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ON THE PROPERTIES OF THE MONAD GRAPHS OF THE GROUP C_n GENERATED BY A CUBIC FUNCTION

H. B. Shelash, H. R. Hashim and A. A. Shukur

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Corresponding Author: H.R. Hashim

Abstract If the vertex of graph Γ belongs to finite group G and linked with its image under the action of linear map f, then this kind of graph was called Monad graph which had been introduced by V. Arnold in 2003. In this paper, we compute the Monad graphs where G is isomorphic to cyclic group C_n of order n and f is basic cubic function, i.e. $f(g) = g^3$. Moreover, some algebraic and dynamical properties of the studied case of Monad graphs were obtained. In fact, these results were indeed obtained and proved using some results and techniques related to the fields of number theory, graph theory and algebra.

1 Introduction

Concerning the dynamical systems defined on group actions, in 2003 V.I. Arnold in [1] introduced a very motivating phenomena termed by *monad.* A discrete dynamical system contains (G, Γ, f_g), where G is an arbitrary finite set, f is mapping each element from G into itself and Γ is Monad graph which scheming of every vertices of elements of the correspond G with its image by directed connected edge under the action of map f.

In [1], V. I. Arnold considered a special case of a discrete dynamical system (G, Γ, f_g), where f_g is the squaring map, i.e. $f(g) = g^2$ and showed that each connected component of a Monad graph is a forest of rooted trees directed towards their roots, which lie on a directed cycle (topologically, a circle) formed by the edges connecting the roots. Besides, it's well-known that *the finite cycle orbit* of an element is the action of element g under the map f_g, i.e.

$$orb_c(g, f_g) = \{g, f(g), f^2(g), \cdots, f^{n-1}(g) : \ n \in \mathbb{N}\}.$$

For $g \in G$, we remind $f^n(g) = g$ for $n \in \mathbb{N}$ is called *periodic point*. We denote by $Pr(g)$ the set of periodic points. For more information, we refer to [2], [3],[4] and the references given there.

Our aim in this paper is to study the Monad graph of discrete dynamical system (G, Γ, f_g), where G is isomorphic to cyclic group C_n of order n and f_g is the cubic map $f(g) = g^3$. Then, we show the dynamical properties of each element in the group C_n. We indeed prove our results using elementary results regarding the fields of number theory, graph theory and algebra. In fact, many papers connecting graph theory with number theory and algebra such as [5] and [6]. Furthermore, for some related results see, e.g. [7], [8] and [9].

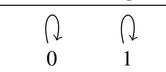
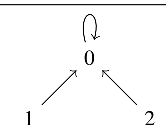
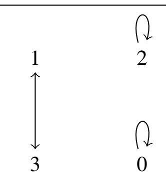
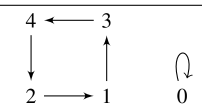
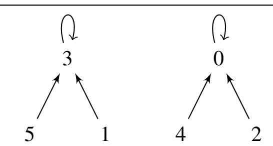
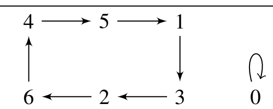
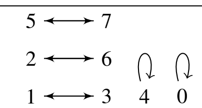
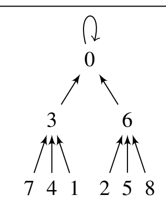
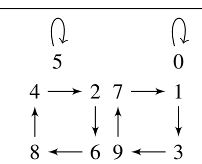
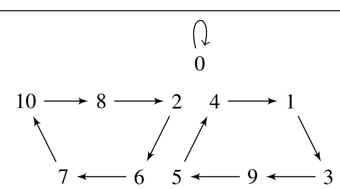
2 Auxiliary Results

For later use, we recall the following results:

Definition 2.1. Let G be a group, $f : G \longrightarrow G$ is a map, $f(g) = g^k$ for $g \in G$ with $k \geq 0$, the Monad graph $\Gamma_k(G)$ is defined by:

1. $V(G) = \{g|g \in G\}$, which represents the set of the vertices;
2. $E(G) = \{a \to a^k|f(a) = a^k \text{ or } f_k(a) \to a^k\}$, which represents the set of edges.

Example 2.2. In the following table, we show the Monad graphs generated by the map $f(g) = g^3$ for the simplest abelian groups of residue classes modulo n with $2 \leq n \leq 11$ under the additive operation:

Group	*Monad Graph*	*Graph's symbol*
Z_2		$O_1 + O_1$
Z_3		T_3
Z_4		$O_2 + 2O_1$
Z_5		$O_4 + O_1$
Z_6		$2T_3$
Z_7		$O_6 + O_1$
Z_8		$3O_2 + 2O_1$
Z_9		T_3^2
Z_{10}		$2O_4 + 2O_1$
Z_{11}		$2O_5 + O_1$

where O_n denotes the directed cycle of length n and T_{3^m} is the rooted tree with 3^m vertices and m branches.

One of the useful results regarding the directed cycle O_n is given by the following lemma due to [1].

Lemma 2.3. *The product of cyclic Monads graph O_n and O_m is a sum of identical cyclic Monads; namely*

$$O_n * O_m = dO_c,$$

where $d = gcd(n, m)$ and $c = \frac{nm}{d}$.

Definition 2.4. Let $n = \prod_{1 \leq i \leq s} p_i^{\alpha_i}$ be a positive integer number, where p denotes a prime number and the integer $\alpha_i \geq 1$, so $\Delta(n)$ represents the set of all divisors of n. Moreover, the Euler Phi Function is defined by

$$\varphi(n) = \prod_{1 \leq i \leq s} \varphi(p_i^{\alpha_i}) = \prod_{1 \leq i \leq s} (p_i^{\alpha_i}(1 - \frac{1}{p_i})) = n \prod_{1 \leq i \leq s} (1 - \frac{1}{p_i}).$$

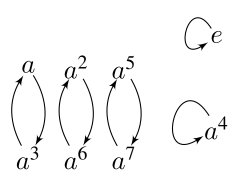
In general, the Euler Phi Function counts the positive integers less than a given integer n that are relatively prime to n. In fact, the Euler Phi Function has many applications and connections to many concepts in number theory such as Diophantine equations, and for more details about these connections and some of these equations one can see e.g. [10] and [11].

3 Main Results

Theorem 3.1. *Let C_n be a cyclic group of order n such that $n = 2^r$ and $r \geq 0$. The following are held:*

1. *If $r = 0$, then $\Gamma_3(C_n) = O_1$;*
2. *If $r = 1$, then $\Gamma_3(C_n) = 2O_1$;*
3. *If $r = 2$, then $\Gamma_3(C_n) = O_2 + 2O_1$;*
4. *If $r = 3$, then $\Gamma_3(C_n) = 3O_2 + 2O_1$;*
5. *If $r \geq 4$, then $\Gamma_3(C_n) = 2\sum_{t=2}^{r-2} O_{2^{r-t}} + 3O_2 + 2O_1$.*

Proof. The proofs of $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are trivial. Let's now start proving $\mathbf{4}$ with which we have $C_8 = \{e, a, a^2, \ldots, a^7\}$ with e represents the identity element of the group C_8 is generated by a. It is clear that the element a^4 is of order 2, and under the Monad mapping $f(g) = g^3$ it goes to itself, namely $f_3(a^4) \to a^4$. Similar idea goes to the identity element. The mappings of the other elements are summarized as follows: $f_3(a) \to a^3, f_3(a^3) \to a$, $f_3(a^2) \to a^6, f_3(a^6) \to a^2$ and $f_3(a^5) \to a^7, f_3(a^7) \to a^5$. The above mappings are presented in the following graph:



Finally, we prove case **(5)**. Indeed, there exist exactly two elements loop to themselves, namely e and $a^{2^{r-1}}$ since $f_3(e) = e$ and

$$f_3(a^{2^{r-1}}) = a^{2^{r-1}} a^{2^{r-1}} a^{2^{r-1}} = a^{32^{r-1}} = a^{(2+1)2^{r-1}} = a^{2^r} a^{2^{r-1}} = a^{2^{r-1}}.$$

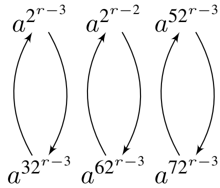
Also, for the elements $a^{2^{r-3}}$ and $a^{3 \cdot 2^{r-3}}$ we see the following:

$$f_3(a^{2^{r-3}}) = a^{32^{r-3}},$$

which means there exists an edge from $v_{a^{2^{r-3}}}$ into $v_{a^{3\cdot 2^{r-3}}}$. For

$$f_3(a^{32^{r-3}}) = a^{92^{r-3}} = a^{(2^3+1)2^{r-3}} = a^{(2^3 2^{r-3}+2^{r-3})} = a^{(2^r+2^{r-3})} = a^{2^{r-3}},$$

that also means there exists an edge from $v_{a^{32^{r-3}}}$ into $v_{a^{2^{r-3}}}$. Similar idea goes to the elements $a^{2^{r-2}}, a^{6\cdot 2^{r-3}}, a^{5\cdot 2^{r-3}}$ and $a^{7\cdot 2^{r-3}}$, and they are presented in the following figure:



In the following, we mention the action of the mapping for some elements and the idea will be the same for the others:

$$f_3(a^{2^{r-4}}) = a^{32^{r-4}},$$
$$f_3(a^{2^{r-1}+2^{r-4}}) = a^{32^{r-1}+32^{r-4}},$$
$$f_3(a^{32^{r-1}+32^{r-4}}) = a^{92^{r-1}+92^{r-4}} = a^{2^{r+2}+2^{r-2}+2^{r-4}} = a^{2^{r-4}},$$
$$f_3(a^{32^{r-4}}) = a^{92^{r-4}} = a^{(2^3+1)2^{r-4}} = a^{2^3 2^{r-4}+2^{r-4}} = a^{2^{r-1}+2^{r-4}}.$$

Now, it remains to consider the remaining 2^{r-1} elements. The number 2^{r-1} is clearly an odd number. Here, this leads to two cyclic graphs of order $O_{2^{r-2}}$ since these elements are clearly splitted up to two cyclic graphs of order $O_{2^{r-2}}$. To finish the proof of this case, it is enough to show that the order of the group is equal to the order of the Monad graph. This can be done as follows:

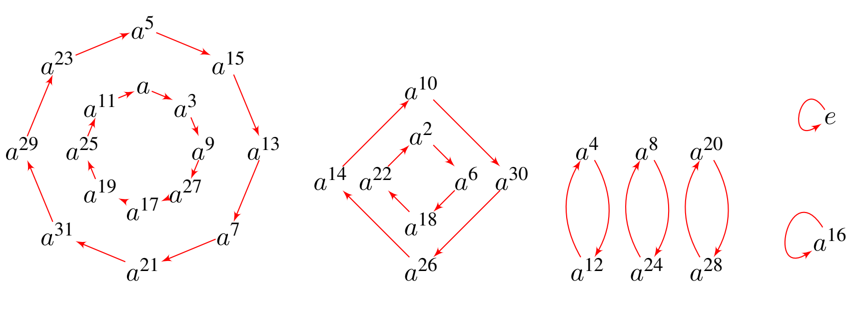
$$2\sum_{t=2}^{r-2} |V(O_{2^{r-t}})| + 3|V(O_2)| + 2|V(O_1)| = 2\sum_{t=2}^{r-2} 2^{r-t} + 6 + 2.$$

Let's now consider the right hand side of the latter equation, we obtain that

$$\begin{aligned} 2\sum_{t=2}^{r-2} 2^{r-t} + 6 + 2 &= 2(2^{r-2} + 2^{r-3} + \cdots + 2^3 + 2^2) + 2^3 \\ &= 2^3(2^{r-4} + 2^{r-5} + \cdots + 2 + 1) + 2^3 \\ &= 2^3(2^{r-3} - 1) + 2^3 (\text{with the sum of geometric series formula}) \\ &= 2^r \\ &= |C_n| = |V(C_n)|, \end{aligned}$$

and this completes the proof of the fifth case. □

Example 3.2. Let's take $n = 2^5$, then the Monad graph $\Gamma_3(C_{2^5})$ is isomorphic to $2O_8 + 2O_4 + 3O_2 + 2O_1$, which is obtained as follows: the set of all vertices is $V(\Gamma_3) = \{a^i : 1 \le i \le 32\}$ and the set of all edges is given by $E(\Gamma_3) = \{\{a^i, f_3(a^i)\} : 1 \le i \le 32\}$. Therefore, the corresponding Monad graph is represented by



$2O_8$ $2O_4$ $3O_2$ $2O_1$

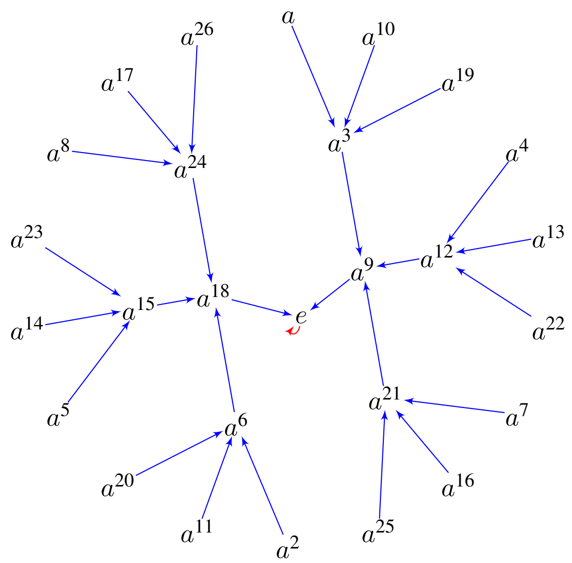
Theorem 3.3. *Let C_n be a cyclic group and $n = 3^\alpha$ is an integer number $\alpha \geq 0$, then*

$$\Gamma_3(C_n) = T_{3^\alpha}.$$

Proof. Let a^{3i} be vertex in $V(\Gamma_3)$ for some some positive integer i. If 3 divides i, there exist three vertices $a^i, a^{i+3^{\alpha-1}}$ and $a^{i+3^{\alpha-1}+3^{\alpha-1}}$ with $f_3(a^i) = a^{3i}$, $f_3(a^{i+3^{\alpha-1}}) = a^{3i}$ and $f_3(a^{i+3^{\alpha-1}+3^{\alpha-1}}) = a^{3i}$. This means that those three vertices $a^i, a^{i+3^{\alpha-1}}$ and $a^{i+3^{\alpha-1}+3^{\alpha-1}}$ have direct edges to vertex a^{3i} for each $3 \mid i$.

If 3 is not divisible i, then the vertices of element have order 3^α and this means there is no more elements which has image under the cubic map. □

Example 3.4. If we take $n = 3^5$, then the Monad graph $\Gamma_3(C_{3^5})$ is isomorphic to T_{81}, which can be summarized as follows: the set of vertices is $V(\Gamma_3) = \{a^i, 1 \leq i \leq 81\}$ and the set of edges is given by $E(\Gamma_3) = \{\{a^i, f_3(a^i)\} : 1 \leq i \leq 81\}$. These are presented in the following figure:



$T_{3^3} * O_1$

Theorem 3.5. *Let C_n be a cyclic group with $n = p^\alpha$, where p is an odd prime number with $p > 3$ and $\alpha \geq 1$, then*

$$\Gamma_3(C_n) = O_{\phi(p^\alpha)} + O_{\phi(p^\alpha - 1)} + \cdots + O_{\phi(p)} + O_1.$$

Proof. First, we see the set of all divisors of n is given by $\{1, p, \cdots, p^\alpha\}$. For each element g of C_n with order p^i where $0 \leq i \leq \alpha$, we can see $(g)^{3^{\varphi(p^{\alpha-i})}} = g$ under mod $n = p^\alpha$. This claim can be justified using the definition of Euler Phi function given in Definition 2.4 with which we can have

$$(g)^{3^{\varphi(p^{\alpha-i})}} = (g)^{3^{p^{-i}\varphi(p^\alpha)}} \equiv g \mod n.$$

This means that all elements have the exact order that gives a cyclic Monod graph of size equals to the order of the elements of C_n. □

Theorem 3.6. *Let C_n be a cyclic group with $n = p^\alpha$ is a positive integer such that p is an odd prime number. Then for all $i \in \Delta(n)$, we have*

$$gcd(2, \varphi(i)) = 2.$$

Proof. Since $n = p^\alpha$, so $\Delta(p^\alpha) = \{p^\alpha, p^{\alpha-1}, \cdots, p, 1\}$. It is clear that by the definition of Euler Phi function given in Definition 2.4, we have that $\varphi(1) = 1$ and $\varphi(p^k) = (p-1)(p^{k-1})$ for all $1 \leq k \leq \alpha$. Since p is an odd prime number, hence $(p-1)$ is divisible by 2. Thus, $gcd(2, \varphi(p^k)) = 2$, and that leads to the proof of the proposition. □

Theorem 3.7. *Let C_n be a cyclic group such that $n = p_1^{\alpha_1} p_2^{\alpha_2}$ is a positive integer and $p_1, p_2 > 3$ are prime numbers, then*

$$\Gamma_3(C_n) = \sum_{i \in \Delta(p_1^{\alpha_1}), j \in \Delta(p_2^{\alpha_2})} gcd(\varphi(i), \varphi(j)) O_{\frac{\varphi(i)\varphi(j)}{gcd(\varphi(i),\varphi(j))}}. \tag{3.1}$$

Proof. It is clear that from Theorem 3.5, the Monad graph of the cyclic group is given by

$$\Gamma_2(C_{p_1^{\alpha_1}}) = \sum_{t \in \Delta(p_1^{\alpha_1})} O_{\varphi(t)}$$

and

$$\Gamma_2(C_{p_2^{\alpha_2}}) = \sum_{k \in \Delta(p_2^{\alpha_2})} O_{\varphi(k)}.$$

Now, by using definition of direct product of two graphs, we obtained the following:

$$\begin{aligned}
\Gamma_2(C_{p_1^{\alpha_1}}) \times \Gamma_2(C_{p_2^{\alpha_2}}) &= \sum_{t \in \Delta(p_1^{\alpha_1})} O_{\varphi(t)} \times \sum_{k \in \Delta(p_2^{\alpha_2})} O_{\varphi(k)} \\
&= (O_{\varphi(p_1^{\alpha_1})} + \cdots + O_1) \times (O_{\varphi(p_2^{\alpha_2})} + \cdots + O_1) \\
&= (O_{\varphi(p_1^{\alpha_1})} + \cdots + O_1) O_{\varphi(p_2^{\alpha_2})} + \cdots + (O_{\varphi(p_1^{\alpha_1})} + \cdots + O_1) O_{p_2} + \\
&\quad (O_{\varphi(p_1^{\alpha_1})} + \cdots + O_1) O_1 \\
&= \sum_{t \in \Delta(p_1^{\alpha_1})} O_{\varphi(t)} O_{\varphi(p_2^{\alpha_2})} + \cdots + \sum_{t \in \Delta(p_1^{\alpha_1})} O_{\varphi(t)} O_{p_2} + \sum_{t \in \Delta(p_1^{\alpha_1})} O_{\varphi(t)} O_1 \\
&= \sum_{t \in \Delta(p_1^{\alpha_1})} gcd(\varphi(t), \varphi(p_2^{\alpha_2})) O_{\frac{\varphi(t)\varphi(p_2^{\alpha_2})}{gcd(\varphi(t),\varphi(p_2^{\alpha_2}))}} + \cdots \\
&\quad + \sum_{t \in \Delta(p_1^{\alpha_1})} gcd(\varphi(t), \varphi(p_2)) O_{\frac{\varphi(t)\varphi(p_2)}{gcd(\varphi(t),\varphi(p_2))}} + \sum_{t \in \Delta(p_1^{\alpha_1})} O_{\varphi(t)} \\
&= \sum_{t \in \Delta(p_1^{\alpha_1}), k \in \Delta(p_2^{\alpha_2}), k \neq 1} gcd(\varphi(t), \varphi(k)) O_{\frac{\varphi(t) O_{\varphi(k)}}{gcd(\varphi(t),\varphi(k))}} + \sum_{t \in \Delta(p_1^{\alpha_1})} O_{\varphi(t)}.
\end{aligned}$$

□

Example 3.8. Let's illustrate the result of Theorem 3.7 in case of $n = 5^2 7 = p_1^{\alpha_1} p_2^{\alpha_2}$. Hence, we have that

$$\Delta(5^2) = \{5^2, 5, 1\} \text{ and } \Delta(7) = \{7, 1\}.$$

In the following, we summarize the details of computations for computing $\varphi(i), \varphi(j), \varphi(i).\varphi(j)$, $gcd(\varphi(i), \varphi(j))$ and $\frac{\varphi(i)\varphi(j)}{gcd(\varphi(i),\varphi(j))}$ in the terms of the summation given in (3.1) for all $i \in \Delta(p_1^{\alpha_1}) = \Delta(5^2) = \{5^2, 5, 1\}$ and $j \in \Delta(p_2^{\alpha_2}) = \Delta(7) = \{7, 1\}$:

i	j	$\varphi(i)$	$\varphi(j)$	$\varphi(i).\varphi(j)$	$\text{gcd}(\varphi(i), \varphi(j))$	$\frac{\varphi(i)\varphi(j)}{\text{gcd}(\varphi(i),\varphi(j))}$
1	1	1	1	1	1	1
1	7	1	6	6	1	6
5	1	4	1	4	1	4
5	7	4	6	24	2	12
5^2	1	100	1	100	1	100
5^2	7	100	6	600	2	300

Now, one can easily determine the Monad graph of the given group under the action of the cubic function, which is given as follows:

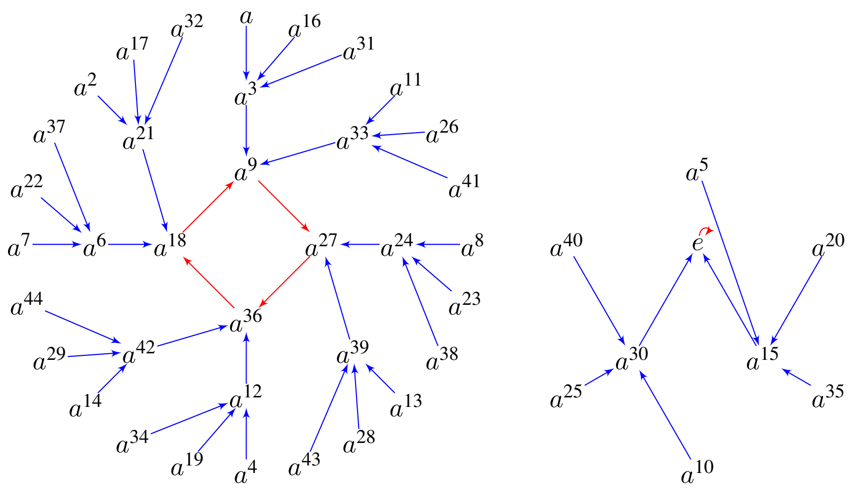
$$\Gamma_3(C_{5^2 7}) = 2O_{300} + O_{100} + 2O_{12} + O_6 + O_4 + O_1.$$

Corollary 3.9. *Suppose that C_n is a cyclic group, the Monod graph is given by the following:*

$$\Gamma_3(C_n) = \begin{cases} T_{3^q} * (2\sum_{r-2\leq t\leq 2} O_{2^{r-t}} + 3O_2 + 2O_1) & if\ n = 2^r 3^\alpha \\ T_{3^q} * (O_{\phi(p^\alpha)} + O_{\phi(p^\alpha - 1)} + \cdots + O_{\phi(p)} + O_1) & if\ n = 2^r p^\alpha, p \neq 2, 3 \end{cases}$$

Proof. The proof is followed directly from Theorems 3.5 and 3.7. □

Example 3.10. We may consider the Monad graph of $\Gamma_3(C_{3^25}) = T_{3^2} * O_4 + T_{3^2} * O_1$; that is



$T_{3^2} * O_4$ $T_{3^2} * O_1$

3.1 The number of orbits

Here, we compute the possible number of orbits of each element belongs to Monad graph of $\Gamma_3(C_n)$. To gain more information, we will define the following concept (see e.g. [12]):

Definition 3.11. Let $\Gamma_k(C_n)$ is a rooted tree attracted by cycle, we say that every element lies on tree branches has orbit of order r bring it to cycle part, i.e.

$$orb_r(g, f_g) = \{g, f(g), f^2(g) \dots f^r(g)\}.$$

Theorem 3.12. *Each element a^i of $\Gamma_3(C_n)$ has the following possible orbits:*

a) If $n = p_1^{\alpha_1} p_2^{\alpha_2}$, then we have only periodic orbit, i.e.

$$orb_c(a^i, f_{a^i}).$$

b) If $n = 2^r 3^q$, then we have

$$\begin{cases} orb_c(a^i, f_{a^i}), & if \quad 2^r | i; \\ \\ orb_r(a^i, f_{a^i}) \cup orb_c(a^i, f_{a^i}), & if \quad 2^r \nmid i. \end{cases}$$

Proof. If the order n of the group of $\Gamma_3(C_n)$ is $p_1^{\alpha_1} p_2^{\alpha_2}$, then from Theorem 3.7 it follows that the orbit of every element of C_n generated by the linear cube map f_g is finite cycle, i.e. $orb_c(g, f_g)$ with finite set of $Pr(g)$.

If the order n of the group of $\Gamma_3(C_n)$ is $2^r 3^q$, then from Corollary 3.9 follows the Monad graph is rooted tree attracted by centered cycle such that if 2^r is not the divisor of i, then a^i lies on cycle part of tree and obviously its orbit $orb_c(a^i, f_{a^i})$. -

If 2^r is the divisor of i, then a^i lies on tree with r branches such that the orbit is $orb_r(g, f_g)$ which bring the element a^i to cycle part, i.e. a^i has $orb_r(a^i, f_{a^i}) \cup orb_c(a^i, f_{a^i})$.

□

References

[1] V. I. Arnold, *Topology of algebra the combinatorics of the squaring operation*, Funct. Anal. Appl., 37 **2**, 1-24, (2003).

[2] V. I. Arnold, *Topology and statistics of arithmetic and algebraic formulae*, Russian Math. Surv., **58**, 3-28, (2003).

[3] R. Uribe-Vargas, *opology of dynamical systems in finite groups and number theory*, Bull. Sci. math., **130**, 377-402, (2006).

[4] R. Uribe-Vargas, *Arithmetics of the numbers of orbits of the Fermat-Euler dynamical systems*, Funct. Anal. Other Math., **1**, 71-83, (2006).

[5] H.R. Hashim, F. Luca, H.B. Shelash and A.A. Shukur, *Generalized Lucas graphs*, Afr. Mat. **34**, 10, (2023).

[6] M. D. Rao, C. Adiga and N. Anitha, *Some Properties of the Fibonacci-difference Graph and Bounds for its Energy*, Palest. J. Math. **12**, 141–153, (2023).

[7] A. Akhmedov, A. Shukur and H. Shelash, *Weighted graphs with an application to returned sequence*, Adv. Studies: Euro-Tbilisi Math. J. **16**, 7–12, (2023).

[8] L. Boro, M. M. Singh and J. Goswam, *On the line graphs associated to the unit graphs of rings*, Palest. J. Math. **11**, 139–145, (2022).

[9] G. G. Belsi, and S. Kavitha, *On the higher order genus of the nilpotent graph of a ring*, Palest. J. Math. **11**, 91–98, (2022).

[10] H. R. Hashim and Sz. Tengely, *Solutions of a generalized Markoff equation in Fibonacci numbers*, Math. Slovaca **34**, 1069–1078, (2020).

[11] K. L. Hua, *Introduction to number theory*, Springer Science & Business Media, (2012).

[12] A. Shukur, H. Shelash and A. Jahanbani, *The behavior of weighted graph's orbit and its energy*, Math. Probl. Eng. **2021**, 6, (2021).

Author information

H. B. Shelash, Department of Mathematics, Faculty of Computer Sciences and Mathematics, University of Kufa, Iraq.
E-mail: `hayder.ameen@uokufa.edu.iq`

H. R. Hashim, Department of Mathematics, Faculty of Computer Sciences and Mathematics, University of Kufa, Iraq.
E-mail: `hayderr.almuswi@uokufa.edu.iq`

A. A. Shukur, Department of Mathematics, Faculty of Computer Sciences and Mathematics, University of Kufa, Iraq.
E-mail: `shukur.math@gmail.com`

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