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THE GEOMETRY OF A NON-CONFORMAL DEFORMATION OF A METRIC AND BI-HARMONICITY

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Abstract This paper introduces a new class of metrics on an m-dimensional Riemannian manifold (M^m, g), obtained by non-conformal deformations of the metric g. We investigate the Levi-Civita connection and characterize the Riemannian curvature of this metric. We also study harmonicity and bi-harmonicity with respect to this metric, and characterize some classes of proper bi-harmonic maps. Finally, we provide examples of proper bi-harmonic maps in the case where (M^m, g) is an Euclidean space.

1 Introduction

Let (M^m, g) be an m-dimensional Riemannian manifold. We denote by R, Ric, and $\widehat{\text{Ric}}$ respectively the Riemannian curvature tensor, the Ricci curvature, and the Ricci tensor of (M, g). They are defined as follows:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$\text{Ric}(X,Y) = \sum_{i=1}^{m} g(R(X,e_i)e_i, Y),$$

$$\widehat{\text{Ric}}(X) = \sum_{i=1}^{m} R(X,e_i)e_i,$$

where, ∇ is the Levi-Civita connection with respect to g, $\{e_i\}_{i=1,..,m}$ is an orthonormal frame, and $X, Y, Z \in \Gamma(TM)$. Given a smooth function f on M, the gradient of f is defined by

$$g(\text{grad}\, f, X) = X(f), \;\; \text{grad}\, f = \sum_{i=1}^{m} e_i(f)e_i,$$

the Hessian of f is defined by

$$\text{Hess}_f(X,Y) = g(\nabla_X \,\text{grad}\, f, Y) = g(\nabla_Y \,\text{grad}\, f, X)$$

and the Laplacian of f is defined as

$$\Delta(f) = \text{trace}\,\text{Hess}_f = \sum_{i=1}^{m} g(\nabla_{e_i} \text{grad}\, f, e_i).$$

(For more details, see for example [10]).
Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds, the tension field of φ is given by

$$\tau(\varphi) = \text{trace}_g \nabla d\varphi = \sum_{i=1}^{m} \nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i).$$

where $\{e_i\}$ is an orthonormal frame on (M, g), and ∇^{φ} denote the pull-back connection on $\varphi^{-1}(TN)$. Then, φ is called harmonic map if the tension field vanishes, i.e. $\tau(\varphi) = 0$. (For more details on the concept of harmonic maps see [1], [2]).
The map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is called a harmonic morphism if, for every harmonic function $u : V \longrightarrow \mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty, the composition $u \circ \varphi$ is harmonic on $\varphi^{-1}(V)$. Furthermore φ is harmonic morphism if and only if for all $u : V \longrightarrow \mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty

$$\Delta^M(u \circ \varphi) = \lambda^2 (\Delta^N u) \circ \varphi,$$

where λ is a positive function on M (for more details see [2]).
The bi-tension field of φ is given by

$$\tau_2(\varphi) = -\,\text{trace}_g[\nabla^{\varphi}\nabla^{\varphi}\tau(\varphi) - \nabla^{\varphi}_{\nabla}\tau(\varphi)] - \text{trace}_g\, R^N(\tau(\varphi), d\varphi)d\varphi,$$

and φ is called bi-harmonic if and only if $\tau_2(\varphi) = 0$.
Clearly, harmonic maps are bi-harmonic. G.Y. Jiang [7] proved that if M is compact without boundary and the sectional curvature of (N, h) is negative, then any bi-harmonic map $\varphi \in \mathcal{C}^\infty(M, N)$ is harmonic. So it is interesting to construct bi-harmonic non-harmonic maps. We refer the reader to (see [6] [1], [9], [12]) for other examples and different approaches to their construction.
In [3], the authors deformed the codomain metric by $\widetilde{h}_\alpha = \alpha h + (1-\alpha) df \otimes df$, where $\alpha \in (0,1)$ and $f \in \mathcal{C}^\infty(N)$, in order to render a map bi-harmonic non-harmonic with respect to the new metric, they gave a necessary and sufficient condition on f and α such that $\varphi : (M, g) \longrightarrow (N, \widetilde{h}_\alpha)$ is bi-harmonic non-harmonic.
In this work, we study the harmonicity and bi-harmonicity with respect to a non conformal deformation of the metric. Let (M^m, g) be a connected Riemannian manifold and $f : M \longrightarrow \mathbb{R}$ a smooth non constant function. On M we consider a deformation of g defined by

$$\widetilde{g}(X, Y) = g(X, Y) + (df \otimes df)(X, Y) = g(X, Y) + X(f)Y(f),$$

and we consider a map $\varphi : (M, g) \longrightarrow (N, h)$. By deforming the metric on M and then the metric on N, we establish the necessary and sufficient conditions in which a map is harmonic or bi-harmonic. so by suitable choices of f, we are able to give examples of proper bi-harmonic maps.

2 Geometry of $(M^m, \widetilde{g})$

Definition 2.1. Let M be a connected Riemannian manifold equipped with Riemannian metric g, and let f a smooth non-constant function on M. We define on M a Riemannian metric denoted $\widetilde{g}$, by

$$\widetilde{g} = g + df \otimes df.$$

For $X, Y \in \Gamma(TM)$, we have the following

$$\widetilde{g}(X, Y)_x = g(X, Y)_x + X(f)_x Y(f)_x, \ \forall x \in M. \tag{2.1}$$

Note that $\widetilde{g}$ is a conformal metric to g on the distribution orthogonal to grad f, or if M is one-dimensional Riemannian manifold.

Remark 2.2. Consider $\{e_i\}_{i=1}^m$ to be an orthonormal frame on manifold M with respect to metric g, where $e_1 = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$. Then, defining $\widetilde{e_1} = \frac{1}{\sqrt{1+|\operatorname{grad} f|^2}} e_1$ and $\widetilde{e_i} = e_i$ for $i = 2, \ldots, m$, we obtain an orthonormal frame on M with respect to the metric $\widetilde{g}$

In the following we put $\alpha = 1 + \|\operatorname{grad} f\|^2$. Consequently, we have

$$e_1 = \frac{\operatorname{grad} f}{\sqrt{\alpha - 1}} \quad and \quad \widetilde{e}_1 = \frac{\operatorname{grad} f}{\sqrt{\alpha(\alpha-1)}}.$$

By employing Kozul's formula, we can establish the relationship between the Levi-Civita connection of the manifold $(M, \widetilde{g})$ and that of (M, g) as follows

Proposition 2.3. *Let (M^m, g) be a connected Riemannian manifold, $f : M \longrightarrow \mathbb{R}$ a smooth non constant function and $\widetilde{g}(X, Y) = g(X, Y) + X(f)Y(f)$. If ∇ (resp. $\widetilde{\nabla}$) denotes the Levi-Civita connection associated to g (resp. $\widetilde{g}$), then*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{\alpha} \operatorname{Hess}_f(X, Y) \operatorname{grad} f, \quad \forall X, Y \in \Gamma(TM). \tag{2.2}$$

Corollary 2.4. *Let (M^m, g) be a connected Riemannian manifold, $f : M \longrightarrow \mathbb{R}$ a smooth non constant function. Then*

$$\widetilde{\nabla}_X \operatorname{grad} f = \nabla_X \operatorname{grad} f + \frac{1}{2\alpha} X(\alpha) \operatorname{grad} f, \quad \forall X \in \Gamma(TM). \tag{2.3}$$

Now, consider the curvature tensor $\widetilde{R}$ of $(M, \widetilde{g})$, writing R for the curvature tensor of (M, g). We have the following result

Proposition 2.5. *Let (M^m, g) be a connected Riemannian manifold and $f : M \longrightarrow \mathbb{R}$ a smooth non constant. Then $\forall X, Y, Z \in \Gamma(TM)$*

$$\begin{aligned}\widetilde{R}(X, Y)Z = {} & R(X, Y)Z - \frac{1}{\alpha} g\big(R(X, Y)Z, \operatorname{grad} f\big) \operatorname{grad} f \\ & + \frac{1}{2\alpha^2} \big[Y(\alpha) \operatorname{Hess}_f(X, Z) - X(\alpha) \operatorname{Hess}_f(Y, Z)\big] \operatorname{grad} f \\ & + \frac{1}{\alpha} \big[\operatorname{Hess}_f(Y, Z) \nabla_X \operatorname{grad} f - \operatorname{Hess}_f(X, Z) \nabla_Y \operatorname{grad} f\big].\end{aligned} \tag{2.4}$$

Proof. By definition of the Riemannian curvature tensor, we have
for all $X, Y, Z \in \Gamma(TM)$.

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z. \tag{2.5}$$

Using Proposition 2.3 and Corollary 2.4, the first term in the right hand of the equation (2.5), becomes

$$\begin{aligned}
\widetilde{\nabla}_X\widetilde{\nabla}_Y Z &= \widetilde{\nabla}_X\big(\nabla_Y Z + \frac{1}{\alpha}\operatorname{Hess}_f(Y,Z)\operatorname{grad} f\big)\\
&= \widetilde{\nabla}_X(\nabla_Y Z) + \widetilde{\nabla}_X\big(\frac{1}{\alpha}\operatorname{Hess}_f(Y,Z)\operatorname{grad} f\big)\\
&= \nabla_X\nabla_Y Z + \frac{1}{\alpha}g(\nabla_X\operatorname{grad} f, \nabla_Y Z)\operatorname{grad} f\\
&+ X[\frac{1}{\alpha}\operatorname{Hess}_f(Y,Z)]\operatorname{grad} f + \frac{1}{\alpha}\operatorname{Hess}_f(Y,Z)\widetilde{\nabla}_X\operatorname{grad} f\\
&= \nabla_X\nabla_Y Z + \frac{1}{\alpha}g(\nabla_X\operatorname{grad} f, \nabla_Y Z)\operatorname{grad} f - \frac{1}{\alpha^2}X(\alpha)\operatorname{Hess}_f(Y,Z)\operatorname{grad} f\\
&+ \frac{1}{\alpha}[g(\nabla_X\nabla_Y\operatorname{grad} f, Z) + g(\nabla_Y\operatorname{grad} f, \nabla_X Z)]\operatorname{grad} f\\
&+ \frac{1}{\alpha}g(\nabla_Y\operatorname{grad} f, Z)\nabla_X\operatorname{grad} f + \frac{1}{2\alpha^2}X(\alpha)g(\nabla_Y\operatorname{grad} f, Z)\operatorname{grad} f\\
&= \nabla_X\nabla_Y Z + \frac{1}{\alpha}g(\nabla_X\nabla_Y\operatorname{grad} f, Z)\operatorname{grad} f\\
&+ \frac{1}{\alpha}\big[g(\nabla_X\operatorname{grad} f, \nabla_Y Z) + g(\nabla_Y\operatorname{grad} f, \nabla_X Z)\big]\operatorname{grad} f\\
&+ \frac{1}{\alpha}\operatorname{Hess}_f(Y,Z)\nabla_X\operatorname{grad} f - \frac{1}{2\alpha^2}X(\alpha)\operatorname{Hess}_f(Y,Z)\operatorname{grad} f. \qquad (2.6)
\end{aligned}$$

The second term in the right hand of the equation (2.5), becomes

$$\begin{aligned}
\widetilde{\nabla}_Y\widetilde{\nabla}_X Z &= \nabla_Y\nabla_X Z + \frac{1}{\alpha}g(\nabla_Y\nabla_X\operatorname{grad} f, Z)\operatorname{grad} f\\
&+ \frac{1}{\alpha}\big[g(\nabla_Y\operatorname{grad} f, \nabla_X Z) + g(\nabla_X\operatorname{grad} f, \nabla_Y Z)\big]\operatorname{grad} f\\
&+ \frac{1}{\alpha}\operatorname{Hess}_f(X,Z)\nabla_Y\operatorname{grad} f - \frac{1}{2\alpha^2}Y(\alpha)\operatorname{Hess}_f(X,Z)\operatorname{grad} f. \qquad (2.7)
\end{aligned}$$

The third term in the right hand of the equation (2.5) is

$$\widetilde{\nabla}_{[X,Y]}Z = \nabla_{[X,Y]}Z + \frac{1}{\alpha}g(\nabla_{[X,Y]}\operatorname{grad} f, Z)\operatorname{grad} f. \qquad (2.8)$$

Substituting (2.6), (2.7) and (2.8) in (2.5), we get the result of Proposition 2.5. □

Corollary 2.6. *Let (M^m, g) be a connected Riemannian manifold and $f : M \longrightarrow \mathbb{R}$ a smooth non constant function, such that $\|\operatorname{grad} f\| = 1$, then*

$$\begin{aligned}
\widetilde{R}(X,Y)Z &= R(X,Y)Z - \frac{1}{2}g\big(R(X,Y)Z, \operatorname{grad} f\big)\operatorname{grad} f\\
&+ \frac{1}{2}\big[\operatorname{Hess}_f(Y,Z)\nabla_X\operatorname{grad} f - \operatorname{Hess}_f(X,Z)\nabla_Y\operatorname{grad} f\big].
\end{aligned} \qquad (2.9)$$

3 Mains results

3.1 Harmonic map

Let (M^m, g), (N^n, h) two Riemannian manifolds, with M is connected and $\widetilde{g} = g + df\otimes df$, where f is a smooth non constant function on M. Then we have the following

Proposition 3.1. *Let $\varphi : (M^m, \widetilde{g}) \longrightarrow (N^n, h)$ be a smooth map. Then the tension field of φ associated to $\widetilde{g}$ is given by*

$$\widetilde{\tau}(\varphi) = \tau(\varphi) - \frac{1}{\alpha}\Big[\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) - \frac{1}{2\alpha}d\varphi(\operatorname{grad}\alpha) + \Big(\frac{1}{2\alpha}\operatorname{grad} f(\alpha) + \Delta(f)\Big)d\varphi(\operatorname{grad} f)\Big], \qquad (3.1)$$

where $\tau(\varphi)$ is the tension field of φ with respect to g.

Proof.
Recall that, $\{e_i\}_{i=1,..,m}$ such that $e_1 = \frac{1}{\sqrt{\alpha-1}}\operatorname{grad} f$ be an orthonormal frame on M with respect to g and $\{\widetilde{e}_i\}_{i=1,..,m}$ such that

$$\widetilde{e}_1 = \frac{1}{\sqrt{\alpha}}e_1 = \frac{1}{\sqrt{\alpha(\alpha-1)}}\operatorname{grad} f, \quad \{\widetilde{e}_i = e_i\}_{i=2,..,m}$$

be an orthonormal frame on M with respect to $\widetilde{g}$.
We have

$$\begin{aligned}
\widetilde{\tau}(\varphi) &= \sum_{i=1}^{m}\big[\nabla^{\varphi}_{\widetilde{e}_i}d\varphi(\widetilde{e}_i) - d\varphi(\widetilde{\nabla}^M_{\widetilde{e}_i}\widetilde{e}_i)\big]\\
&= \nabla^{\varphi}_{\widetilde{e}_1}d\varphi(\widetilde{e}_1) - d\varphi(\widetilde{\nabla}^M_{\widetilde{e}_1}\widetilde{e}_1) + \sum_{i=2}^{m}[\nabla^{\varphi}_{e_i}d\varphi(e_i) - d\varphi(\widetilde{\nabla}^M_{e_i}e_i)].
\end{aligned} \qquad (3.2)$$

We calculate the first term in the right hand of (3.2). To simplify calculations, we denote by $\beta = \frac{1}{\sqrt{\alpha(\alpha-1)}}$, then

$$\begin{aligned}\nabla^{\varphi}_{\widetilde{e}_1} d\varphi(\widetilde{e}_1) - d\varphi(\widetilde{\nabla}^M_{\widetilde{e}_1}\widetilde{e}_1) &= \nabla^{\varphi}_{\beta\,\text{grad}\,f} d\varphi(\beta\,\text{grad}\,f) - d\varphi(\widetilde{\nabla}^M_{\beta\,\text{grad}\,f}\beta\,\text{grad}\,f)\\ &= \beta^2\big[\nabla^{\varphi}_{\text{grad}\,f} d\varphi(\text{grad}\,f)\\ &- d\varphi(\widetilde{\nabla}^M_{\text{grad}\,f}\,\text{grad}\,f)\big].\end{aligned} \tag{3.3}$$

From the corollary 2.4, we get

$$\widetilde{\nabla}^M_{\text{grad}\,f}\,\text{grad}\,f = \nabla^M_{\text{grad}\,f}\,\text{grad}\,f + \frac{1}{2\alpha}\,\text{grad}\,f(\alpha)\,\text{grad}\,f,$$

then

$$\begin{aligned}\nabla^{\varphi}_{\widetilde{e}_1} d\varphi(\widetilde{e}_1) - d\varphi(\widetilde{\nabla}^M_{\widetilde{e}_1}\widetilde{e}_1) &= \beta^2\big[\nabla^{\varphi}_{\text{grad}\,f} d\varphi(\text{grad}\,f) - d\varphi(\widetilde{\nabla}^M_{\text{grad}\,f}\,\text{grad}\,f)\big]\\ &= \beta^2\Big[\nabla^{\varphi}_{\text{grad}\,f} d\varphi(\text{grad}\,f) - d\varphi\Big(\nabla^M_{\text{grad}\,f}\,\text{grad}\,f + \frac{1}{2\alpha}\,\text{grad}\,f(\alpha)\,\text{grad}\,f\Big)\Big]\\ &= \frac{1}{\alpha(\alpha-1)}\Big[\nabla^{\varphi}_{\text{grad}\,f} d\varphi(\text{grad}\,f) - \frac{1}{2}d\varphi(\text{grad}\,\alpha)\\ &- \frac{1}{2\alpha}\,\text{grad}\,f(\alpha) d\varphi(\text{grad}\,f)\Big].\end{aligned} \tag{3.4}$$

For the second term in the right hand of (3.2), we have

$$\begin{aligned}\sum_{i=2}^{m}[\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\widetilde{\nabla}^M_{e_i}e_i)] &= \sum_{i=2}^{m}[\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i}e_i) - \frac{1}{\alpha}\,\text{Hess}_f(e_i,e_i) d\varphi(\text{grad}\,f)]\\ &= \tau(\varphi) - \nabla^{\varphi}_{e_1} d\varphi(e_1) + d\varphi(\nabla^M_{e_1}e_1)\\ &- \frac{1}{\alpha}\big[\Delta(f) - g(\nabla_{e_1}\,\text{grad}\,f, e_1)\big] d\varphi(\text{grad}\,f)\\ &= \tau(\varphi) - \frac{1}{1-\alpha}\nabla^{\varphi}_{\text{grad}\,f} d\varphi(\text{grad}\,f) + \frac{1}{2(\alpha-1)} d\varphi(\text{grad}\,\alpha)\\ &+ \Big(\frac{1}{2\alpha(\alpha-1)}\,\text{grad}\,f(\alpha) - \frac{1}{\alpha}\Delta(f)\Big) d\varphi(\text{grad}\,f).\end{aligned} \tag{3.5}$$

By substituting (3.5) and (3.4) in (3.2), we get the result of the Proposition 3.1. □

Corollary 3.2. *The map $\varphi : (M^m, \widetilde{g}) \longrightarrow (N^n, h)$ is harmonic if and only if*

$$\tau(\varphi) - \frac{1}{\alpha}\Big[\nabla^{\varphi}_{\text{grad}\,f} d\varphi(\text{grad}\,f) - \frac{1}{2\alpha} d\varphi(\text{grad}\,\alpha) + \Big(\frac{1}{2\alpha}\,\text{grad}\,f(\alpha) + \Delta(f)\Big) d\varphi(\text{grad}\,f)\Big] = 0.$$

Corollary 3.3. *Let $\varphi : (M^m, \widetilde{g}) \longrightarrow (N^n, h)$ a smooth map. If $\|\text{grad}\,f\|$ is constant then, φ is harmonic if and only if*

$$\tau(\varphi) - \frac{1}{\alpha}\Big(\Delta(f) d\varphi(\text{grad}\,f) + \nabla^{\varphi}_{\text{grad}\,f} d\varphi(\text{grad}\,f)\Big) = 0$$

From the Corollary 3.2, we get the following

Remark 3.4. 1) The identity map $\varphi = Id_M : (M^m, \widetilde{g}) \longrightarrow (M^m, g)$ is harmonic if and only if

$$\frac{\alpha-1}{2\alpha}\,\text{grad}(\alpha) + \Big[\frac{1}{2\alpha}\,\text{grad}\,f(\alpha) + \Delta(f)\Big]\,\text{grad}\,f = 0.$$

2) If $\|\text{grad}\,f\|$ is constant, then the identity map $\varphi = Id_M : (M^m, \widetilde{g}) \longrightarrow (M^m, g)$ is harmonic if and only if, f is harmonic function.

Example 3.5. Let $M = \mathbb{H}^2 \times \mathbb{R}$ equipped with the Riemannian metric

$$g = \frac{1}{y^2}(dx^2 + dy^2) + dt^2,$$

where $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 / y > 0\}$ is a 2-dimensional hyperbolic space and let

$$f(x,y,t) = at + b, \quad for \quad (a,b) \in \mathbb{R}^* \times \mathbb{R}.$$

According to Remarks 3.4, the identity map $\varphi = Id_M : (M^m, \widetilde{g}) \longrightarrow (M^m, g)$ is harmonic, with

$$\widetilde{g} = \frac{1}{y^2}(dx^2 + dy^2) + (1 + a^2)dt^2.$$

Now we will study the harmonicity by deforming the metric of the codomain.
Let (M^m, g), (N^n, h) two Riemannian manifolds, with N is connected and $\widetilde{h} = h + df \otimes df$, where $f : N \longrightarrow \mathbb{R}$ is a smooth non constant function, then we have the following

Lemma 3.6. *Let (M^m, g), (N^n, h) two Riemannian manifolds, with N is connected, $\varphi : (M, g) \longrightarrow (N, \widetilde{h})$ be a smooth map and $f : N \longrightarrow \mathbb{R}$ is a smooth non constant function, then*

$$\widetilde{\nabla}^{\varphi}_X V = \nabla^{\varphi}_X V + \frac{1}{\alpha} h\big(\nabla^{\varphi}_X (\operatorname{grad}^N f) \circ \varphi, V\big)(\operatorname{grad}^N f) \circ \varphi,$$

where $\alpha = 1 + \| \operatorname{grad}^N f\|^2_h$

Proof. Let $\overline{X}, \overline{V} \in \Gamma(TN)$, such that $\overline{V} \circ \varphi = V$ and $\overline{X} \circ \varphi = d\varphi(X)$. Then

$$\begin{aligned}\widetilde{\nabla}^{\varphi}_X V &= (\widetilde{\nabla}^N_{\overline{X}} \overline{V}) \circ \varphi \\ &= \big[\nabla^N_{\overline{X}} \overline{V} + \frac{1}{\alpha} h(\nabla^N_{\overline{X}} \operatorname{grad}^N f, \overline{V}) \operatorname{grad}^N f\big] \circ \varphi \\ &= \nabla^{\varphi}_X V + \frac{1}{\alpha} h(\nabla^{\varphi}_X \operatorname{grad}^N f \circ \varphi, V)(\operatorname{grad}^N f) \circ \varphi\end{aligned}$$

□

Proposition 3.7. *Let (M^m, g), (N^n, h) two Riemannian manifolds, with N is connected, $\varphi : (M, g) \longrightarrow (N, \widetilde{h})$ be a smooth map and $f : N \longrightarrow \mathbb{R}$ is a smooth non constant function. Then, the tension field of φ with respect to $\widetilde{h}$ is given by*

$$\widetilde{\tau}(\varphi) = \tau(\varphi) + \frac{1}{\alpha} \operatorname{trace} h\Big(\nabla^{\varphi}(\operatorname{grad}^N f) \circ \varphi, d\varphi\Big)(\operatorname{grad}^N f) \circ \varphi.$$

Proof. Let $\{e_i\}_{i=1,..,m}$ be an orthonormal frame on M with respect to g. Then using Lemma 3.6 we get

$$\begin{aligned}\widetilde{\tau}(\varphi) &= \sum_{i=1}^{m} \widetilde{\nabla}^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i) \\ &= \sum_{i=1}^{m} \big[\nabla^{\varphi}_{e_i} d\varphi(e_i) + \frac{1}{\alpha} h\Big(\nabla^{\varphi}_{e_i}(\operatorname{grad}^N f) \circ \varphi, d\varphi(e_i)\Big)(\operatorname{grad}^N f) \circ \varphi - d\varphi(\nabla^M_{e_i} e_i)\big] \\ &= \tau(\varphi) + \frac{1}{\alpha} \operatorname{trace} h\Big(\nabla^{\varphi}(\operatorname{grad}^N f) \circ \varphi, d\varphi\Big)(\operatorname{grad}^N f) \circ \varphi.\end{aligned}$$

□

3.2 Harmonic Morphism

Let (M^m, g), (N^n, h) two Riemannian manifolds, with M is connected $\widetilde{g} = g + df \otimes df$ be a deformation of g, where $f : M \longrightarrow \mathbb{R}$ is a smooth non constant function, then we have the following

Proposition 3.8. *Let $\varphi : (M^m, \widetilde{g}) \longrightarrow (N^n, h)$ be a smooth map, then for all harmonic function $u : V \longrightarrow \mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty, we have*

$$\begin{aligned}\widetilde{\Delta}^M(u \circ \varphi) = \Delta^M(u \circ \varphi) &- \frac{1}{\alpha} \operatorname{Hess}_{u\circ\varphi}(\operatorname{grad} f, \operatorname{grad} f) \\ &- \frac{1}{\alpha}\big[\Delta(f) - \frac{1}{2\alpha} \operatorname{grad} f(\alpha)\big] \operatorname{grad} f(u \circ \varphi).\end{aligned}$$

Proof. Recall that $\{e_i\}_{i=1,..,m}$ such that $e_1 = \frac{1}{\sqrt{\alpha-1}} \operatorname{grad} f$ be an orthonormal frame on M with respect to g and $\{\widetilde{e}_i\}_{i=1,..,m}$, such that $\widetilde{e}_1 = \frac{1}{\sqrt{\alpha}} e_1 = \frac{1}{\sqrt{\alpha(\alpha-1)}} \operatorname{grad} f$ and $\widetilde{e}_i = e_i$ *for* $i = 2, .., m$ be an orthonormal frame on M with respect to $\widetilde{g}$. We have

$$\begin{aligned}\widetilde{\Delta}^M(u \circ \varphi) &= \sum_{i=1}^{m} \big[\widetilde{e}_i(\widetilde{e}_i(u \circ \varphi)) - \big(\widetilde{\nabla}^M_{\widetilde{e}_i} \widetilde{e}_i\big)(u \circ \varphi)\big] \\ &= \widetilde{e}_1(\widetilde{e}_1(u \circ \varphi)) - \big(\widetilde{\nabla}^M_{\widetilde{e}_1} \widetilde{e}_1\big)(u \circ \varphi) \\ &+ \sum_{i=2}^{m} \big[e_i(e_i(u \circ \varphi)) - \big(\widetilde{\nabla}^M_{e_i} e_i\big)(u \circ \varphi)\big] \\ &= \frac{1}{\sqrt{\alpha}} e_1(\frac{1}{\sqrt{\alpha}} e_1(u \circ \varphi)) - \frac{1}{\sqrt{\alpha}}\big(\widetilde{\nabla}^M_{e_1} \frac{1}{\sqrt{\alpha}} e_1\big)(u \circ \varphi) \\ &+ \sum_{i=1}^{m} \big[e_i(e_i(u \circ \varphi)) - \big(\widetilde{\nabla}^M_{e_i} e_i\big)(u \circ \varphi)\big] - e_1(e_1(u \circ \varphi)) + \big(\widetilde{\nabla}^M_{e_1} e_1\big)(u \circ \varphi) \\ &= (\frac{1}{\alpha} - 1) e_1(e_1(u \circ \varphi)) + (1 - \frac{1}{\alpha})\big(\widetilde{\nabla}^M_{e_1} e_1\big)(u \circ \varphi) \\ &+ \sum_{i=1}^{m} \big[e_i(e_i(u \circ \varphi)) - \big(\widetilde{\nabla}^M_{e_i} e_i\big)(u \circ \varphi)\big].\end{aligned}$$

By using the Proposition 2.3, we get

$$\begin{aligned}
\widetilde{\Delta}^M(u\circ\varphi) &= \frac{\alpha-1}{\alpha}\Big[-e_1(e_1(u\circ\varphi))+(\nabla^M_{e_1}e_1)(u\circ\varphi)+\frac{1}{\alpha}g(\nabla_{e_1}\operatorname{grad}f,e_1)\operatorname{grad}f(u\circ\varphi)\Big]\\
&+ \sum_{i=1}^{m}\big[e_i(e_i(u\circ\varphi))-\big(\nabla^M_{e_i}e_i\big)(u\circ\varphi)-\frac{1}{\alpha}g(\nabla_{e_i}\operatorname{grad}f,e_i)\operatorname{grad}f(u\circ\varphi)\big]\\
&= -\frac{\alpha-1}{\alpha}\Big[\frac{1}{\alpha-1}\operatorname{grad}f(\operatorname{grad}f(u\circ\varphi))\\
&- \frac{1}{\alpha-1}(\nabla^M_{\operatorname{grad}f}\operatorname{grad}f)(u\circ\varphi)-\frac{1}{\alpha(\alpha-1)}g(\nabla_{\operatorname{grad}f}\operatorname{grad}f,\operatorname{grad}f)\operatorname{grad}f(u\circ\varphi)\Big]\\
&- \frac{1}{\alpha}\Delta(f)\operatorname{grad}f(u\circ\varphi)+\Delta^M(u\circ\varphi)\\
&= \Delta^M(u\circ\varphi)-\frac{1}{\alpha}\big[\operatorname{grad}f\big(g(\operatorname{grad}f,\operatorname{grad}(u\circ\varphi))\big)-g(\nabla_{\operatorname{grad}f}\operatorname{grad}f,\operatorname{grad}(u\circ\varphi))\big]\\
&+ \frac{1}{2\alpha^2}\operatorname{grad}f(\alpha)\operatorname{grad}f(u\circ\varphi)-\frac{1}{\alpha}\Delta(f)\operatorname{grad}f(u\circ\varphi)\\
&= \Delta^M(u\circ\varphi)-\frac{1}{\alpha}\operatorname{Hess}_{u\circ\varphi}(\operatorname{grad}f,\operatorname{grad}f)\\
&- \frac{1}{\alpha}\big[\Delta(f)-\frac{1}{2\alpha}\operatorname{grad}f(\alpha)\big]\operatorname{grad}f(u\circ\varphi)
\end{aligned}$$

□

Corollary 3.9. *Let (M^m,g), (N^n,h) two Riemannian manifolds, with M is connected, $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be an harmonic morphism and $\widetilde{g}=g+df\otimes df$, where f is a smooth non constant function on M, then $\varphi:(M^m,\widetilde{g})\longrightarrow(N^n,h)$ is harmonic morphism if and only if*

$$\operatorname{Hess}_{u\circ\varphi}(\operatorname{grad}f,\operatorname{grad}f)+\big[\Delta(f)-\frac{1}{2\alpha}\operatorname{grad}f(\alpha)\big]\operatorname{grad}f(u\circ\varphi)=0,$$

for all harmonic function $u:V\longrightarrow\mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty.

Corollary 3.10. *Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be an harmonic morphism and $\widetilde{g}=g+df\otimes df$, where f is a smooth non constant function on M. If $\|\operatorname{grad}f\|$ is constant, then $\varphi:(M^m,\widetilde{g})\longrightarrow(N^n,h)$ is harmonic morphism if and only if*

$$\operatorname{Hess}_{u\circ\varphi}(\operatorname{grad}f,\operatorname{grad}f)+\Delta(f)\operatorname{grad}f(u\circ\varphi)=0,$$

for all harmonic function $u:V\longrightarrow\mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty.

From the Corollary 3.10, we get the following

Corollary 3.11. *Let $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be an harmonic morphism and $\widetilde{g}=g+df\otimes df$, where f is a smooth non constant function on M, such that $\|\operatorname{grad}f\|$ is constant, then the identity map $\varphi=Id_M:(M^m,\widetilde{g})\longrightarrow(M^m,g)$ is harmonic morphism if and only if*

$$\operatorname{Hess}_u(\operatorname{grad}f,\operatorname{grad}f)+\Delta(f)\operatorname{grad}f(u)=0,$$

for all $u:V\longrightarrow\mathbb{R}$ defined on an open subset V (non-empty) of N.

3.3 Bi-harmonic map

Theorem 3.12. *Let (M^m,g), (N^n,h) two Riemannian manifolds, with M is connected, $\varphi:(M^m,g)\longrightarrow(N^n,h)$ be a smooth map and $f:M\longrightarrow\mathbb{R}$ a smooth non constant function such that $\|\operatorname{grad}f\|=1$ and*

$$\widetilde{g}=g+df\otimes df,$$

then the bi-tension field of φ is given by

$$\begin{aligned}
\widetilde{\tau}_2(\varphi) &= \tau_2(\varphi)+\frac{1}{2}\nabla^\varphi_{\operatorname{grad}f}\nabla^\varphi_{\operatorname{grad}f}\tau(\varphi)+\frac{1}{2}R^N(\tau(\varphi),d\varphi(\operatorname{grad}f))d\varphi(\operatorname{grad}f)+\frac{1}{2}\Delta(f)\nabla^\varphi_{\operatorname{grad}f}\tau(\varphi)\\
&+\frac{1}{2}\Big[\Delta(\Delta f)-\frac{1}{2}\operatorname{grad}f(\operatorname{grad}f(\Delta f))-\frac{1}{2}\Delta(f)\operatorname{grad}f(\Delta(f))\Big]d\varphi(\operatorname{grad}f)\\
&-\frac{1}{2}\Delta(f)\,\mathrm{J}_\varphi(d\varphi(\operatorname{grad}f))-\frac{1}{2}\mathrm{J}_\varphi(\nabla^\varphi_{\operatorname{grad}f}d\varphi(\operatorname{grad}f))\\
&-\frac{1}{4}R^N(\nabla^\varphi_{\operatorname{grad}f}d\varphi(\operatorname{grad}f),d\varphi(\operatorname{grad}f))d\varphi(\operatorname{grad}f)+\nabla^\varphi_{\operatorname{grad}(\Delta(f))}d\varphi(\operatorname{grad}f)\\
&-\frac{1}{2}\Big[\operatorname{grad}f(\Delta(f))+\frac{1}{2}(\Delta(f))^2\Big]\nabla^\varphi_{\operatorname{grad}f}d\varphi(\operatorname{grad}f),
\end{aligned}$$

where $\mathrm{J}_\varphi(V)=\Delta^\varphi(V)+\operatorname{trace}_g R^N(V,d\varphi)d\varphi$, $\forall V\in\varphi^{-1}(TN)$ is the Jacobi operator of φ.

Proof. Let $\{e_i\}_{i=1,\dots,m}$ such that $e_1 = \operatorname{grad} f$ be a local orthnormal frame on M with respect to g and $\{\widetilde{e}_i\}_{i=1,\dots,m}$ such that $\widetilde{e}_1 = \frac{\operatorname{grad} f}{\sqrt{2}}$ and $\widetilde{e}_i = e_i \ for\ i = 2, \dots, m$ be a local orthnormal frame on M with respect to $\widetilde{g}$. The bi-tension field of φ with respect to $\widetilde{g}$ is given by

$$\begin{aligned}\widetilde{\tau}_2(\varphi) &= -\Big[\nabla^{\varphi}_{\widetilde{e}_i}\nabla^{\varphi}_{\widetilde{e}_i}\widetilde{\tau}(\varphi) - \nabla^{\varphi}_{\widetilde{\nabla}^M_{\widetilde{e}_i}\widetilde{e}_i}\widetilde{\tau}(\varphi)\Big] - R^N(\widetilde{\tau}(\varphi), d\varphi(\widetilde{e}_i))d\varphi(\widetilde{e}_i) \\ &= -\nabla^{\varphi}_{\widetilde{e}_i}\nabla^{\varphi}_{\widetilde{e}_i}\widetilde{\tau}(\varphi) + \nabla^{\varphi}_{\widetilde{\nabla}^M_{\widetilde{e}_i}\widetilde{e}_i}\widetilde{\tau}(\varphi) - R^N(\widetilde{\tau}(\varphi), d\varphi(\widetilde{e}_i))d\varphi(\widetilde{e}_i)\end{aligned} \tag{3.6}$$

For the first term in the right hand of (3.6), we have

$$\begin{aligned}-\nabla^{\varphi}_{\widetilde{e}_i}\nabla^{\varphi}_{\widetilde{e}_i}\widetilde{\tau}(\varphi) &= -\nabla^{\varphi}_{\widetilde{e}_1}\nabla^{\varphi}_{\widetilde{e}_1}\widetilde{\tau}(\varphi) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\widetilde{\tau}(\varphi) \\ &= -\frac{1}{2}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}\widetilde{\tau}(\varphi) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\widetilde{\tau}(\varphi).\end{aligned} \tag{3.7}$$

Recall that $\widetilde{\tau}(\varphi) = \tau(\varphi) - \frac{1}{2}\Delta(f)d\varphi(\operatorname{grad} f) - \frac{1}{2}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f)$, then

$$\begin{aligned}\nabla^{\varphi}_{\operatorname{grad} f}\widetilde{\tau}(\varphi) &= \nabla^{\varphi}_{\operatorname{grad} f}\tau(\varphi) - \frac{1}{2}\operatorname{grad} f(\Delta(f))d\varphi(\operatorname{grad} f) - \frac{1}{2}\Delta(f)\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) \\ &\quad - \frac{1}{2}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f),\end{aligned}$$

therefore,

$$\begin{aligned}-\frac{1}{2}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}\widetilde{\tau}(\varphi) &= -\frac{1}{2}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}\tau(\varphi) + \frac{1}{4}\operatorname{grad} f(\operatorname{grad} f(\Delta(f)))d\varphi(\operatorname{grad} f) \\ &\quad + \frac{1}{2}\operatorname{grad} f(\Delta(f))\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) + \frac{1}{4}(\Delta(f))\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) \\ &\quad + \frac{1}{4}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f).\end{aligned} \tag{3.8}$$

on the other hand

$$\begin{aligned}\nabla^{\varphi}_{e_i}\widetilde{\tau}(\varphi) &= \nabla^{\varphi}_{e_i}\tau(\varphi) - \frac{1}{2}e_i(\Delta(f))d\varphi(\operatorname{grad} f) - \frac{1}{2}\Delta(f)\nabla^{\varphi}_{e_i}d\varphi(\operatorname{grad} f) \\ &\quad - \frac{1}{2}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f),\end{aligned}$$

then

$$\begin{aligned}-\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\widetilde{\tau}(\varphi) &= -\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi) + \frac{1}{2}e_i(e_i(\Delta(f)))d\varphi(\operatorname{grad} f) + e_i(\Delta(f))\nabla^{\varphi}_{e_i}d\varphi(\operatorname{grad} f) \\ &\quad + \frac{1}{2}\Delta(f)\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}d\varphi(\operatorname{grad} f) + \frac{1}{2}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f).\end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) in (3.7), we get

$$\begin{aligned}-\nabla^{\varphi}_{\widetilde{e}_i}\nabla^{\varphi}_{\widetilde{e}_i}\widetilde{\tau}(\varphi) &= -\frac{1}{2}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}\tau(\varphi) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi) + \frac{1}{4}(\Delta(f))\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) \\ &\quad + \Big[\frac{1}{4}\operatorname{grad} f(\operatorname{grad} f(\Delta(f))) + \frac{1}{2}e_i(e_i(\Delta(f)))\Big]d\varphi(\operatorname{grad} f) \\ &\quad + \frac{1}{2}\operatorname{grad} f(\Delta(f))\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) + e_i(\Delta(f))\nabla^{\varphi}_{e_i}d\varphi(\operatorname{grad} f) \\ &\quad + \frac{1}{4}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) + \frac{1}{2}\Delta(f)\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}d\varphi(\operatorname{grad} f) \\ &\quad + \frac{1}{2}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f).\end{aligned} \tag{3.10}$$

For the second term in the right hand of (3.6), we recall that

$$\begin{aligned}\widetilde{\nabla}^M_{\widetilde{e}_i}\widetilde{e}_i &= \frac{1}{2}\widetilde{\nabla}^M_{\operatorname{grad} f}\operatorname{grad} f + \widetilde{\nabla}^M_{e_i}e_i \\ &= \widetilde{\nabla}^M_{e_i}e_i \\ &= \nabla^M_{e_i}e_i + \frac{1}{2}\Delta(f)\operatorname{grad} f,\end{aligned}$$

then

$$\begin{aligned}\nabla^{\varphi}_{\widetilde{\nabla}^M_{\widetilde{e}_i}\widetilde{e}_i}\widetilde{\tau}(\varphi) &= \nabla^{\varphi}_{\nabla^M_{e_i}e_i}\widetilde{\tau}(\varphi) + \frac{1}{2}\Delta(f)\nabla^{\varphi}_{\operatorname{grad} f}\widetilde{\tau}(\varphi) \\ &= \nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi) - \frac{1}{2}(\nabla^M_{e_i}e_i)(\Delta(f))d\varphi(\operatorname{grad} f) \\ &\quad - \frac{1}{2}\Delta(f)\nabla^{\varphi}_{\nabla^M_{e_i}e_i}d\varphi(\operatorname{grad} f) - \frac{1}{2}\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) \\ &\quad + \frac{1}{2}\Delta(f)\Big[\nabla^{\varphi}_{\operatorname{grad} f}\tau(\varphi) - \frac{1}{2}\operatorname{grad} f(\Delta(f))d\varphi(\operatorname{grad} f) - \frac{1}{2}\Delta(f)\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f) \\ &\quad - \frac{1}{2}\nabla^{\varphi}_{\operatorname{grad} f}\nabla^{\varphi}_{\operatorname{grad} f}d\varphi(\operatorname{grad} f)\Big].\end{aligned} \tag{3.11}$$

Now, we calculate the third term in the right hand of (3.6)

$$\begin{aligned} -R^N(\widetilde{\tau}(\varphi), d\varphi(\widetilde{e}_i))d\varphi(\widetilde{e}_i) &= -\frac{1}{2}R^N(\widetilde{\tau}(\varphi), d\varphi(\text{grad } f)d\varphi(\text{grad } f) - R^N(\widetilde{\tau}(\varphi), d\varphi(e_i))d\varphi(e_i) \\ &= -\frac{1}{2}R^N(\tau(\varphi), d\varphi(\text{grad } f))d\varphi(\text{grad } f) - R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) \\ &+ \frac{1}{4}R^N(\nabla^{\varphi}_{\text{grad } f} d\varphi(\text{grad } f), d\varphi(\text{grad } f))d\varphi(\text{grad } f) \\ &+ \frac{1}{2}\Delta(f)R^N(d\varphi(\text{grad } f), d\varphi(e_i))d\varphi(e_i) \\ &+ \frac{1}{2}R^N(\nabla^{\varphi}_{\text{grad } f} d\varphi(\text{grad } f), d\varphi(e_i))d\varphi(e_i). \end{aligned} \tag{3.12}$$

Recall that

$$\tau_2(\varphi) = -\Big[\nabla^{\varphi}_{\text{grad } f}\nabla^{\varphi}_{\text{grad } f}\tau(\varphi) + \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi) - \nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi)\Big]$$
$$- R^N(\tau(\varphi), d\varphi(\text{grad } f))d\varphi(\text{grad } f) - R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i),$$

the Jacobi operator is defined by

$$\mathrm{J}_{\varphi}(V) = \Delta^{\varphi}(V) - \text{trace}_g\, R^N(V, d\varphi)d\varphi,$$

and

$$\Delta(\Delta(f)) = \Big[\text{grad } f(\text{grad } f(\Delta(f))) + e_i(e_i(\Delta(f))) - (\nabla^M_{e_i}e_i)(\Delta(f))\Big],$$

then, by substituting (3.12) , (3.11) and (3.10) in (3.6), we deduce the result of Theorem 3.12. □

4 Construction of proper bi-harmonic maps

From Theorem 3.12, we deduce the following results

Corollary 4.1. *Let (M^m, g), (N^n, h) two Riemannian manifolds, with M is connected, $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be an harmonic map, $f : M \longrightarrow \mathbb{R}$ a smooth non constant function such that $\|\text{grad } f\| = 1$ and $\widetilde{g} = g + df \otimes df$. Then $\varphi : (M^m, \widetilde{g}) \longrightarrow (N^n, h)$ is bi-harmonic if and only if*

$$\begin{aligned} &+ \frac{1}{2}\Big[\Delta(\Delta f) - \frac{1}{2}\text{grad } f(\text{grad } f(\Delta f)) - \frac{1}{2}\Delta(f)\text{ grad } f(\Delta(f))\Big]d\varphi(\text{grad } f) \\ &- \frac{1}{2}\Delta(f)\,\mathrm{J}_{\varphi}(d\varphi(\text{grad } f)) - \frac{1}{2}\mathrm{J}_{\varphi}(\nabla^{\varphi}_{\text{grad } f}d\varphi(\text{grad } f)) - \frac{1}{4}R^N(\nabla^{\varphi}_{\text{grad } f}d\varphi(\text{grad } f), d\varphi(\text{grad } f))d\varphi(\text{grad } f), \\ &+ \nabla^{\varphi}_{\text{grad}(\Delta(f))}d\varphi(\text{grad } f) - \frac{1}{2}\Big[\text{grad } f(\Delta(f)) + \frac{1}{2}(\Delta(f))^2\Big]\nabla^{\varphi}_{\text{grad } f}d\varphi(\text{grad } f) = 0 \end{aligned}$$

Example 4.2. Let $n \geq 2$, $M = \mathbb{R}^n$ equipped with the canonical metric $g = dx_1^2 + dx_2^2 + ... + dx_n^2$ and $N = \mathbb{H} = \{(y_1, y_2, .., y_n)\ /y_n > 0\}$ be a n-dimensional hyperbolic space, equipped with the metric

$$h = y_n^2(dy_1^2 + dy_2^2 + .. + dy_n^2).$$

Consider the harmonic map

$$\varphi : (M, g) \longrightarrow (N, h),$$
$$\varphi(x_1, x_2, .., x_n) = \big(x_1, x_2, .., x_{n-1}, \sqrt{n-1}\sqrt{x_n^2 + 1}\big),$$

and let the function $f(x_1, x_2, .., x_n) = x_n$. Thus with respect to the Riemannian metric

$$\widetilde{g} = g + df \otimes df = g + dx_1^2 + dx_2^2 + ... + 2dx_n^2,$$

the map $\varphi : (M, \widetilde{g}) \longrightarrow (N, h)$ is bi-harmonic non harmonic.

From Corollary 4.1 and Remark 3.4, we get the following

Corollary 4.3. *Let (M^m, g) be a connected Riemannian manifold, $f : M \longrightarrow \mathbb{R}$ a non constant and non harmonic smooth function such that $\|\text{grad } f\| = 1$ and $\widetilde{g} = g + df \otimes df$, then the identity map $\varphi = Id_M : (M, \widetilde{g}) \longrightarrow (M, g)$ is a proper bi-harmonic if and only if*

$$\Big[\Delta(\Delta f) - \frac{1}{2}\text{grad } f(\text{grad } f(\Delta f)) - \frac{1}{2}\Delta(f)\text{ grad } f(\Delta(f))\Big]\text{ grad } f$$
$$- \Delta(f)\,\mathrm{J}_{Id_M}(\text{grad } f) + 2\nabla^M_{\text{grad}(\Delta(f))}\text{ grad } f = 0.$$

Remark 4.4. If $f : M \longrightarrow \mathbb{R}$ is non harmonic function such that $\|\text{grad } f\| = 1$ and $\triangle(f) = k \neq 0$, then the identity map $\varphi = Id_M : (M, \widetilde{g}) \longrightarrow (M, g)$ is a proper bi-harmonic if and only if

$$\Delta(\text{grad } f) + \widehat{\text{Ric}}(\text{grad } f) = 0.$$

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