Palestine Journal of Mathematics

Vol 13(4)(2024) , 1427–1435

$k-$LEONARDO NUMBERS

Hasan Gökbaş

Communicated by Peter Larcombe

MSC 2010 Classifications: Primary 11B39; Secondary 15A23.

Keywords and phrases: Fibonacci numbers, Leonardo numbers, k-Fibonacci numbers, k-Leonardo numbers.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract In this study, we define a new family of Leonardo numbers and establish some properties of the relation to the ordinary Leonardo numbers. These numbers are introduced from the Leonardo numbers. We give some identities k-Leonardo numbers. Moreover, we obtain the Binet's formula, generating function formula and some formulas for this new type numbers. Morever, we give the matrix representation of the k-Leonardo numbers.

1 Introduction

Fibonacci sequence has delighted mathematicians and scientists alike for centuries with their beauty and their propensity to pop up in quite unexpected places. Leonardo de Pisa has not even guess that the number sequences would be so adventurous with the rabbit problem. However, the Fibonacci numbers found in Pascal's triangle, Pythagorean triples, computer algorithms, graph theory and many other areas of mathematics. They also occur in variety of other fields such as physics, finance, architecture, computer sciences, color image processing, geostatics music and art. There have been many studies in literature about this special number sequence because of its numerous applications. There are many generalizations on this sequence some of which can be seen in [2, 4, 5, 6, 12, 13, 14, 17, 20, 21, 22, 28, 30].

The Leonardo sequence, also known as Leonardo numbers, is a linear, recurrence sequence of integers. It is a sequence similar to the Fibonacci sequence, the third term of which is a constant number. It is thought that these numbers were examined by Leonardo of Pisa and could not be proven in any study in the literature due to the scarcity of relevant studies. It is emphasized that this sequence is related to the Fibonacci sequence [3, 7, 23, 31, 32].

Leonardo's recurring non-homogeneous sequence, which we shall denote by Le_n is a linear and recurrent sequence, having its characteristic recurrence formula defined as

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2$$

with Le_n the nth term of the Leonardo sequence and initial terms indicated by $Le_0 = Le_1 = 1$. It also performed algebraic manipulations on this recurrence, obtaining a new homogeneous recurrence.

$$Le_n = 2Le_{n-1} - Le_{n-3}, \quad n \geq 3$$

with Le_n the nth term of the Leonardo sequence and initial terms indicated by $Le_0 = Le_1 = 1$ and $Le_2 = 3$. Also, the characteristic equation can be given as: $x^3 - 2x^2 + 1 = 0$. $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2}$ and $x_3 = 1$, it has three real roots. x_1 and x_2 are the roots of the characteristic equation of the Fibonacci sequence [11, 26].

The k-Fibonacci numbers defined by [16] for any real number k as follows. For any positive real number k, the k-Fibonacci sequence, say $F_{k,n}$, $n \in \mathbb{N}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1$$

with $F_{k,n}$ the nth term of the k-Fibonacci sequence and initial terms indicated by $F_{k,0} = 0$ and $F_{k,1} = 1$. This number sequence has been one of the number sequences studied over the years [1, 8, 9, 10, 18, 19, 24, 25, 27].

$F_{k,0} = 0$
$F_{k,1} = 1$
$F_{k,2} = k$
$F_{k,3} = k^2 + 1$
$F_{k,4} = k^3 + 2k$
$F_{k,5} = k^4 + 3k^2 + 1$
$F_{k,6} = k^5 + 4k^3 + 3k$
$F_{k,7} = k^6 + 5k^4 + 6k^2 + 1$
$F_{k,5} = k^7 + 6k^5 + 10k^3 + 4k$

Table 1. The first terms of the k-Fibonacci numbers

$HM_n, n \in \mathbb{N}$ is defined as follows [29], $HM_n = rHM_{n-1} + sHM_{n-2} + tHM_{n-3}$, $HM_0 = a$, $HM_1 = b$, $HM_2 = c$, $n \geq 1$ where HM_0, HM_1, HM_2 are arbitrary complex or real numbers and r, s, t are real numbers. So, in this case, using Binet's formula

$$HM_n = \frac{m_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{m_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{m_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.1}$$

where

$$m_1 = HM_2 - (\beta+\gamma)HM_1 + \beta\gamma HM_0,$$
$$m_2 = HM_2 - (\alpha+\gamma)HM_1 + \alpha\gamma HM_0,$$
$$m_3 = HM_2 - (\alpha+\beta)HM_1 + \alpha\beta HM_0.$$

In this work, a variety of algebraic properties of the k-Leonardo numbers will be presented. Some identities will be given for k-Leonardo sequence such as Binet's formula, generating function formula and some formulas.

2 k-Leonardo numbers

In this section, a new generalization of the Leonardo numbers is introduced. We give some properties of the k-Leonardo numbers.

Definition 2.1. Let $k > 0$ be real number. The numbers $Le_{k,n}$ be recursively defined by

$$Le_{k,n} = kLe_{k,n-1} + Le_{k,n-2} + 1, n \geq 2$$

$$Le_{k,0} = Le_{k,1} = 1$$

We also call them k-Leonardo numbers.

$Le_{k,0} = 1$
$Le_{k,1} = 1$
$Le_{k,2} = k + 2$
$Le_{k,3} = k^2 + 2k + 2$
$Le_{k,4} = k^3 + 2k^2 + 3k + 3$
$Le_{k,5} = k^4 + 2k^3 + 4k^2 + 5k + 3$
$Le_{k,6} = k^5 + 2k^4 + 5k^3 + 7k^2 + 6k + 4$
$Le_{k,7} = k^6 + 2k^5 + 6k^4 + 9k^3 + 10k^2 + 9k + 4$
$Le_{k,8} = k^7 + 2k^6 + 7k^5 + 11k^4 + 15k^3 + 16k^2 + 10k + 5$

Table 2. The first terms of the k-Leonardo numbers

Sequence $Le_{1,n}$, $n \in \mathbb{N}$ is the classical Leonardo sequence.

Note that if k is a real variable x, the Leonardo polynomials defined by

$$Le_{n+1}(x) = \left\{ \begin{array}{rl} 1, & ifn = 0 \\ 1, & ifn = 1 \\ xLe_n(x) + Le_{n-1}(x), & ifn > 1 \end{array} \right\}.$$

$Le_0 = 1$
$Le_1 = 1$
$Le_2 = x + 2$
$Le_3 = x^2 + 2x + 2$
$Le_4 = x^3 + 2x^2 + 3x + 3$
$Le_5 = x^4 + 2x^3 + 4x^2 + 5x + 3$
$Le_6 = x^5 + 2x^4 + 5x^3 + 7x^2 + 6x + 4$
$Le_7 = x^6 + 2x^5 + 6x^4 + 9x^3 + 10x^2 + 9x + 4$
$Le_8 = x^7 + 2x^6 + 7x^5 + 11x^4 + 15x^3 + 16x^2 + 10x + 5$

Table 3. The first terms of the Leonardo polynomials

Corollary 2.2. *From the two equations*

$$Le_{k,n} = kLe_{k,n-1} + Le_{k,n-2} + 1$$

$$Le_{k,n+1} = kLe_{k,n} + Le_{k,n-1} + 1$$

we obtained by subtraction the recursion formula

$$Le_{k,n+1} = (k+1)Le_{k,n} + (1-k)Le_{k,n-1} - Le_{k,n-2}, n \geq 2$$

where $Le_{k,2} = k + 2$ is an additional value. The associated characteristic polynomial

$$p(X) = X^3 - (k+1)X^2 - (1-k)X + 1 = (X-1)(X^2 - kX - 1).$$

$p(X)$ has the roots $t_1 = \frac{k+\sqrt{k^2+4}}{2}$, $t_2 = \frac{k-\sqrt{k^2+4}}{2}$ and $t_3 = 1$.

Definition 2.3. $Le_{k,n}$, $n \in \mathbb{N}$ is defined as follows

$$Le_{k,n} = (k+1)Le_{k,n-1} + (1-k)Le_{k,n-2} - Le_{k,n-3}, n \geq 3,$$

$$Le_{k,0} = Le_{k,1} = 1, Le_{k,2} = k + 2.$$

The sequence $Le_{k,n}$, $n \in \mathbb{N}$ be extended to negative subscripts by defined

$$Le_{k,-n} = (1-k)Le_{k,-n+1} + (k+1)Le_{k,-n+2} - Le_{k,-n+3}.$$

$Le_{k,-1} = -k$
$Le_{k,-2} = k^2$
$Le_{k,-3} = -k^3 - k - 1$
$Le_{k,-4} = k^4 + 2k^2 + k - 1$
$Le_{k,-5} = -k^5 - 3k^3 - k^2 - 2$

Table 4. The first terms of the negative k-Leonardo numbers

Theorem 2.4.

$$t_1 = \frac{k + \sqrt{k^2+4}}{2},$$

$$t_2 = \frac{k - \sqrt{k^2+4}}{2},$$
$$t_3 = 1,$$
$$\alpha = \frac{(k^2+1)(k^2+4-(k-2)\sqrt{k^2+4})}{2k(k^2+4)},$$
$$\beta = \frac{(k^2+1)(k^2+4+(k-2)\sqrt{k^2+4})}{2k(k^2+4)},$$
$$\gamma = -\frac{1}{k}.$$

Then, the Binet formula of the $Le_{k,n}$ number is

$$Le_{k,n} = \alpha t_1^n + \beta t_2^n + \gamma t_3^n, n \geq 0.$$

Moreover,

$$t_1 + t_2 + t_3 = k + 1$$
$$t_1 t_2 t_3 = -1$$
$$t_1 t_2 + t_1 t_3 + t_2 t_3 = k - 1.$$

Proof. The proof is performed using the equation (1.1). □

Theorem 2.5. *The generating function formula of the $Le_{k,n}$ number is*

$$\sum_{n=0}^{\infty} Le_{k,n} t^n = \frac{1 - kt + kt^2}{1 - (k+1)t - (1-k)t^2 + t^3}.$$

Proof. Let $h(t)$ be the generating function for k-Leonardo numbers as $h(t) = \sum_{n=0}^{\infty} Le_{k,n} t^n$. We get the following equations, $(k+1)th(t) = (k+1)\sum_{n=0}^{\infty} Le_{k,n} t^{n+1}$, $(1-k)t^2 h(t) = (1-k)\sum_{n=0}^{\infty} Le_{k,n} t^{n+2}$ and $-t^3 h(t) = -\sum_{n=0}^{\infty} Le_{k,n} t^{n+3}$. After the needed calculations, the generating function for k-Leonardo numbers is obtained as

$$\sum_{n=0}^{\infty} Le_{k,n} t^n = \frac{1 - kt + kt^2}{1 - (k+1)t - (1-k)t^2 + t^3}.$$

□

Theorem 2.6. *For $n \geq 1$, the following identity holds*

$$Le_{k,n} = \frac{1}{k}\left[F_{k,n+1} + F_{k,n} - 1\right] + F_{k,n-1}$$

where $Le_{k,n}$ is nth k-Leonardo number and $F_{k,n}$ is nth k-Fibonacci number.

Proof. Using the principle of finite induction, the equality holds for $n = 1$,

$$Le_{k,1} = \frac{1}{k}\left[F_{k,2} + F_{k,1} - 1\right] + F_{k,0} = \frac{k+1-1}{k} + 0 = 1.$$

Now suppose that the equality is true for $n > 1$. Then, we can verify for $n+1$ as follows

$$\begin{aligned} kLe_{k,n} + Le_{k,n-1} + 1 &= k\left(\frac{1}{k}\left[F_{k,n+1} + F_{k,n} - 1\right] + F_{k,n-1}\right) + \frac{1}{k}\left[F_{k,n} + F_{k,n-1} - 1\right] + F_{k,n-2} + 1 \\ &= F_{k,n+1} + F_{k,n} - 1 + kF_{k,n-1} + \frac{1}{k}\left[F_{k,n} + F_{k,n-1} - 1\right] + F_{k,n-2} + 1 \\ &= \frac{1}{k}\left[kF_{k,n+1} + F_{k,n} + kF_{k,n} + F_{k,n-1} - 1\right] + kF_{k,n-1} + F_{k,n-2} \\ &= \frac{1}{k}\left[F_{k,n+2} + F_{k,n+1} - 1\right] + F_{k,n} = Le_{k,n+1}. \end{aligned}$$

□

$Le_{k,n}$	$\frac{1}{k}\left[F_{k,n+1}+F_{k,n}-1\right]+F_{k,n-1}$
$Le_{k,0}$	$\frac{1}{k}\left[F_{k,1}+F_{k,0}-1\right]+F_{k,-1}=\frac{1}{k}(1+0-1)+1=1$
$Le_{k,1}$	$\frac{1}{k}\left[F_{k,2}+F_{k,1}-1\right]+F_{k,0}=\frac{1}{k}(k+1-1)+0=1$
$Le_{k,2}$	$\frac{1}{k}\left[F_{k,3}+F_{k,2}-1\right]+F_{k,1}=\frac{1}{k}(k^2+1+k-1)+1=k+2$
$Le_{k,3}$	$\frac{1}{k}\left[F_{k,4}+F_{k,3}-1\right]+F_{k,2}=\frac{1}{k}(k^3+2k+k^2+1-1)+k=k^2+k+2$
$Le_{k,4}$	$\frac{1}{k}\left[F_{k,5}+F_{k,4}-1\right]+F_{k,3}=\frac{1}{k}(k^4+3k^2+1+k^3+2k-1)+k^2+1=k^3+2k^2+3k+3$

Table 5. The first terms of the k-Leonardo numbers from k-Fibonacci numbers

The table below examples for different values of n and k.

	$k=1$	$k=2$	$k=3$
$n=0$	$Le_{1,0}=1$	$Le_{2,0}=1$	$Le_{3,0}=1$
$n=1$	$Le_{1,1}=1$	$Le_{2,1}=1$	$Le_{3,1}=1$
$n=2$	$Le_{1,2}=3$	$Le_{2,2}=4$	$Le_{3,2}=5$
$n=3$	$Le_{1,3}=5$	$Le_{2,3}=10$	$Le_{3,3}=17$
$n=4$	$Le_{1,4}=9$	$Le_{2,4}=19$	$Le_{3,4}=57$
$n=5$	$Le_{1,5}=15$	$Le_{2,5}=61$	$Le_{3,5}=189$

Table 6. The first terms of the k-Leonardo numbers

3 The matrix form of k-Leonardo numbers

In this section, we will give the matrix representation of the k-Leonardo numbers. The k-Leonardo number' properties will be obtained using matrix representation.

Definition 3.1. The basic matrix of the k-Leonardo sequence is

$$Q=\begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

Based on the Cayley-Hamilton Theorem, k-Leonardo's characteristic polynomial is given as

$$p(\lambda)=det(\lambda I-Q)$$

where

$$\lambda I=\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & \\ 0 & 0 & \lambda \end{bmatrix} \quad and \quad Q=\begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

$$p(\lambda)=det(\lambda I-Q)=\begin{vmatrix} \lambda-k-1 & -1 & 0 \\ k-1 & \lambda & -1 \\ 1 & 0 & \lambda \end{vmatrix}=\lambda^3-(k+1)\lambda^2-(1-k)\lambda+1=0.$$

Theorem 3.2. *Let $n>0$ be an integer. The following equality holds*

$$a)\begin{bmatrix} Le_{k,n+3} & Le_{k,n+2} & Le_{k,n+1} \\ Le_{k,n+2} & Le_{k,n+1} & Le_{k,n} \\ Le_{k,n+1} & Le_{k,n} & Le_{k,n-1} \end{bmatrix}=\begin{bmatrix} Le_{k,3} & Le_{k,2} & Le_{k,1} \\ Le_{k,2} & Le_{k,1} & Le_{k,0} \\ Le_{k,1} & Le_{k,0} & Le_{k,-1} \end{bmatrix}\cdot\begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n$$

$$b)\begin{bmatrix} Le_{k,-n+3} & Le_{k,-n+2} & Le_{k,-n+1} \\ Le_{k,-n+2} & Le_{k,-n+1} & Le_{k,-n} \\ Le_{k,-n+1} & Le_{k,-n} & Le_{k,-n-1} \end{bmatrix}=\begin{bmatrix} Le_{k,3} & Le_{k,2} & Le_{k,1} \\ Le_{k,2} & Le_{k,1} & Le_{k,0} \\ Le_{k,1} & Le_{k,0} & Le_{k,-1} \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & k+1 \\ 0 & 1 & 1-k \end{bmatrix}^n$$

Proof. a) For the proof, we use induction method on n. The equality holds for $n = 1$. Now suppose that the equality is true for $n > 1$. Then, we can verify for $n+1$ as follows

$$\begin{bmatrix} Le_{k,3} & Le_{k,2} & Le_{k,1} \\ Le_{k,2} & Le_{k,1} & Le_{k,0} \\ Le_{k,1} & Le_{k,0} & Le_{k,-1} \end{bmatrix} Q^{n+1} = \begin{bmatrix} Le_{k,3} & Le_{k,2} & Le_{k,1} \\ Le_{k,2} & Le_{k,1} & Le_{k,0} \\ Le_{k,1} & Le_{k,0} & Le_{k,-1} \end{bmatrix} Q^n Q$$

$$\begin{bmatrix} Le_{k,n+3} & Le_{k,n+2} & Le_{k,n+1} \\ Le_{k,n+2} & Le_{k,n+1} & Le_{k,n} \\ Le_{k,n+1} & Le_{k,n} & Le_{k,n-1} \end{bmatrix} \cdot \begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} Le_{k,n+4} & Le_{k,n+3} & Le_{k,n+2} \\ Le_{k,n+3} & Le_{k,n+2} & Le_{k,n+1} \\ Le_{k,n+2} & Le_{k,n+1} & Le_{k,n} \end{bmatrix}.$$

Thus, the theorem can be proved easily.

b) Similarly, the proof is seen by induction on n.

□

Theorem 3.3. *Let $n > 0$ be an integer. The following equality holds*

$$a) \begin{bmatrix} Le_{k,n+2} & Le_{k,n+1} & Le_{k,n} \end{bmatrix} = \begin{bmatrix} Le_{k,2} & Le_{k,1} & Le_{k,0} \end{bmatrix} \cdot \begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n$$

$$b) \begin{bmatrix} Le_{k,-n+2} & Le_{k,-n+1} & Le_{k,-n} \end{bmatrix} = \begin{bmatrix} Le_{k,2} & Le_{k,1} & Le_{k,0} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & k+1 \\ 0 & 1 & 1-k \end{bmatrix}^n$$

Proof. a) Using the principle of finite induction, the equality holds for $n = 1$. Now suppose that the equality is true for $n > 1$. Then, we can verify for $n+1$ as follows

$$\begin{bmatrix} Le_{k,2} \\ Le_{k,1} \\ Le_{k,0} \end{bmatrix}^T \cdot \begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^{n+1} = \begin{bmatrix} Le_{k,2} \\ Le_{k,1} \\ Le_{k,0} \end{bmatrix}^T \cdot \begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} Le_{k,n+2} \\ Le_{k,n+1} \\ Le_{k,n} \end{bmatrix}^T \cdot \begin{bmatrix} k+1 & 1 & 0 \\ 1-k & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} Le_{k,n+3} & Le_{k,n+2} & Le_{k,n+1} \end{bmatrix}.$$

Thus, the theorem can be proved easily.

b) Similarly, the proof is seen by induction on n. □

Corollary 3.4. *(Simson's Identity) Let $n > 0$ be an integer. The following equality holds*

$$\begin{vmatrix} Le_{k,n+3} & Le_{k,n+2} & Le_{k,n+1} \\ Le_{k,n+2} & Le_{k,n+1} & Le_{k,n} \\ Le_{k,n+1} & Le_{k,n} & Le_{k,n-1} \end{vmatrix} = \begin{vmatrix} Le_{k,-n+3} & Le_{k,-n+2} & Le_{k,-n+1} \\ Le_{k,-n+2} & Le_{k,-n+1} & Le_{k,-n} \\ Le_{k,-n+1} & Le_{k,-n} & Le_{k,-n-1} \end{vmatrix} = (-1)^n (Le_{k,3} - 1).$$

4 Sums of k-Leonardo numbers

In this section, we present some results concerning sums of terms of the k-Leonardo sequence by using some results of k-Fibonacci sequences. We can obtaine the following results for sums of k-Leonardo numbers.

Lemma 4.1. *Let $F_{k,n}$ be the nth k-Fibonacci numbers. In this case [15]*

$$a) \sum_{i=0}^{n} F_{k,i} = \frac{F_{k,n+1} + F_{k,n} - 1}{k}$$

$$b)\sum_{i=0}^{n}F_{k,2i}=\frac{F_{k,2n+2}-F_{k,2n}-k}{k^2}$$

$$c)\sum_{i=0}^{n}F_{k,2i+1}=\frac{F_{k,2n+3}-F_{k,2n+1}}{k^2}$$

Theorem 4.2. *Let $Le_{k,n}$ be the nth k-Leonardo numbers. In this case*

$$a)\sum_{i=0}^{n}Le_{k,i}=\frac{1}{k}\left[\frac{F_{k,n+2}+2F_{k,n+1}+F_{k,n}-k(n+1)-2}{k}+F_{k,n}+F_{k,n-1}+k-1\right]$$

$$b)\sum_{i=0}^{n}Le_{k,2i}=\frac{1}{k}\left[\frac{F_{k,2n+3}+F_{k,2n+2}-F_{k,2n+1}-F_{k,2n}-k^2(n+1)-k}{k^2}+\frac{F_{k,2n+1}-F_{k,2n-1}+k^2}{k}\right]$$

$$c)\sum_{i=0}^{n}Le_{k,2i-1}=\frac{1}{k}\left[\frac{F_{k,2n+2}+F_{k,2n+1}-F_{k,2n}-F_{k,2n-1}-nk^2-k}{k^2}+\frac{F_{k,2n}-F_{k,2n-2}-k^3-k}{k}\right]$$

Proof.

$$\begin{aligned}
a)\sum_{i=0}^{n}Le_{k,i} &= \sum_{i=0}^{n}\left[\frac{1}{k}\left(F_{k,i+1}+F_{k,i}-1\right)F_{k,i-1}\right]\\
&= \frac{1}{k}\left(\sum_{i=0}^{n}F_{k,i+1}\right)+\frac{1}{k}\left(\sum_{i=0}^{n}F_{k,i}\right)-\frac{1}{k}\left(\sum_{i=0}^{n}1\right)+\sum_{i=0}^{n}F_{k,i-1}\\
&= \frac{1}{k}\left[\frac{F_{k,n+2}+2F_{k,n+1}+F_{k,n}-k(n+1)-2}{k}+F_{k,n}+F_{k,n-1}+k-1\right]
\end{aligned}$$

Other sums are proven through the same method. □

Find the value of the Theorem 4.2.a for $k=1$ and $n=5$.

$$Le_{1,0}+Le_{1,1}+Le_{1,2}+Le_{1,3}+Le_{1,4}+Le_{1,5}=1+1+3+5+9+15=34,$$

$$\begin{aligned}
\sum_{i=0}^{5}Le_{1,i} &= \frac{1}{1}\left[\frac{F_{1,7}+2F_{1,6}+F_{1,5}-(5+1)-2}{1}+F_{1,5}+F_{1,4}+1-1\right]\\
&= 13+16+5-8+5+3=34.
\end{aligned}$$

5 Conclusion remarks

This study presents the k-Leonardo number sequence. We obtain this new sequence not defined in the literature before. We generate Binet's formula, generating function formula and matrix representation. Also, these identities have beautiful application to graph theory. Since this study includes some new results, it contributes to literature by providing essential information concerning the number sequences. For further studies, we intend to find some properties, particularly combinational properties, for these new numbers. Therefore, we hope that this new number system and properties that we have found will offer a new perspective to the researchers.

References

[1] W. M. Abd-Elhameed, N. A. Zeyada, *New formulas including convolution, connection and radicals formulas of k-Fibpnacci and k-Lucas polynomials*, Indian Journal of Pure and Applied Mathematics, doi.org/10.1007/s13226-021-00214-5, (2022).

[2] S. L. Adler, *Quaternionic quantum mechanics and quantum fields*, New York Oxford Univ. Press., (1994).

[3] Y. Alp, E. G. Koçer, *Hybrid Leonardo numbers*, Chaos, Solitons, Fractals, doi.org/10.1016/j.chaos.2021.111128, (2021).

[4] F. Alves, R. Vieira, *The Newton Fractal's Leonardo Sequence Study with the Google Colab*, International Electronic Journal of Mathematics Education, **15(2)**, 1–9, (2020).

[5] J. Baez, *The Octonians*, Bull. Amer. Math. Soc., **145(39)**, 2, 145–205, (2001).

[6] B.D. Bitim, N. Topal, *Binomial sum formulas from the exponential generating functions of* (p, q)*-Fibonacci and* (p, q)*-Lucas quaternions*, Palestine Journal of Mathematics, **10(1)**, 279–289, (2021).

[7] J. Bravo, C. A. Gomez, J. L. Herrera, *On The Intersection of k-Fibonacci and Pell Numbers*, Bulletin of the Korean Mathematical Society, **56(2)**, 535–547, (2019).

[8] J. J. Bravo, C. A. Gomez, F. Luca, *Powers Of The As Sums Of Two* k*-Fibonacci Numbers*, Miskolc Mathematical Notes, **17(1)**, 85–100, (2016).

[9] C. Bolat, H. Köse, *On the Properties of* k*-Fibonacci Numbers*, Int. J. Contemp. Math. Sciences, **5(22)**, 1097–1105, (2010).

[10] P. Catarino, *On Some Identities for* k*-Fibonacci Sequence*, Int. J. Contemo. Math. Sciences, **9(1)**, 37–42, (2014).

[11] P. Catarino, A. Borges, *On Leonardo Numbers*, Acta Math. Univ. Comenianae, **1**, 75–86, (2020).

[12] P. Dhanya, K.M. Nagaraja and P.S.K. Reddy, *A note on d'Ocagne's identity on generalized Fibonacci and Lucas numbers*, Palestine Journal of Mathematics, **10(2)**, 751–755, (2021).

[13] M. Edson, O. Yayenie, *A new generalization of Fibonacci sequences and the extended Binet's formula*, Inyegers Electron. J. Comb. Number Theor., **9**, 639–654, (2009).

[14] S. Falcon, A. Plaza, *On the Fibonacci* k*-numbers*, Chaos, Solitions, Fractals, **32(5)**, 1615–1624, (2007).

[15] S. Falcon, A. Plaza, *On* k*-Fibonacci numbers of arithmetic indexes*, Applied Mathematics and Computation, **208**, 180–185, (2009).

[16] S. Falcon, A. Plaza, k*-Fibonacci sequences modulo* m, Chaos, Solitions, Fractals, **41**, 497–504, (2009).

[17] A. H. George, *Some formula for the Fibonacci sequence with generalization*, Fibonacci Quart., **7**, 113–130, (1969).

[18] A. D. Godase, M. B. Dhakne, *On the properties of* k*-Fibonacci and* k*-Lucas numbers*, International Journal of Advances in Applied Mathematics and Mechanics, **2(1)**, 100–106, (2014).

[19] A. Gueye, S. E. Rihane, A. Togbe, *Coincidence Between* k*-Fibonacci Numbers and Product of Two Fermat Numbers*, Bulletin of the Brazilian Mathematical Society, New Series, doi.org/10.1007/s00574-021-00269-2, (2022).

[20] W. R. Hamilton, *Li on Quaternions; or on a New System of Imaginaries in Algebra*, Philos. Mag. Ser. Taylor, Francis, **25(163)**, 10, (1844).

[21] C. J. Harman, *Complex Fibonacci Numbers*, The Fibonacci Quart., **19(1)**, 82–86, (1981).

[22] A. F. Horadam, *A generalized Fibonacci sequence*, Math. Mag., **68**, 455–459, (1961).

[23] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience Publication, New York, (2001).

[24] G. Y. Lee, J. S. Kim, *The linear algebra of the* k*-Fibonacci matrix*, Linear Algebra and its Applications, **373**, 75–87, (2003).

[25] G. Lee, k*-Fibonacci numbers and* k*-Lucas numbers and associated bipartite graphs*, Turkish Journal of Mathematics, **46(3)**, 884–893, (2022).

[26] M. Mangueria, F. Alves, P. Catarino, *Os Biquternions Elipticos de Leonardo*, Revista Eletronica Paulista de Matematica, **21**, 130–139, (2021).

[27] A. Pakapongpun, J. Kongson, *Three combined sequences related to* k*-Fibonacci sequences*, International Journal of Mathematics and Computer Science, **14(2)**, 551–559, (2022).

[28] L. Ramirez, *Some combinatorial properties of the* k*-Fibonacci and the* k*-Lucas quaternions*, An. S t. Univ. Ovidius Constanta, Ser. Mat., **23(2)**, 201–212, (2015).

[29] Y. Soykan, *On* k*-circulant matrices with the generalized third-order Pell numbers*, Notes on Number Theory and Discrete Mathematics, **27(4)**, 187–206, (2021).

[30] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Ellis Horwood Limited Publ., England, (1989).

[31] R. Vieira, M. Mangueria, F. Alves, P. Catarino, *A forma matricial dos numeros de Leonardo*, Universidade Federal de Santa Maria, **42**, 1–12, (2020).

[32] T. Yaying, B. Hazarika, et al., *On New Banach Sequence Spaces Involving Leonardo Numbers and the Associated Mapping Ideal*, Hindawi Journal of Function Spaces, doi.org/10.1155/2022/8269000, (2022).

Author information

Hasan Gökbaş, Department of Mathematics, Faculty of Science and Arts, University of Bitlis Eren, 13000, Bitlis, Turkey.
E-mail: hgokbas@beu.edu.tr

Received: 2023-08-10
Accepted: 2023-11-28