$-bi-module. Now I is a $(2, \mathcal{J})$-ideal of R if and only if $I \boxplus M$ is a $(2, \mathcal{J})$-ideal of $R \boxplus M$.*

Proof. This follows from Proposition 3.1 by taking ρ equal to $\mathcal{J}$. □

Theorem 5.15. *Let the $\mathcal{J}$ be the Jacobson radical radical and R a noncommutative ring with identity. Let I be a $(2, \mathcal{J})$-ideal of R and N an $R-R$-bi-submodule of the $R-R$-bi-module M. Then, if $(\mathcal{J}(R)M : M) = \mathcal{J}(R)$ and N is a $\mathcal{J}$-submodule of M with $IM + MI \subseteq N$, then $I \boxplus N$ is a $(2, \mathcal{J})$-ideal of $R \boxplus M$.*

Proof. This follows from Theorem 3.3 by taking ρ equal to $\mathcal{J}$. □

Proposition 5.16. *Let the $\mathcal{J}$ be the Jacobson radical and R a noncommutative ring with identity. Let I be an ideal of R and N a proper $R-R$-bi-submodule of the $R-R$-bi-module M. Suppose $\mathcal{J}(R)$ is a prime ideal. If $I \subseteq \mathcal{J}(R)$ then $I \boxplus M$ is a $(2, J)$-ideal of $R \boxplus M$.*

Proof. This follows from Proposition 3.4 by taking ρ equal to $\mathcal{J}$. □

6 $(2.\mathcal{P})$-ideals

In this section the special radical will be the prime radical. In [9] Tekir et al. introduced the notion of n-ideals for commutative rings with identity element. In [8] Tamekkante et al. introduced the concept of $(2, n)$-ideals as a generalization of an n-ideal. They investigate many properties of $(2, n)$-ideals. We show that for the prime radical many of the results proved by Tamekkante et al. are also true for noncommutative rings.

In what follows for the noncommutative ring R, $\mathcal{P}(R)$ will denote the prime radical of the ring R.

Definition 6.1. A proper ideal I of a ring R is a $\mathcal{P}$-ideal if whenever $a, b \in R$ such that $aRb \subseteq I$ and $a \notin \mathcal{P}(R)$, then $b \in I$.

If R is a commutative ring, then the notion of a $\mathcal{P}$-ideal coincides with a n-ideal as been defined by Tekir et al. in [9].

Definition 6.2. A proper ideal I of a ring R is a $(2, \mathcal{P})$-ideal if whenever $a, b, c \in R$ such that $aRbRc \subseteq I$ then $ab \in I$ or $ac \in \mathcal{P}(R)$, or $bc \in \mathcal{P}(R)$. If R is a commutative ring, then the notion of a $(2, \mathcal{P})$-ideal coincides with a $(2, n)$-ideal as been defined by Tamekkante et al. in [8].

Proposition 6.3. *(See [8, Theorem 2.4])Let $\mathcal{P}$ be the prime radical and P a proper ideal of the ring R.The following statements are equivalent:*

(i) P is a $(2, \mathcal{P})$-primary ideal of the ring R and $\mathcal{P}(P) = \mathcal{P}(R)$.

(ii) P is a $(2, \mathcal{P})$-ideal of R.

Proof. This follows from Proposition2.17 by taking ρ equal to $\mathcal{P}$. □

Proposition 6.4. *(See [8, Corollary 2.5]) Let $\mathcal{P}$ be the prime radical and I a proper of the ring R. If I is a prime ideal, then the following are equivalent:*

(i) I is a $(2, \mathcal{P})$-ideal of R.

(ii) $I = \mathcal{P}(R)$.

(iii) I is a $\mathcal{P}$-ideal of R.

Proof. This follows from Proposition 2.11 by taking ρ equal to $\mathcal{P}$. □

Theorem 6.5. *(See [8, Proposition 2.7]) Let $\mathcal{P}$ be the prime radical and P a proper ideal of the ring R.The following are equivalent:*

(i) P is a $(2, \mathcal{P})$-ideal of R.

(ii) $(P : xRyR) \subseteq (\mathcal{P}(R) : xR) \cup (\mathcal{P}(R) : yR)$ for all $x, y \in R$ with $xy \notin P$.

(iii) $(P : xRyR) \subseteq (\mathcal{P}(R) : xR)$ or $(P : xRyR) \subseteq (\mathcal{P}(R) : yR)$ for all $x, y \in R$ with $xy \notin P$.

(iv) For all $x, y \in R$ and each ideal J of R, $xRyJ \subseteq P$ implies either $xy \in P$ or $xJ \subseteq \mathcal{P}(R)$ or $yJ \subseteq \mathcal{P}(R)$.

(v) For all $x \in R$ and ideals J and K of R, $xJK \subseteq P$ implies either $xJ \subseteq P$ or $xK \subseteq \mathcal{P}(R)$ or $JK \subseteq \mathcal{P}(R)$.

(vi) For all ideals I, J, K of R such that $IJK \subseteq P$ either $IJ \subseteq P$ or $IK \subseteq \mathcal{P}(R)$ or $JK \subseteq \mathcal{P}(R)$.

Proof. This follows from Theorem 2.14 by taking ρ equal to $\mathcal{P}$. □

Proposition 6.6. *(See [8, Proposition 2.8]Let $\mathcal{P}$ be the prime radical.*

(i) The intersection of any set of $(2, \mathcal{P})$-ideals of the ring R is a $(2, \mathcal{P})$-ideal.

(ii) Let R be a ring. $\mathcal{P}(R)$ is a $(2, \mathcal{P})$-ideal of R if and only if it is a 2-absorbing ideal of R.

Proof. This follows from Proposition2.15 by taking ρ equal to $\mathcal{P}$. □

Theorem 6.7. *(See [8, Proposition 3.1]) Let R and S be rings and $f : R \to S$ be a surjective ring-homomorphism. For the prime radical $\mathcal{P}$, the following statements hold:*

(i) If I is a $(2, \mathcal{P})$-ideal of R and $\ker(f) \subseteq I$, then $f(I)$ is a$(2, \mathcal{P})$-ideal of S.

(ii) If J is a $(2, \mathcal{P})$-ideal of S and $\ker(f) \subseteq \mathcal{P}(R)$, then $f^{-1}(J)$ is a$(2, \mathcal{P})$-ideal of R.

Proof. This follows from Theorem 2.18 by taking ρ equal to $\mathcal{P}$. □

Proposition 6.8. *(See [8, Corollary 3.3])Let $\mathcal{P}$ be the prime radical and let R be a ring and let I, K be two ideals of R with $K \subseteq I$. Then the following hold.*

(i) If I is a $(2, \mathcal{P})$-ideal of R, then I/K is a $(2, \mathcal{P})$-ideal of R/K.

(ii) If I/K is a $(2, \mathcal{P})$-ideal of R/K and $K \subseteq \mathcal{P}(R)$, then I is a $(2, \mathcal{P})$-ideal of R.

(iii) If I/K is a $(2, \mathcal{P})$-ideal of R/K and K is a $(2, \mathcal{P})$-ideal of R, then I is a $(2, \mathcal{P})$-ideal of R.

Proof. This follows from Corollary2.19 by taking ρ equal to $\mathcal{P}$. □

Proposition 6.9. *Let $\mathcal{P}$ be the prime radical and let R be a noncommutative ring with identity. Let I be a proper ideal of R. Let M be a $R-R$-bi-module. Now I is a $(2,\mathcal{P})$-ideal of R if and only if $I \boxplus M$ is a $(2,\mathcal{P})$-ideal of $R \boxplus M$.*

Proof. This follows from Proposition 3.1 by taking ρ equal to $\mathcal{P}$. □

Theorem 6.10. *Let the $\mathcal{P}$ be the prime radical and R a noncommutative ring with identity. Let I be a $(2,\mathcal{P})$-ideal of R and N an $R-R$-bi-submodule of the $R-R$-bi-module M. If $(\mathcal{P}(R)M : M) = \mathcal{P}(R)$ and N is a $\mathcal{P}$-submodule of M with $IM + MI \subseteq N$, then $I \boxplus N$ is a $(2,\mathcal{P})$-ideal of $R \boxplus M$.*

Proof. This follows from Theorem 3.3 by taking ρ equal to $\mathcal{P}$. □

Proposition 6.11. *Let the $\mathcal{P}$ be the prime radical radical and R a noncommutative ring with identity. Let I be an ideal of R and N a proper $R-R$-bi-submodule of the $R-R$-bi-module M. Suppose $\mathcal{P}(R)$ is a prime ideal. If $I \subseteq \mathcal{P}(R)$ then $I \boxplus M$ is a $(2,J)$-ideal of $R \boxplus M$.*

Proof. This follows from Proposition 3.4 by taking ρ equal to $\mathcal{P}$. □

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