Palestine Journal of Mathematics

Vol 13(4)(2024) , 144–149

ON CONJECTURE OF KEOGH AND PROBLEM 4.21 BY HAYMAN

Lande Ma and Zhaokun Ma

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 30C10; Secondary 30C50,30C55.

Keywords and phrases: polynomials, constant, modulus, inequality.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract *The conjecture proposed by F.R. Keogh and collected in 'Research Problems in Function Theory' by Hayman(1967) as Problem 4.21 suggests the equality* $|f(z)|^2 = n + 2\sum_{k=1}^{n} b_k cos(k\theta)$*. We disprove this equality. We disprove this equality and present clear evidence refuting Keogh's conjecture.*

1 Introduction

Problem 4.21[1] by Hayman is a conjecture which was posed by F.R. Keogh[2, 3, 4]. The conjecture is as: If $a_k = \pm 1$*,* $k = 0, 1, \cdots, n$ *and* $b_k = a_n a_{n-k} + a_{n-1}a_{n-k-1} + \cdots + a_k a_0$*, is it true that* $\sum_{k=1}^{n} |b_k|^2 > An^2$*, where* A *is an absolute constant? If* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$*, then,* $|f(z)|^2 = n + 2\sum_{k=1}^{n} b_k cos(k\theta)$*. Therefore, the truth of the above inequality would imply* $\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^4 d\theta \geq n^2(1 + A)$*.*

According to the new version of 'Research Problems in Function Theory'[5], over fifty years, no results have been found regarding the conjecture posed by F.R. Keogh. The conjecture is considered an isolated problem in function theory with little relation to other main problems. Its difficulty lies in the complexity of its computation. In this paper, we disprove the conjecture using two usual polynomials and six selected values. We employ various techniques and computations to demonstrate that $|f(z)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$*. This inequality is crucial to proving the truth of* $\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^4 d\theta \geq n^2(1+A)$*, which becomes impossible if* $|f(z)|^2 = n+2\sum_{k=1}^{n} b_k cos(k\theta)$ *is disproved. There are some recent related study see[6, 7]. Our proof provides the first result on Keogh's conjecture after 56 years, establishing its invalidity.*

To establish the truth of a result, it is necessary to verify it in general cases. In this paper, we utilize two general polynomials to calculate three special values and find that the equality proposed by Keogh does not hold. Further tests with additional general values reveal that the equality remains untrue. Even when computing six values, employing two general polynomials fails to confirm Keogh's conjecture. Therefore, our proof using two polynomials adequately demonstrates the invalidity of Keogh's conjecture and highlights the importance of examining a problem in its most general form.

In this paper, we prove the following conclusion: the equality $|f(z)|^2 = n+2\sum_{k=1}^{n} b_k cos(k\theta)$ *is not true, and Keogh's conjecture is also not true.*

2 The results and proofs

Example 2.1. $f_1(z) = 1 + p_1(z)$. $a_k = 1$, $k = 0, 1, \cdots, n$. $p_1(z) = \sum_{k=1}^{n} z^k$.

For all a_j *and* a_l*,* $j = 0, 1, \cdots, n$*,* $l = 0, 1, \cdots, n$*,* $a_j a_l = 1$*.* $b_k = a_n a_{n-k} + a_{n-1}a_{n-k-1} + \cdots + a_k a_0 = 1 + 1 + \cdots + 1$*. So, we have:*

Lemma 2.2. $b_k = n + 1 - k$.

Lemma 2.3. *When $z = 1$, there is $|f_1(1)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$.*

Proof. When $z = 1$, $p_1(1) = \sum_{k=1}^{n} 1^k = \sum_{k=1}^{n} 1 = n$. $f_1(1) = 1 + P_1(1) = n + 1$. $|f_1(1)|^2 = (n+1)^2 = n^2 + 2n + 1$.

When $z = 1, \theta = 0$. $cos(k\theta) = cos(0) = 1$. $n+2\sum_{k=1}^{n} b_k cos(k\theta) = n+2\sum_{k=1}^{n}(n+1-k) = n + 2n(n+1) - 2\sum_{k=1}^{n} k = n + 2n(n+1) - n(n+1) = n^2 + 2n$. □

Lemma 2.4. *When $z = -1$, there is $|f_1(-1)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$.*

Proof. When $z = -1$, $n = 2m$. $p_1(-1) = \sum_{k=1}^{2m}(-1)^k = \sum_{l=0}^{m-1}(-1)^{2l+1} + \sum_{l=0}^{m-1}(-1)^{2l+2} = \sum_{l=0}^{m-1}(-1) + \sum_{l=0}^{m-1} 1 = 0$. $f_1(-1) = 1 + P_1(-1) = 0 + 1 = 1$. $z = -1, \theta = \pi$. $k = 2l + 1$, $l = 0, 1, \cdots, m-1$. $cos((2l+1)\pi) = cos(2l\pi + \pi) = -1$. $k = 2l + 2$. $cos((2l+2)\pi) = cos(2l\pi + 2\pi) = 1$.

$2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m-1}(-b_{2l+1}) + 2\sum_{l=0}^{m-1} b_{2l+2} = 2\sum_{l=0}^{m-1}(b_{2l+2} - b_{2l+1}) = 2\sum_{l=0}^{m-1}(n + 1 - 2l - 2 - n - 1 + 2l + 1) = 2\sum_{l=0}^{m-1}(-1) = -2m = -n$. $n + 2\sum_{k=1}^{n} b_k cos(k\theta) = 0$. $|f_1(-1)|^2 = 1$. $|f_1(-1)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$.

When $n = 2m+1$, $p_1(-1) = \sum_{k=1}^{2m+1}(-1)^k = \sum_{l=0}^{m}(-1)^{2l+1} + \sum_{l=0}^{m-1}(-1)^{2l+2} = \sum_{l=0}^{m}(-1) + \sum_{l=0}^{m-1} 1 = -1$. $f_1(-1) = 1 + P_1(-1) = 1 - 1 = 0$. $2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m}(-b_{2l+1}) + 2\sum_{l=0}^{m-1} b_{2l+2} = -2b_{2m+1} + 2\sum_{l=0}^{m-1}(b_{2l+2} - b_{2l+1}) = -2b_n + 2\sum_{l=0}^{m-1}(n + 1 - 2l - 2 - n - 1 + 2l + 1) = -2 + 2\sum_{l=0}^{m-1}(-1) = -2 - 2m = -n - 1$. $n + 2\sum_{k=1}^{n} b_k cos(k\theta) = -1$. $|f_1(-1)|^2 = 0$. $|f_1(-1)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$. □

Lemma 2.5. *When $z = i$, there is $|f_1(i)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$.*

Proof. When $z = i$, $n = 4m$, $p_1(i) = \sum_{k=1}^{4m} i^k = \sum_{l=0}^{m-1} i^{4l+1} + \sum_{l=0}^{m-1} i^{4l+2} + \sum_{l=0}^{m-1} i^{4l+3} + \sum_{l=0}^{m-1} i^{4l+4} = \sum_{l=0}^{m-1} i + \sum_{l=0}^{m-1}(-1) + \sum_{l=0}^{m-1}(-i) + \sum_{l=0}^{m-1} 1 = 0$. $f_1(i) = 1 + P_1(i) = 0 + 1 = 1$. $\theta = \frac{\pi}{2}$. $k = 4l + 1$, $l = 0, 1, \cdots, m-1$. $cos((4l+1)\frac{\pi}{2}) = cos(2l\pi + \frac{\pi}{2}) = 0$. $k = 4l + 2$, $cos((4l+2)\frac{\pi}{2}) = cos(2l\pi + \pi) = -1$. $k = 4l + 3$, $cos((4l+3)\frac{\pi}{2}) = cos(2l\pi + \frac{3\pi}{2}) = 0$. $k = 4l + 4$, $cos((4l+4)\frac{\pi}{2}) = cos(2l\pi + 2\pi) = 1$.

$2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m-1}(-b_{4l+2}) + 2\sum_{l=0}^{m-1} b_{4l+4} = 2\sum_{l=0}^{m-1}(b_{4l+4} - b_{4l+2}) = 2\sum_{l=0}^{m-1}(n + 1 - 4l - 4 - n - 1 + 4l + 2) = 2\sum_{l=0}^{m-1}(-2) = -4m = -n$. $n + 2\sum_{k=1}^{n} b_k cos(k\theta) = 0$. $|f_1(i)|^2 = 1$. $|f_1(i)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$.

When $n = 4m + 1$, $p_1(i) = \sum_{k=1}^{4m+1} i^k = \sum_{l=0}^{m} i^{4l+1} + \sum_{l=0}^{m-1} i^{4l+2} + \sum_{l=0}^{m-1} i^{4l+3} + \sum_{l=0}^{m-1} i^{4l+4} = \sum_{l=0}^{m} i + \sum_{l=0}^{m-1}(-1) + \sum_{l=0}^{m-1}(-i) + \sum_{l=0}^{m-1} 1 = i$. $f_1(i) = 1 + P_1(i) = i + 1$. $2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m-1}(-b_{4l+2}) + 2\sum_{l=0}^{m-1} b_{4l+4} = 2\sum_{l=0}^{m-1}(b_{4l+4} - b_{4l+2}) = 2\sum_{l=0}^{m-1}(n + 1 - 4l - 4 - n - 1 + 4l + 2) = 2\sum_{l=0}^{m-1}(-2) = -4m - 1 + 1 = -n + 1$. $n + 2\sum_{k=1}^{n} b_k cos(k\theta) = n - n + 1 = 1$. $|f_1(i)|^2 = 2$. $|f_1(i)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$.

When $n = 4m+2$, $p_1(i) = \sum_{k=1}^{4m+2} i^k = \sum_{l=0}^{m} i^{4l+1} + \sum_{l=0}^{m} i^{4l+2} + \sum_{l=0}^{m-1} i^{4l+3} + \sum_{l=0}^{m-1} i^{4l+4} = \sum_{l=0}^{m} i + \sum_{l=0}^{m}(-1) + \sum_{l=0}^{m-1}(-i) + \sum_{l=0}^{m-1} 1 = i - 1$. $f_1(i) = 1 + P_1(i) = i$. $2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m}(-b_{4l+2}) + 2\sum_{l=0}^{m-1} b_{4l+4} = -2b_{4m+2} + 2\sum_{l=0}^{m-1}(b_{4l+4} - b_{4l+2})$, $n = 4m+2$, $b_{4m+2} = b_n = 1$, $2\sum_{k=1}^{n} b_k cos(k\theta) = -2 + 2\sum_{l=0}^{m-1}(n + 1 - 4l - 4 - n - 1 + 4l + 2) = -2 + 2\sum_{l=0}^{m-1}(-2) = -2 - 4m = -n$. $n + 2\sum_{k=1}^{n} b_k cos(k\theta) = n - n = 0$. $|f_1(i)|^2 = 1$. $|f_1(i)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$.

When $n = 4m+3$, $p_1(i) = \sum_{k=1}^{4m+3} i^k = \sum_{l=0}^{m} i^{4l+1} + \sum_{l=0}^{m} i^{4l+2} + \sum_{l=0}^{m} i^{4l+3} + \sum_{l=0}^{m-1} i^{4l+4} = \sum_{l=0}^{m} i + \sum_{l=0}^{m}(-1) + \sum_{l=0}^{m}(-i) + \sum_{l=0}^{m-1} 1 = i - 1 - i = -1$. $f_1(i) = 1 + P_1(i) = 0$. $2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m}(-b_{4l+2}) + 2\sum_{l=0}^{m-1} b_{4l+4} = -2b_{4m+2} + 2\sum_{l=0}^{m-1}(b_{4l+4} - b_{4l+2})$, $n = 4m + 3$, $b_{4m+2} = b_{n-1} = 2$, $2\sum_{k=1}^{n} b_k cos(k\theta) = -4 + 2\sum_{l=0}^{m-1}(n + 1 - 4l - 4 - n - 1 + 4l + 2) = -4 + 2\sum_{l=0}^{m-1}(-2) = -4 - 4m = -n - 1$. $n + 2\sum_{k=1}^{n} b_k cos(k\theta) = n - n - 1 = -1$. $|f_1(i)|^2 = 0$. $|f_1(i)|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$. □

Lemma 2.6. *When $n = 8m_1$, $n = 4m_2$, $m_2 = 2m_1$, $f_{12}(z) = \sum_{k=1}^{n} z^k = (1+z)(1+z^2)(1+z^4)\sum_{l=0}^{m_1-1} z^{8l+1}$.*

Proof. When $n=8m_1, n=4m_2, m_2=2m_1, f_{12}(z)=\sum_{k=1}^{n} z^k=\sum_{l=0}^{m_2-1} z^{4l+1}+\sum_{l=0}^{m_2-1} z^{4l+2}+\sum_{l=0}^{m_2-1} z^{4l+3}+\sum_{l=0}^{m_2-1} z^{4l+4}=\sum_{l=0}^{m_2-1} z^{4l+1}+z\sum_{l=0}^{m_2-1} z^{4l+1}+z^2\sum_{l=0}^{m_2-1} z^{4l+1}+z^3\sum_{l=0}^{m_2-1} z^{4l+1}=(1+z)(1+z^2)\sum_{l=0}^{m_2-1} z^{4l+1}$.

$\sum_{l=0}^{m_2-1} z^{4l+1}=\sum_{k=1}^{m_2} z^{4k-3}=\sum_{l=0}^{m_1-1} z^{8l+1}+\sum_{l=0}^{m_1-1} z^{8l+5}=\sum_{l=0}^{m_1-1} z^{8l+1}+z^4\sum_{l=0}^{m_1-1} z^{8l+1}=(1+z^4)\sum_{l=0}^{m_1-1} z^{8l+1}$. □

Lemma 2.7. *When* $n=8m_1$, $n=4m_2$, $m_2=2m_1$, *for* $z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ *and* $f_1(z)$, $2\sum_{k=1}^{n} b_k cos(k\theta)=-8m_1$.

Proof. When $n=8m_1$, $n=4m_2$, $m_2=2m_1$, $z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$, $\theta=\frac{\pi}{4}$. $k=8l+1$, $l=0,1,\cdots,m_1-1$. $cos((8l+1)\frac{\pi}{4})=cos(2l\pi+\frac{\pi}{4})=\frac{\sqrt{2}}{2}$. $k=8l+2$, $cos((8l+2)\frac{\pi}{4})=cos(2l\pi+\frac{\pi}{2})=0$. $k=8l+3$, $cos((8l+3)\frac{\pi}{4})=cos(2l\pi+\frac{3\pi}{4})=-\frac{\sqrt{2}}{2}$. $k=8l+4$, $cos((8l+4)\frac{\pi}{4})=cos(2l\pi+\pi)=-1$. $k=8l+5$, $cos((8l+5)\frac{\pi}{4})=cos(2l\pi+\frac{5\pi}{4})=-\frac{\sqrt{2}}{2}$. $k=8l+6$, $cos((8l+6)\frac{\pi}{4})=cos(2l\pi+\frac{3\pi}{2})=0$. $k=8l+7$, $cos((8l+7)\frac{\pi}{4})=cos(2l\pi+\frac{7\pi}{4})=\frac{\sqrt{2}}{2}$. $k=8l+8$, $cos((8l+8)\frac{\pi}{4})=cos(2l\pi+2\pi)=1$.

$2\sum_{k=1}^{n} b_k cos(k\theta)=2\sum_{l=0}^{m_1-1} b_{8l+1}cos((8l+1)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+2}cos((8l+2)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+3}cos((8l+3)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+4}cos((8l+4)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+5}cos((8l+5)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+6}cos((8l+6)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+7}cos((8l+7)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+8}cos((8l+8)\frac{\pi}{4})=\sqrt{2}\sum_{l=0}^{m_1-1} b_{8l+1}-\sqrt{2}\sum_{l=0}^{m_1-1} b_{8l+3}-2\sum_{l=0}^{m_1-1} b_{8l+4}-\sqrt{2}\sum_{l=0}^{m_1-1} b_{8l+5}+\sqrt{2}\sum_{l=0}^{m_1-1} b_{8l+7}+2\sum_{l=0}^{m_1-1} b_{8l+8}=\sqrt{2}\sum_{l=0}^{m_1-1}(b_{8l+1}-b_{8l+3}-b_{8l+5}+b_{8l+7})+2\sum_{l=0}^{m_1-1}(b_{8l+8}-b_{8l+4})=\sqrt{2}\sum_{l=0}^{m_1-1}(n+1-8l-1-n-1+8l+3-n-1+8l+5+n+1-8l-7)+2\sum_{l=0}^{m_1-1}(n+1-8l-8-n-1+8l+4)=-8m_1$. □

Lemma 2.8. *When* $n=8m_1$, *there is* $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2\neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$.

Proof. According to Lemma 1.6, when $z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$, $f_{12}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=0$. So, $f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=1+f_{12}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=1+0=1$.

According to Lemma 1.7, for $z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$. $2\sum_{k=1}^{n} b_k cos(k\theta)=-8m_1=-n$. $n+2\sum_{k=1}^{n} b_k cos(k\theta)=n-n=0$. $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2=1$. $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2\neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$. □

Lemma 2.9. *When* $n=8m_1+1$, *there is* $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2\neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$.

Proof. According to Lemma 1.6, when $z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$, $f_{12}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=0$. So, $f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=1+f_{12}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1}=1+0+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1}=1+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1}$.

According to Lemma 1.7, for $z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$. $2\sum_{k=1}^{n} b_k cos(k\theta)=\sqrt{2}b_{8m_1+1}-8m_1$, $n=8m_1+1$, $b_{8m_1+1}=b_n=1$, $2\sum_{k=1}^{n} b_k cos(k\theta)=\sqrt{2}+1-n$. $n+2\sum_{k=1}^{n} b_k cos(k\theta)=n-n+\sqrt{2}+1=\sqrt{2}+1$. $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2$ is a very complex expression, but, it is not equal to $\sqrt{2}+1$. So, $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2\neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$. □

Lemma 2.10. *When* $n=8m_1+2$, *there is* $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2\neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$.

Proof. According to Lemma 1.6, when $z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$, $f_{12}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=0$. So, $f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=1+f_{12}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1}+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+2}=1+0+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1}+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+2}=1+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1}+(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+2}$.

According to Lemma 1.7, for $z=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$. $2\sum_{k=1}^{n} b_k cos(k\theta)=\sqrt{2}b_{8m_1+1}-8m_1$, $n=8m_1+2$, $b_{8m_1+1}=b_{n-1}=2$, $2\sum_{k=1}^{n} b_k cos(k\theta)=2\sqrt{2}+2-n$. $n+2\sum_{k=1}^{n} b_k cos(k\theta)=n-n+2\sqrt{2}+2=2\sqrt{2}+2$. $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2$ is a very complex expression, but, it is not equal to $2\sqrt{2}+2$. So, $|f_1(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2\neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$. □

Example 2.11. $f_2(z)=1+p_2(z)$. $a_k=(-1)^k$, $k=0,1,\cdots,n$. $p_2(z)=\sum_{k=1}^{n}(-1)^k z^k$.

Lemma 2.12. $b_k = (-1)^k(n+1-k)$.

Proof. In the expression of b_k which is constituted by a_j, if $(-1)^{j+l}a_j a_l$ is the former term, then its back adjacent term is $(-1)^{j+l-2}a_j a_l$. So, in the expression of b_k, the positive symbol $+$ or the negative symbol $-$ of all terms are all same. So, as same as the coefficients of b_k in $f_1(z)$, b_k in $f_2(z)$ is the sum of $n+1-k$ terms. But, b_k in $f_2(z)$ has the change of positive or negative symbol. Because $(-1)^{2n-k} = (-1)^k$. $b_k = a_n a_{n-k} + a_{n-1}a_{n-k-1} + \cdots + a_k a_0 = (-1)^k(1+1+\cdots+1) = (-1)^k(n+1-k)$. □

Lemma 2.13. *When $z=-1$, there is $|f_2(-1)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$.*

Proof. When $z=-1$, $p_2(-1) = \sum_{k=1}^n (-)^k(-1)^k = \sum_{k=1}^n 1 = n$. $f_2(-1) = 1 + P_2(-1) = n+1$. $|f_2(-1)|^2 = (n+1)^2 = n^2+2n+1$.

When $z=-1$, $\theta=\pi$. $k = 2l+1$, $l = 0,1,\cdots,m-1$. $cos((2l+1)\pi) = cos(2l\pi+\pi) = -1$. $k=2l+2$. $cos((2l+2)\pi) = cos(2l\pi+2\pi) = 1$.

When $n=2m$, $2\sum_{k=1}^n b_k cos(k\theta) = 2\sum_{l=0}^{m-1}(-b_{2l+1}) + 2\sum_{l=0}^{m-1} b_{2l+2} = 2\sum_{l=0}^{m-1}(b_{2l+2} - b_{2l+1}) = 2\sum_{l=0}^{m-1}(n+1-2l-2+n+1-2l-1) = 2\sum_{l=0}^{m-1} 2(n+1) - 2\sum_{l=0}^{m-1}(4l) - 2\sum_{l=0}^{m-1} 3 = 2n(n+1) - 4m(m-1) - 6m = 2n^2+2n-4m^2+4m-3n = 8m^2-4m^2+n = n^2+n$. $n + 2\sum_{k=1}^n b_k cos(k\theta) = n^2+2n$. $|f_2(-1)|^2 = n^2+2n+1$. $|f_2(-1)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$.

When $n = 2m+1$, $2\sum_{k=1}^n b_k cos(k\theta) = 2\sum_{l=0}^{m}(-b_{2l+1}) + 2\sum_{l=0}^{m-1} b_{2l+2} = -2b_{2m+1} + 2\sum_{l=0}^{m-1}(b_{2l+2} - b_{2l+1})$, $b_{2m+1} = b_n = -1$, $2\sum_{k=1}^n b_k cos(k\theta) = -2b_n + 2\sum_{l=0}^{m-1}(b_{2l+2}-b_{2l+1}) = 2 + 2\sum_{l=0}^{m-1}(n+1-2l-2+n+1-2l-1) = 2 + 2\sum_{l=0}^{m-1} 2(n+1) - 2\sum_{l=0}^{m-1}(4l) - 2\sum_{l=0}^{m-1} 3 = 2n(n+1) - 4m(m-1) - 6m = 2 + 8m^2 + 4m - 4m^2 + 4m - 6m = 4m^2+2m+2 = (2m+1)^2 - 2m + 1 = n^2 - n + 2$. $n + 2\sum_{k=1}^n b_k cos(k\theta) = n^2+2$. $|f_2(-1)|^2 = n^2+2n+1$. $|f_2(-1)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$. □

Lemma 2.14. *When $z=1$, there is $|f_2(1)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$.*

Proof. When $z=1$, $n=2m$. $p_2(1) = \sum_{k=1}^{2m}(-1)^k = \sum_{l=0}^{m-1}(-1)^{2l+1} + \sum_{l=0}^{m-1}(-1)^{2l+2} = \sum_{l=0}^{m-1}(-1) + \sum_{l=0}^{m-1} 1 = 0$. $f_2(1) = 1 + P_2(1) = 0+1 = 1$. $\theta = 0$. $cos(k\theta) = cos(0) = 1$. $2\sum_{k=1}^n b_k cos(k\theta) = 2\sum_{l=0}^{m-1}(b_{2l+1}) + 2\sum_{l=0}^{m-1} b_{2l+2} = 2\sum_{l=0}^{m-1}(n+1-2l-2-n-1+2l+1) = 2\sum_{l=0}^{m-1}(-1) = -2m = -n$. $n + 2\sum_{k=1}^n b_k cos(k\theta) = 0$. $|f_2(1)|^2 = 1$. $|f_2(1)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$.

When $n = 2m+1$, $p_2(1) = \sum_{k=1}^{2m+1}(-1)^k = \sum_{l=0}^{m}(-1)^{2l+1} + \sum_{l=0}^{m-1}(-1)^{2l+2} = \sum_{l=0}^{m}(-1) + \sum_{l=0}^{m-1} 1 = -1$. $f_2(1) = 1 + P_2(1) = 1 - 1 = 0$. $2\sum_{k=1}^n b_k cos(k\theta) = 2\sum_{k=1}^n b_k = 2\sum_{l=0}^{m}(b_{2l+1}) + 2\sum_{l=0}^{m-1} b_{2l+2} = 2b_{2m+1} + 2\sum_{l=0}^{m-1}(b_{2l+2}+b_{2l+1})$, $b_{2m+1} = b_n = -1$, $2\sum_{k=1}^n b_k cos(k\theta) = 2b_n + 2\sum_{l=0}^{m-1}(b_{2l+2}+b_{2l+1}) = -2 + 2\sum_{l=0}^{m-1}(n+1-2l-2-n-1+2l+1) = -2 + 2\sum_{l=0}^{m-1}(-1) = -2-2m = -n-1$. $n + 2\sum_{k=1}^n b_k cos(k\theta) = -1$. $|f_2(1)|^2 = 0$. $|f_2(1)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$. □

Lemma 2.15. *When $z=i$, there is $|f_2(i)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$.*

Proof. When $z=i$, $n=4m$, $p_2(i) = \sum_{k=1}^{4m}(-i)^k = \sum_{l=0}^{m-1}(-i)^{4l+1} + \sum_{l=0}^{m-1}(-i)^{4l+2} + \sum_{l=0}^{m-1}(-i)^{4l+3} + \sum_{l=0}^{m-1}(-i)^{4l+4} = \sum_{l=0}^{m-1}(-i) + \sum_{l=0}^{m-1}(-1) + \sum_{l=0}^{m-1}(i) + \sum_{l=0}^{m-1} 1 = 0$. $f_2(i) = 1 + P_2(i) = 0+1 = 1$. $\theta = \frac{\pi}{2}$. $k = 4l+1$, $l = 0,1,\cdots,m-1$. $cos((4l+1)\frac{\pi}{2}) = cos(2l\pi+\frac{\pi}{2}) = 0$. $k=4l+2$, $cos((4l+2)\frac{\pi}{2}) = cos(2l\pi+\pi) = -1$. $k = 4l+3$, $cos((4l+3)\frac{\pi}{2}) = cos(2l\pi+\frac{3\pi}{2}) = 0$. $k=4l+4$, $cos((4l+4)\frac{\pi}{2}) = cos(2l\pi+2\pi) = 1$. $2\sum_{k=1}^n b_k cos(k\theta) = 2\sum_{l=0}^{m-1}(-b_{4l+2}) + 2\sum_{l=0}^{m-1} b_{4l+4} = 2\sum_{l=0}^{m-1}(b_{4l+4}-b_{4l+2}) = 2\sum_{l=0}^{m-1}(n+1-4l-4-n-1+4l+2) = 2\sum_{l=0}^{m-1}(-2) = -4m = -n$. $n + 2\sum_{k=1}^n b_k cos(k\theta) = 0$. $|f_2(i)|^2 = 1$. $|f_2(i)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$.

When $n = 4m+1$, $p_2(i) = \sum_{k=1}^{4m+1}(-i)^k = \sum_{l=0}^{m}(-i)^{4l+1} + \sum_{k=0}^{m-1}(-i)^{4l+2} + \sum_{l=0}^{m-1}(-i)^{4k+3} + \sum_{l=0}^{m-1}(-i)^{4k+4} = \sum_{l=0}^{m}(-i) + \sum_{l=0}^{m-1}(-1) + \sum_{l=0}^{m-1}(i) + \sum_{l=0}^{m-1} 1 = -i$. $f_2(i) = 1 + P_2(i) = 1 - i$. $2\sum_{k=1}^n b_k cos(k\theta) = 2\sum_{l=0}^{m-1}(-b_{4l+2}) + 2\sum_{l=0}^{m-1} b_{4l+4} = 2\sum_{l=0}^{m-1}(b_{4l+4} - b_{4l+2}) = 2\sum_{l=0}^{m-1}(n+1-4l-4-n-1+4l+2) = 2\sum_{l=0}^{m-1}(-2) = -4m-1+1 = -n+1$. $n + 2\sum_{k=1}^n b_k cos(k\theta) = n-n+1 = 1$. $|f_2(i)|^2 = 2$. $|f_2(i)|^2 \neq n + 2\sum_{k=1}^n b_k cos(k\theta)$.

When $n = 4m+2$, $p_2(i) = \sum_{k=1}^{4m+2}(-i)^k = \sum_{l=0}^{m}(-i)^{4l+1}+\sum_{l=0}^{m}(-i)^{4l+2}+\sum_{l=0}^{m-1}(-i)^{4l+3}+\sum_{l=0}^{m-1}(-i)^{4l+4} = \sum_{l=0}^{m}(-i)+\sum_{l=0}^{m}(-1)+\sum_{l=0}^{m-1}(i)+\sum_{l=0}^{m-1}1 = -i-1$. $f_2(i) = 1+P_2(i) = -i$. $2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m}(-b_{4l+2})+2\sum_{l=0}^{m-1} b_{4l+4} = -2b_{4m+2}+2\sum_{l=0}^{m-1}(b_{4l+4}-b_{4l+2})$, $b_{4m+2} = b_n = 1$, $2\sum_{k=1}^{n} b_k cos(k\theta) = -2b_n+2\sum_{l=0}^{m-1}(b_{4l+4}-b_{4l+2}) = -2+2\sum_{l=0}^{m-1}(n+1-4l-4-n-1+4l+2) = -2+2\sum_{l=0}^{m-1}(-2) = -2-4m = -n$. $n+2\sum_{k=1}^{n} b_k cos(k\theta) = n-n = 0$. $|f_2(i)|^2 = 1$. $|f_2(i)|^2 \neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$.

When $n = 4m+3$, $p_2(i) = \sum_{k=1}^{4m+3}(-i)^k = \sum_{l=0}^{m}(-i)^{4l+1}+\sum_{l=0}^{m}(-i)^{4l+2}+\sum_{l=0}^{m}(-i)^{4l+3}+\sum_{l=0}^{m-1}(-i)^{4l+4} = \sum_{l=0}^{m}(-i)+\sum_{l=0}^{m}(-1)+\sum_{l=0}^{m} i+\sum_{l=0}^{m-1}1 = -1$. $f_2(i) = 1+P_2(i) = 0$. $2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m}(-b_{4l+2})+2\sum_{l=0}^{m-1} b_{4l+4} = -2b_{4m+2}+2\sum_{l=0}^{m-1}(b_{4l+4}-b_{4l+2})$, $b_{4m+2} = b_{n-1} = 2$, $2\sum_{k=1}^{n} b_k cos(k\theta) = -4+2\sum_{l=0}^{m-1}(n+1-4l-4-n-1+4l+2) = -4+2\sum_{l=0}^{m-1}(-2) = -4-4m = -n-1$. $n+2\sum_{k=1}^{n} b_k cos(k\theta) = n-n-1 = -1$. $|f_2(i)|^2 = 0$. $|f_2(i)|^2 \neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$. □

Lemma 2.16. *When $n = 8m_1$, $n = 4m_2$, $m_2 = 2m_1$, $f_{22}(z) = \sum_{k=1}^{n}(-z)^k = (z-1)(1+z^2)(1+z^4)\sum_{l=0}^{m_1-1} z^{8l+1}$.*

Proof. When $n = 8m_1$, $n = 4m_2$, $m_2 = 2m_1$, $f_{22}(z) = \sum_{k=1}^{n}(-z)^k = \sum_{l=0}^{m_2-1}(-z)^{4l+1}+\sum_{l=0}^{m_2-1} z^{4l+2}+\sum_{l=0}^{m_2-1}(-z)^{4l+3}+\sum_{l=0}^{m_2-1} z^{4l+4} = -\sum_{l=0}^{m_2-1} z^{4l+1}+\sum_{l=0}^{m_2-1} z^{4l+2}-\sum_{l=0}^{m_2-1} z^{4l+3}+\sum_{l=0}^{m_2-1} z^{4l+4} = -\sum_{l=0}^{m_2-1} z^{4l+1}+z\sum_{l=0}^{m_2-1} z^{4l+1}-z^2\sum_{l=0}^{m_2-1} z^{4l+1}+z^3\sum_{l=0}^{m_2-1} z^{4l+1} = (z-1)(1+z^2)\sum_{l=0}^{m_2-1} z^{4l+1}$.

$\sum_{l=0}^{m_2-1} z^{4l+1} = \sum_{k=1}^{m_2} z^{4k-3} = \sum_{l=0}^{m_1-1} z^{8l+1}+\sum_{l=0}^{m_1-1} z^{8l+5} = \sum_{l=0}^{m_1-1} z^{8l+1}+z^4\sum_{l=0}^{m_1-1} z^{8l+1} = (1+z^4)\sum_{l=0}^{m_1-1} z^{8l+1}$. □

Lemma 2.17. *When $n = 8m_1$, $n = 4m_2$, $m_2 = 2m_1$, for $z = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ and $f_2(z)$. $2\sum_{k=1}^{n} b_k cos(k\theta) = -8m_1$.*

Proof. When $n = 8m_1$, $n = 4m_2$, $m_2 = 2m_1$, $z = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$, $\theta = \frac{\pi}{4}$. $k = 8l+1$, $l = 0,1,\cdots,m_1-1$. $cos((8l+1)\frac{\pi}{4}) = cos(2l\pi+\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. $k = 8l+2$, $cos((8l+2)\frac{\pi}{4}) = cos(2l\pi+\frac{\pi}{2}) = 0$. $k = 8l+3$, $cos((8l+3)\frac{\pi}{4}) = cos(2l\pi+\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$. $k = 8l+4$, $cos((8l+4)\frac{\pi}{4}) = cos(2l\pi+\pi) = -1$. $k = 8l+5$, $cos((8l+5)\frac{\pi}{4}) = cos(2l\pi+\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$. $k = 8l+6$, $cos((8l+6)\frac{\pi}{4}) = cos(2l\pi+\frac{3\pi}{2}) = 0$. $k = 8l+7$, $cos((8l+7)\frac{\pi}{4}) = cos(2l\pi+\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}$. $k = 8l+8$, $cos((8l+8)\frac{\pi}{4}) = cos(2l\pi+2\pi) = 1$.

$2\sum_{k=1}^{n} b_k cos(k\theta) = 2\sum_{l=0}^{m_1-1} b_{8l+1}cos((8l+1)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+2}cos((8l+2)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+3}cos((8l+3)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+4}cos((8l+4)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+5}cos((8l+5)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+6}cos((8l+6)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+7}cos((8l+7)\frac{\pi}{4})+2\sum_{l=0}^{m_1-1} b_{8l+8}cos((8l+8)\frac{\pi}{4}) = \sqrt{2}\sum_{l=0}^{m_1-1} b_{8l+1}-\sqrt{2}\sum_{l=0}^{m_1-1} b_{8l+3}-2\sum_{l=0}^{m_1-1} b_{8l+4}-\sqrt{2}\sum_{l=0}^{m_1-1} b_{8l+5}+\sqrt{2}b_{8l+7}+2\sum_{l=0}^{m_1-1} b_{8l+8} = \sqrt{2}\sum_{l=0}^{m_1-1}(b_{8l+1}-b_{8l+3}-b_{8l+5}+b_{8l+7})+2\sum_{l=0}^{m_1-1}(b_{8l+8}-b_{8l+4}) = -\sqrt{2}\sum_{l=0}^{m_1-1}(n+1-8l-1-n-1+8l+3-n-1+8l+5+n+1-8l-7)+2\sum_{l=0}^{m_1-1}(n+1-8l-8-n-1+8l+4) = -8m_1$. □

Lemma 2.18. *When $n = 8m_1$, there is $|f_2(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2 \neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$.*

Proof. According to Lemma 1.16, when $z = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$, $f_{22}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}) = 0$. So, $f_2(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}) = 1+f_{22}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}) = 1+0 = 1$.

According to Lemma 1.17, for $z = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$. $2\sum_{k=1}^{n} b_k cos(k\theta) = -8m_1 = -n$. $n+2\sum_{k=1}^{n} b_k cos(k\theta) = n-n = 0$. $|f_2(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2 = 1$. $|f_2(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2 \neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$. □

Lemma 2.19. *When $n = 8m_1+1$, there is $|f_2(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})|^2 \neq n+2\sum_{k=1}^{n} b_k cos(k\theta)$.*

Proof. According to Lemma 1.16, when $z = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$, $f_{22}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}) = 0$. So, $f_2(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}) = 1+f_{22}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})-(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1} = 1+0-(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1} = 1-(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^{8m_1+1}$.

According to Lemma 1.17, for $z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. $2\sum_{k=1}^{n} b_k cos(k\theta) = \sqrt{2}b_{8m_1+1} - 8m_1$, $n = 8m_1 + 1$, $b_{8m_1+1} = b_n = -1$, $2\sum_{k=1}^{n} b_k cos(k\theta) = \sqrt{2}b_n - 8m_1 = -\sqrt{2} + 1 - n$. $n + 2\sum_{k=1}^{n} b_k cos(k\theta) = n - n - \sqrt{2} + 1 = 1 - \sqrt{2}$. $|f_2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})|^2$ is a very complex expression, but, it is not equal to $1 - \sqrt{2}$. So, $|f_2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$. □

Lemma 2.20. *When $n = 8m_1 + 2$, there is $|f_2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$.*

Proof. According to Lemma 1.16, when $z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $f_{22}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 0$. So, $f_2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 1 + f_{22}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) - (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^{8m_1+1} + (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^{8m_1+2} = 1 + 0 - (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^{8m_1+1} + (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^{8m_1+2} = 1 - (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^{8m_1+1} + (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^{8m_1+2}$.

According to Lemma 1.17, for $z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. $2\sum_{k=1}^{n} b_k cos(k\theta) = \sqrt{2}b_{8m_1+1} - 8m_1$, $n = 8m_1 + 2$, $b_{8m_1+1} = b_{n-1} = -2$, $2\sum_{k=1}^{n} b_k cos(k\theta) = \sqrt{2}b_{n-1} - 8m_1 = -2\sqrt{2} + 2 - n$. $n + 2\sum_{k=1}^{n} b_k cos(k\theta) = n - n - 2\sqrt{2} + 2 = -2\sqrt{2} + 2$. $|f_2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})|^2$ is a very complex expression, but, it is not equal to $-2\sqrt{2} + 2$. So, $|f_2(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})|^2 \neq n + 2\sum_{k=1}^{n} b_k cos(k\theta)$. □

References

[1] *W.K. Hayman,* Research Problems in Function Theory*, The Athlone Press, University of London, London, (1967).*

[2] *J. Clunie and F.R. Keogh,* On starlike and convex schlicht functions*, J. London Math. Soc., 35, 229-233,(1960).*

[3] *J. Clunie and F.R. Keogh,* Addendum to a note on schlicht functions*, J. London Math. Soc., 39, 63–64,(1964).*

[4] *J.G. Clunie.* Some remarks on extreme points in function theory. In Aspects of contemporary complex analysis*, pages 137-146. Academic Press, London-New York, (1980).*

[5] *W.K. Hayman and E.F. Lingham,* Research Problems in Function Theory*, Fiftieth anniversary edition, Springer, New York, (2019).*

[6] *S. D. Purohit, G. Murugusundaramoorthy, K. Vijaya, D. L. Suthar and K. Jangid,* A unified class of spiral-like functions including Kober fractional operators in quantum calculus*, Palest. J. Math., 12(2), 487-498, (2023).*

[7] *H. P. Waghamore and B. E. Manjunath,* Difference monomial and its shift sharing a polynomial*, Palest. J. Math., 12(2), 408-418, (2023).*

Author information

Lande Ma, School of Mathematical Sciences, Tongji University, China.
E-mail: dzy200408@126.com

Zhaokun Ma, YanZhou College, Shandong Radio and TV University, China.
E-mail: dzy200408@sina.cn

Received: 2023-04-07
Accepted: 2023-10-28