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# On DNA codes over additive skew cyclic codes

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**Abstract** This paper presents the study of DNA codes over the chain ring $\mathcal{R} = \mathbb{F}_4 + v\mathbb{F}_4$ with $v^2 = 0$ and mixed alphabet $\mathbb{F}_4\mathcal{R}$. First, we characterize skew cyclic codes over these rings and study reversible and complement conditions to obtain the DNA codes. Then, we use a Gray map to get DNA strings corresponding to the elements of $\mathcal{R}$ and $\mathbb{F}_4\mathcal{R}$, respectively. Finally, we provide some examples based on the results established in the preceding sections.

## 1 Introduction

In 1994, Adleman performed the first successful experiment to solve a computationally difficult problem using DNA molecules [2]. Since then, due to the uses of DNA molecules in parallel computing, researchers have found it to be one of the most compelling methods to tackle computationally challenging problems. Due to DNA molecules' high density and long-term durability, DNA data storage is a promising medium for storing data. It uses sophisticated processes and specific physical conditioning to perform such operations, which makes it prone to errors. Some common errors that occur in DNA computing are the long run length of an oligonucleotide, undesirable hybridization, common prefix codewords, and others. To overcome such errors, some DNA constraints, such as no higher run-length of bases, balanced GC-content, and Hamming distance, are applied in finding DNA codes. Here, we illustrate algebraic coding schemes to find error-correcting code in the practical implementation of algorithms in the different domains of DNA computing. Mathematically, a DNA code of length $n$ is a set of $n$-length strings over the alphabets $\{A, T, G, C\}$.

Interestingly, for the last two decades, algebraic methods have been widely used to construct DNA codes compared to other methods, such as the greedy approach and the lexicographic method. Towards this, researchers ideally consider the algebraic structures (rings) of characteristic 2 or 4. On the other hand, being rich algebraic structures, cyclic codes are extensively examined for reversible and DNA code construction over some finite commutative rings [1, 3, 8, 17, 18]. Besides, a polynomial ring $\mathbb{F}_q[x]$ is a unique factorization domain( UFD) that provides limited polynomials for code construction. However, in the case of the non-unique factorization domain, being many factorizations of a polynomial, we have more generating polynomials. Hence, we may have a high chance of getting codes with better parameters. Here, we recall a few initial works on the constructions of skew-cyclic codes (non-UFD) [5, 6, 7]. More recently, some works have been presented on the construction of quantum codes using skew-cyclic and generalized skew-cyclic codes [16, 20, 21]. The study by Dinh et al. [8] pertains to the study of DNA encoding via the utilization of cyclic codes over a non-chain ring and a mixed alphabet. Notably, the examination of classical codes or DNA codes over chain rings has yet to undergo extensive exploration. The present study analyzes DNA codes using skew cyclic codes over a chain ring and mixed alphabets (ring). The work of this paper is arranged as follows: Section 2 contains basic definitions and results related to the study. In Section 3, we give the structure of skew cyclic codes and find reversibility conditions in terms of their generating polynomials. Section 4 studies the reversible skew cyclic codes over $\mathbb{F}_4\mathcal{R}$ while the complementary condition and DNA codes over $\mathbb{F}_4$ and $\mathbb{F}_4\mathcal{R}$ are studied in Section 5. Section 6 concludes the work. Let $\mathbb{F}_4 = \mathbb{F}_2[\omega]/\langle\omega^2+\omega+1\rangle$ be the Galois field of order 4. Suppose $\mathcal{R} := \mathbb{F}_4 + v\mathbb{F}_4$ where $v^2 = 0$, then any element $\mathfrak{r} \in \mathcal{R}$ can be uniquely represented as $\mathfrak{r} = r_1 + r_2 v$ where $r_1, r_2 \in \mathbb{F}_4$. Also, $\mathfrak{r}$ is a unit if and only if $r_1$ is a unit. Moreover, $\mathcal{R}$ has unique maximal ideal $\langle 2, v\rangle$. Thus, $\mathcal{R}$ is a finite commutative local ring with characteristic 2 and order 16. Now, we define two sets $\mathbb{F}_4\mathcal{R} = \{(s,t) : s \in \mathbb{F}_4, t \in \mathcal{R}\}$ and $\mathbb{F}_4^n\mathcal{R}^m = \{(\boldsymbol{r}, \boldsymbol{s}) : \boldsymbol{r} \in \mathbb{F}_4^n, \boldsymbol{s} \in \mathcal{R}^m\}$. The set $\mathbb{F}_4^n\mathcal{R}^m$ is not an $\mathcal{R}$-module under the standard multiplication. To make $\mathbb{F}_4^n\mathcal{R}^m$ an $\mathcal{R}$-module, we define a projection map $\Gamma : \mathcal{R} \longrightarrow \mathbb{F}_4$ by $r_1 + r_2 v \longmapsto r_1$ for all $\mathfrak{r} \in \mathcal{R}$. Thus, we can define a multiplication

$$* : \mathcal{R} \times \mathbb{F}_4^n\mathcal{R}^m \longrightarrow \mathbb{F}_4^n\mathcal{R}^m$$

by

$$r * (\boldsymbol{s}, \boldsymbol{t}) = (\Gamma(r)\boldsymbol{s}, r\boldsymbol{t}),$$

where $\boldsymbol{s} = (s_0, s_1, \ldots, s_{n-1}) \in \mathbb{F}_4^n$ and $\boldsymbol{t} = (t_0, t_1, \ldots, t_{m-1}) \in \mathcal{R}^m$. It can be easily verified that the set $\mathbb{F}_4^n\mathcal{R}^m$ is an $\mathcal{R}$-module under componentwise sum and the multiplication '$*$'.
Let $\mathfrak{r} = r_1 + r_2 v \in \mathcal{R}$ and $\Theta : \mathcal{R} \longrightarrow \mathcal{R}$ be a map defined by $\mathfrak{r} \longmapsto r_1^2 + r_2^2 v$. Then $\Theta$ is an automorphism over $\mathcal{R}$ of order 2. We represent the set of automorphisms of $\mathcal{R}$ by $Aut(\mathcal{R})$ and order of $\Theta$ by $o(\Theta)$.

**Definition 1.1.** Let $\Theta \in Aut(\mathcal{R})$. Then the skew polynomial ring over $\mathcal{R}$ with respect to $\Theta$ is the set

$$\mathcal{R}[\mathfrak{z}; \Theta] = \{\mathfrak{b}_0 + \mathfrak{b}_1\mathfrak{z} + \cdots + \mathfrak{b}_\ell \mathfrak{z}^\ell \mid \mathfrak{b}_i \in \mathcal{R}, \forall i = 0, 1, \ldots, \ell\}$$

endowed with addition as the usual addition of polynomials, and multiplication of polynomials is defined under the rule $(\mathfrak{b}_i\mathfrak{z})(\mathfrak{b}_j\mathfrak{z}) = \mathfrak{b}_i\Theta(\mathfrak{b}_j)\mathfrak{z}^2$.

From the multiplication of skew polynomials, it is easy to verify that ring $\mathcal{R}[\mathfrak{z}; \Theta]$ is noncommutative unless $\Theta$ is the identity automorphism.

Consider a linear code $\mathfrak{C}$ of length $n$ over $\mathcal{R}$. This code is essentially an $\mathcal{R}$-submodule of $\mathcal{R}^n$; its elements are called codewords. The Hamming weight of an element $\mathfrak{d}$, denoted by $w_H(\mathfrak{d})$, is defined as $w_H(\mathfrak{d}) = \mid \{i : \mathfrak{d}_i \neq 0\} \mid$ and Hamming distance $d_H(\mathfrak{d}, \mathfrak{d}')$ between any two elements $\mathfrak{d} = (\mathfrak{d}_0, \mathfrak{d}_1, \ldots \mathfrak{d}_n)$ and $\mathfrak{d}' = (\mathfrak{d}'_0, \mathfrak{d}'_1, \ldots \mathfrak{d}'_n)$ in $\mathfrak{C}$ is defined as

$$d_H(\mathfrak{d}, \mathfrak{d}') = w_H(\mathfrak{d} - \mathfrak{d}').$$

Also, the minimum Hamming distance of the code $\mathfrak{C}$, denoted as $d_H(\mathfrak{C})$, is given by the smallest Hamming distance among all pairs of distinct codewords in $\mathfrak{C}$. It has been ascertained that the concept of edit distance holds significant utility within the field of bioinformatics. This metric finds relevance in gauging the similarity between two strings. When considering a pair of strings denoted as $\mathfrak{z}_1$ and $\mathfrak{z}_2$, the edit distance $e(\mathfrak{z}_1, \mathfrak{z}_2)$ is precisely characterized as the minimum count of operations, encompassing insertions, deletions, or substitutions, necessary to transform $\mathfrak{z}_2$ into $\mathfrak{z}_1$. For example, edit distance between $\mathfrak{z}_1$ = "$TTATTATTA$" and $\mathfrak{z}_2$ = "$ATTATTATT$" is 2 whereas Hamming distance of $\mathfrak{z}_1$ and $\mathfrak{z}_2$ is 6.
Now, we define a Gray map $\Phi : \mathcal{R} \longrightarrow \mathbb{F}_4^2$ as

$$\Phi(a_0 + a_1 v) = (a_0, a_0 + a_1), \tag{1.1}$$

where $a_i \in \mathbb{F}_4$ for $i = 0, 1$. Here, the function $\Phi$ is an $\mathbb{F}_4$-linear distance preserving map, and this map is extendable component-wise to $\mathcal{R}^n$. Further, we map $A$, $T$, $G$ and $C$ to 0, 1, $\omega^2$ and $\omega$, respectively. Furthermore, to find the codons corresponding to the elements of $\mathbb{F}_4\mathbb{F}_4[v]/\langle v^2\rangle$, we define a Gray map $\Phi'$ from $\mathbb{F}_4\mathbb{F}_4[v]/\langle v^2\rangle$ to $\mathbb{F}_4^3$ by

$$\Phi'(a, a_0 + a_1 v) = (a, a_0, a_0 + a_1), \quad \text{where } a, a_0, a_1 \in \mathbb{F}_4.$$

For a cyclic code to have the property of reversibility over $\mathbb{F}_4$, it is necessary that the corresponding generator polynomial exhibit self-reciprocal characteristics.

| | | | |
|---|---|---|---|
| $0$ | AA | $v$ | AT |
| $\omega v$ | AC | $\omega^2 v$ | AG |
| $1$ | TT | $v+1$ | TA |
| $\omega v+1$ | TG | $\omega^2 v+1$ | TC |
| $\omega$ | CC | $v+\omega$ | CG |
| $\omega v+\omega$ | CA | $\omega^2 v+\omega$ | CT |
| $\omega^2$ | GG | $v+\omega^2$ | GC |
| $\omega v+\omega^2$ | GT | $\omega^2 v+\omega^2$ | GA |

**Table 1.** Codons correspondence to ring($\mathcal{R}$) elements using map $\Phi$

**Definition 1.2.** Let $f(\mathfrak{z}) = f_0 + f_1\mathfrak{z} + \ldots + f_m\mathfrak{z}^m$ be a polynomial in $\mathbb{F}_4[\mathfrak{z}, \theta]$. Then polynomial $f(\mathfrak{z})$ is said to palindromic and $\theta$-palindromic if for each $i$ in $\{1, 2, \ldots, m\}$, we have $f_i = f_{m-i}$ and $\theta(f_i) = f_{m-i}$, respectively.

**Lemma 1.3.** *(i) For any $a = (a_0 + a_1 v) \in \mathcal{R}$, we have* $\Phi(a_0 + a_1 v)^r = \Phi[(a_0 + a_1) + a_1 v]$, *where* $a_0, a_1 \in \mathbb{F}_4$.

*(ii)* $\Phi(a_0 + a_1)^r = \Phi(a_0)^r + \Phi(a_1)^r$, *where* $a_0, a_1 \in \mathbb{F}_4$.

*Proof.* Using the definition of the Gray map over $\mathcal{R}$, we have the following statements.

(i) Let $a = (a_0 + a_1 v) \in \mathcal{R}$, then

$$\begin{aligned}\Phi(a_0 + a_1 v)^r &= (a_0, a_0 + a_1)^r \\ &= (a_0 + a_1, a_0) \\ &= \Phi[(a_0 + a_1) + a_1 v].\end{aligned}$$

(ii) The proof is similar to the above part.

□

## 2 Reversible skew cyclic code over $\mathcal{R}$

This section delves into the structure of skew cyclic codes over the ring $\mathcal{R}$. Initially, we establish a necessary and sufficient condition that characterises the reversibility of the codes. Towards this, we begin with the skew cyclic code of length $m$.

**Definition 2.1.** A non-empty subset $\mathfrak{C}$ of $\mathcal{R}^m$ is said to be skew cyclic code of length $m$ if

$(i)$ $\mathfrak{C}$ is a left $\mathcal{R}$ submodule of module $\mathcal{R}^m$;

$(ii)$ $\tau : \mathcal{R}^m \longrightarrow \mathcal{R}^m$ skew cyclic shift operator, then $\mathfrak{C}$ is closed under $\tau$, i.e., for $c = (c_0, c_1, \ldots, c_{m-1}) \in \mathfrak{C}$
$\Longrightarrow \tau(c) = (\Theta(c_{m-1}), \Theta(c_0), \ldots, \Theta(c_{m-2})) \in \mathfrak{C}$.

Similar to the commutative case, here also we can identify each element $(r_0, r_1, \ldots, r_{m-1}) \in \mathcal{R}^m$ to a skew polynomial $r_0 + r_1\mathfrak{z} + \cdots + r_{m-1}\mathfrak{z}^{m-1}$ in the quotient $\mathcal{R}[\mathfrak{z}; \Theta]/\langle \mathfrak{z}^m - 1\rangle$. Since the skew polynomial ring $\mathcal{R}[\mathfrak{z}; \Theta]$ is noncommutative, $\langle \mathfrak{z}^m - 1\rangle$ need not be a two-sided ideal. But Jitman et al. [12] have shown that if $m$ is a multiple of $o(\Theta)$, then $\langle \mathfrak{z}^m - 1\rangle$ forms the center of $\mathcal{R}[\mathfrak{z}, \Theta]$, hence a two-sided ideal. In this case, the quotient $\mathcal{R}[\mathfrak{z}; \Theta]/\langle \mathfrak{z}^m - 1\rangle$ is a ring

| | | | |
|---|---|---|---|
| $(0,0)$ | AAA | $(1,0)$ | TAA |
| $(\omega,0)$ | CAA | $(\omega^2,0)$ | GAA |
| $(0,1)$ | ATT | $(1,1)$ | TTT |
| $(\omega,1)$ | CTT | $(\omega^2,1)$ | GTT |
| $(0,\omega)$ | ACC | $(1,\omega)$ | TCC |
| $(\omega,\omega)$ | CCC | $(\omega^2,\omega)$ | GCC |
| $(0,\omega^2)$ | AGG | $(1,\omega^2)$ | TGG |
| $(\omega,\omega^2)$ | CGG | $(\omega^2,\omega^2)$ | GGG |
| $(0,v)$ | AAT | $(1,v)$ | TAT |
| $(\omega,v)$ | CAT | $(\omega^2,v)$ | GAT |
| $(0,v+1)$ | ATA | $(1,v+1)$ | TTA |
| $(\omega,v+1)$ | CTA | $(\omega^2,v+1)$ | GTA |
| $(0,v+\omega)$ | ACG | $(1,v+\omega)$ | TCG |
| $(\omega,v+\omega)$ | CCG | $(\omega^2,v+\omega)$ | GCG |
| $(0,v+\omega^2)$ | AGC | $(1,v+\omega^2)$ | TGC |
| $(\omega,v+\omega^2)$ | CGC | $(\omega^2,v+\omega^2)$ | GGC |
| $(0,\omega v)$ | AAC | $(1,\omega v)$ | TAC |
| $(\omega,\omega v)$ | CAC | $(\omega^2,\omega v)$ | GAC |
| $(0,\omega v+1)$ | ATG | $(1,\omega v+1)$ | TTG |
| $(\omega,\omega v+1)$ | CTG | $(\omega^2,\omega v+1)$ | GTG |
| $(0,\omega v+\omega)$ | ACA | $(1,\omega v+\omega)$ | TCA |
| $(\omega,\omega v+\omega)$ | CCA | $(\omega^2,\omega v+\omega)$ | GCA |
| $(0,\omega v+\omega^2)$ | AGT | $(1,\omega v+\omega^2)$ | TGT |
| $(\omega,\omega v+\omega^2)$ | CGT | $(\omega^2,\omega v+\omega^2)$ | GGT |
| $(0,\omega^2 v)$ | AAG | $(1,\omega^2 v)$ | TAG |
| $(\omega,\omega^2 v)$ | CAG | $(\omega^2,\omega^2 v)$ | GAG |
| $(0,\omega^2 v+1)$ | ATC | $(1,\omega^2 v+1)$ | TTC |
| $(\omega,\omega^2 v+1)$ | CTC | $(\omega^2,\omega^2 v+1)$ | GTC |
| $(0,\omega^2 v+\omega)$ | ACT | $(1,\omega^2 v+\omega)$ | TCT |
| $(\omega,\omega^2 v+\omega)$ | CCT | $(\omega^2,\omega^2 v+\omega)$ | GCT |
| $(0,\omega^2 v+\omega^2)$ | AGA | $(1,\omega^2 v+\omega^2)$ | TGA |
| $(\omega,\omega^2 v+\omega^2)$ | CGA | $(\omega^2,\omega^2 v+\omega^2)$ | GGA |

**Table 2.** Codons correspondence to ring($\mathbb{F}_4R$) elements using map $\phi'$

and a linear code $\mathfrak{C}$ of length $m$ over $\mathcal{R}$ is defined as left ideal of $\mathcal{R}[\mathfrak{z};\Theta]/\langle \mathfrak{z}^m - 1\rangle$. However, if $m$ is not a multiple of $o(\Theta)$, then the quotient $\mathcal{R}[\mathfrak{z};\Theta]/\langle \mathfrak{z}^m - 1\rangle$ is a left $\mathcal{R}[\mathfrak{z};\Theta]$-module with left multiplication defined by $r(\mathfrak{z})(t(\mathfrak{z})+\langle \mathfrak{z}^m - 1\rangle) = r(\mathfrak{z})t(\mathfrak{z})+\langle \mathfrak{z}^m - 1\rangle$, where $r(\mathfrak{z}), t(\mathfrak{z}) \in \mathcal{R}[\mathfrak{z};\Theta]$. In this case, a skew cyclic code $\mathfrak{C}$ of length $m$ over $\mathcal{R}$ is defined as left submodule of $\mathcal{R}[\mathfrak{z};\Theta]$-module $\mathcal{R}[\mathfrak{z};\Theta]/\langle \mathfrak{z}^m - 1\rangle$.

In the following theorem, we consider the ring $\frac{\mathcal{R}[\mathfrak{z};\Theta]}{\langle \mathfrak{z}^n - 1\rangle}$ to give the structure of skew cyclic codes of even length $n$ over the chain ring $\mathcal{R}$.

**Theorem 2.2.** *Let $\mathfrak{C}$ be a skew cyclic code in $\frac{\mathcal{R}[\mathfrak{z};\Theta]}{\langle \mathfrak{z}^n - 1\rangle}$. Then the code $\mathfrak{C}$ can be one of the following forms:*

*(i)* $\mathfrak{C} = \langle g_0(\mathfrak{z}) + v g_1(\mathfrak{z})\rangle$ *with* $deg(g_1(\mathfrak{z})) < deg(g_0(\mathfrak{z}))$ *and* $g_0(\mathfrak{z})|_r(\mathfrak{z}^n - 1)$ *in* $\mathbb{F}_4[\mathfrak{z},\theta]$.

*(ii)* $\mathfrak{C} = \langle v g_1(\mathfrak{z})\rangle$ *with* $deg(g_1(\mathfrak{z})) < n$ *and* $g_1(\mathfrak{z})|_r(\mathfrak{z}^n - 1)$ *in* $\mathbb{F}_4[\mathfrak{z},\theta]$.

*(iii)* $\mathfrak{C} = \langle g_0(\mathfrak{z})+v g_1(\mathfrak{z}), v a(\mathfrak{z})\rangle$ *with* $deg(g_1(\mathfrak{z})) < deg(a(\mathfrak{z})) < deg(g_0(\mathfrak{z})) < n$ *and* $a(\mathfrak{z})|_r g_0(\mathfrak{z})|_r(\mathfrak{z}^n - 1)$ *in* $\mathbb{F}_4[\mathfrak{z},\theta]$.

*where polynomials $g_0(\mathfrak{z}), g_1(\mathfrak{z})$ and $a(\mathfrak{z})$ are from $\mathbb{F}_4[\mathfrak{z},\theta]$.*

**Definition 2.3.** Given a code $\mathfrak{C} = \langle g_0(\mathfrak{z})+v g_1(\mathfrak{z}), v a(\mathfrak{z})\rangle$ in $\mathcal{R}[\mathfrak{z},\Theta]$, we define $\mathfrak{C}_v$ by $\{q(\mathfrak{z})|\ v q(\mathfrak{z}) \in \mathfrak{C}\}$. In particular, since $a(\mathfrak{z})|_r g_0(\mathfrak{z})$, $\mathfrak{C}_v = \langle a(\mathfrak{z})\rangle$.

Here, we give a result to get the distance of the code $\mathfrak{C}$ using the above definition in terms of $\mathfrak{C}_v$.

**Theorem 2.4.** *Given a code $\mathfrak{C} = \langle g_0(\mathfrak{z}) + v g_1(\mathfrak{z}), v a(\mathfrak{z})\rangle$ in $\mathcal{R}[\mathfrak{z},\Theta]$, we have*

$$d_H(\mathfrak{C}) = d_H(\mathfrak{C}_v).$$

*Proof.* Let $k(\mathfrak{z}) = k_0(\mathfrak{z}) + v k_1(\mathfrak{z}) \in \mathfrak{C}$. Then $v k(\mathfrak{z}) = v k_0(\mathfrak{z})$. We know that $w_H(v k(\mathfrak{z})) \leqslant w_H(k(\mathfrak{z}))$ and $v\mathfrak{C}$ is a subcode of $\mathfrak{C}$ implies $w_H(v\mathfrak{C}) \leqslant w_H(\mathfrak{C})$. Since Hamming distance is given by a vector of minimum weight in the given code, $w_H(\mathfrak{C}) \leqslant w_H(v\mathfrak{C})$. Therefore, $w_H(\mathfrak{C}) = w_H(\mathfrak{C}_v)$. □

**Example 2.5.** Consider a code of length $n = 8$ over $\mathcal{R}$. Then

$$\mathfrak{z}^8 - 1 = (\mathfrak{z} + \omega^2)(\mathfrak{z} + 1)(\mathfrak{z} + \omega^2)^2(\mathfrak{z} + \omega)^2(\mathfrak{z} + 1)(\mathfrak{z} + \omega);$$

and

$$\mathfrak{z}^8 - 1 = (\mathfrak{z} + \omega^2)^4(\mathfrak{z} + \omega)^4.$$

Now, consider $g_0(\mathfrak{z}) = \mathfrak{z}^2 + \omega\mathfrak{z} + \omega$ and $a(\mathfrak{z}) = \mathfrak{z} + \omega$. Then, we get a skew cyclic code $\langle \mathfrak{z}^2 + \omega\mathfrak{z} + \omega, v(\mathfrak{z} + \omega)\rangle$ over $\mathcal{R}$ of parameters $[16, 13, 2]$.

For an odd length $n$, skew cyclic code and cyclic are the same because the order of the automorphism is relative prime to the length of the code [19]. Thus, we have the following result for an odd length.

**Theorem 2.6.** *Let $\mathfrak{C}$ be a skew cyclic code in $\frac{\mathcal{R}[\mathfrak{z};\Theta]}{\langle \mathfrak{z}^n - 1\rangle}$ of odd length $n$. Then $\mathfrak{C} = \langle g_0(\mathfrak{z}), v g_1(\mathfrak{z})\rangle = \langle g_0(\mathfrak{z}) + v g_1(\mathfrak{z})\rangle$ with $g_1(\mathfrak{z})|_r g_0(\mathfrak{z})|_r(\mathfrak{z}^n - 1)$ in $\mathbb{F}_4[\mathfrak{z};\theta]$.*

Now, with the help of the above structure, we give the following lemmas to check the reversibility of skew cyclic code of different lengths.

**Lemma 2.7.** *Let $\mathfrak{C} = \langle g(\mathfrak{z})\rangle$ be a skew cyclic code of even length $n$, where $g(\mathfrak{z}) = 1 + g_1\mathfrak{z} + \ldots + g_{m-1}\mathfrak{z}^{m-1} + \mathfrak{z}^m$ is a monic right divisor of $(\mathfrak{z}^n - 1)$ in $\mathbb{F}_4[\mathfrak{z},\theta]$ with $deg(g(\mathfrak{z})) = m$ is even. Then code $\mathfrak{C}$ is reversible iff skew polynomial $g(\mathfrak{z})$ is $\theta$-palindromic.*

*Proof.* Let $\mathfrak{C}$ be a skew cyclic code of even length generated by $\theta$-palindromic polynomial $g(\mathfrak{z})$ of even degree $m$ over the ring $\mathbb{F}_4$. Then elements of the generated code are given by $\sum_{i=0}^{k-1} \alpha_i \mathfrak{z}^i g(\mathfrak{z})$. By repetitive use of lemma 1.3, for $c = \phi(\sum_{i=0}^{k-1} \alpha_i \mathfrak{z}^i g(\mathfrak{z})) \in \mathfrak{C}$, we get

$$(\phi(\sum_{i=0}^{k-1} \alpha_i \mathfrak{z}^i g(\mathfrak{z})))^r = \phi(\sum_{i=0}^{k-1} \alpha_i \mathfrak{z}^{k-i-1} g(\mathfrak{z})) \in \mathfrak{C}.$$

Where $\alpha \in \mathbb{F}_4$ and $k = n - m$. Since the codeword $c^r$ belongs to $\mathfrak{C}$, the code $\mathfrak{C}$ is a reversible code.
Conversely, let $\mathfrak{C}$ be a reversible code generated by $g(\mathfrak{z}) = 1 + g_1\mathfrak{z} + \ldots + g_{m-1}\mathfrak{z}^{m-1} + \mathfrak{z}^m$. Then, because $n - m - 1$ is odd, implies that

$$\mathfrak{z}^{n-m-1} g(\mathfrak{z}) = \mathfrak{z}^{n-m-1} + \theta(g_1)\mathfrak{z}^{n-m} + \ldots + \theta(g_{m-1})\mathfrak{z}^{n-2} + \mathfrak{z}^{n-1}.$$

Since $\mathfrak{C}$ is a reversible code,

$$[\mathfrak{z}^{n-m-1} g(\mathfrak{z})]^r = 1 + \theta(g_{m-1})\mathfrak{z} + \theta(g_{m-2})\mathfrak{z}^2 + \ldots + \theta(g_1)\mathfrak{z}^{m-1} + \mathfrak{z}^m \in \mathfrak{C}.$$

Further we get $deg(g(\mathfrak{z}) - [\mathfrak{z}^{n-m-1} g(\mathfrak{z})]^r) < m$, which is contradiction to the fact that $g(\mathfrak{z})$ is a minimal degree polynomial in $\mathfrak{C}$ implies $g(\mathfrak{z}) - [\mathfrak{z}^{n-m-1} g(\mathfrak{z})]^r = 0$. Comparing coefficients we get $[g_i - \theta(g_{m-i})] = 0$ for $i = 1, \ldots, m-1$. $g_i = \theta(g_{m-i})$. Thus the polynomial $g(\mathfrak{z})$ is $\theta$-palindromic. □

**Lemma 2.8.** *Let $\mathfrak{C}$ be a skew cyclic code of even length generated by $g(\mathfrak{z}) = 1 + g_1\mathfrak{z} + \ldots + g_{m-1}\mathfrak{z}^{m-1} + \mathfrak{z}^m$ where $g(\mathfrak{z})|_r(\mathfrak{z}^n - 1)$ in $\mathbb{F}_4[\mathfrak{z}, \theta]$ and $m$ is odd. Then the code $\mathfrak{C}$ is reversible if and only if skew polynomial $g(\mathfrak{z})$ is palindromic.*

*Proof.* Let $\mathfrak{C}$ be a skew cyclic code of even length generated by a palindromic polynomial $g(\mathfrak{z})$ of odd degree $m$ over the ring $\mathbb{F}_4$. Then elements of the generated code are given by $\sum_{j=0}^{k-1} \alpha_j \mathfrak{z}^j g(\mathfrak{z})$. Further, applying the lemma [1.3] and the property of palindromic polynomial, for $c = \phi(\sum_{j=0}^{k-1} \alpha_j \mathfrak{z}^j g(\mathfrak{z})) \in \mathfrak{C}$, we get

$$(\phi(\sum_{j=0}^{k-1} \alpha_j \mathfrak{z}^j g(\mathfrak{z})))^r = \phi(\sum_{j=0}^{k-1} \alpha_j \mathfrak{z}^{k-j-1} g(\mathfrak{z})) \in \mathfrak{C}.$$

Where $\alpha \in \mathbb{F}_4$ and $k = n - m$. Since the codeword $c^r$ belongs to $\mathfrak{C}$, the code $\mathfrak{C}$ is a reversible code.
Conversely, let $\mathfrak{C}$ be a reversible code generated by $g(\mathfrak{z}) = 1 + g_1\mathfrak{z} + \ldots + g_{m-1}\mathfrak{z}^{m-1} + \mathfrak{z}^m$. Since $n - m - 1$ is even, implies that

$$\mathfrak{z}^{n-m-1} g(\mathfrak{z}) = \mathfrak{z}^{n-m-1} + g_1\mathfrak{z}^{n-m} + \ldots + g_{m-1}\mathfrak{z}^{n-2} + \mathfrak{z}^{n-1}.$$

Also, $\mathfrak{C}$ is a skew cyclic and reversible code, then $[\mathfrak{z}^{n-m-1} g(\mathfrak{z})]^r$ belong to $\mathfrak{C}$ and we have

$$[\mathfrak{z}^{n-m-1} g(\mathfrak{z})]^r = 1 + g_{m-1}\mathfrak{z} + g_{m-2}\mathfrak{z}^2 + \ldots + g_1\mathfrak{z}^{m-1} + \mathfrak{z}^m.$$

Further, we get $deg(g(\mathfrak{z}) - [\mathfrak{z}^{n-m-1} g(\mathfrak{z})]^r) < m$, which contradicts the fact that $g(\mathfrak{z})$ is a minimal degree polynomial in $\mathfrak{C}$ implies $g(\mathfrak{z}) - [\mathfrak{z}^{n-m-1} g(\mathfrak{z})]^r = 0$. Hence, by comparing coefficients, we get $[g_i - g_{m-i}] = 0$ for $i = 0, 1, \ldots, m$ implies $g_i = g_{m-i}$. Thus, the polynomial $g(\mathfrak{z})$ is palindromic. □

The next theorem follows from the above two lemmas where reversible code is given by palindromic and $\theta$-palindromic polynomials, depending upon their degrees.

**Theorem 2.9.** *Let $\mathfrak{C} = \langle g_0(\mathfrak{z}), vg_1(\mathfrak{z}) \rangle$ be a skew cyclic code of even length where $g_0(\mathfrak{z}),\ g_1(\mathfrak{z})$ and $g_2(\mathfrak{z})$ right divide $(\mathfrak{z}^n - 1)$ in $\mathbb{F}_4[\mathfrak{z}, \theta]$ and $deg(g_i(\mathfrak{z}))$ is even(odd). Then code $\mathfrak{C}$ is reversible if and only if skew polynomials $g_i(\mathfrak{z}),\ i = 0, 1$ is $\theta$-palindromic (palindromic).*

**Example 2.10.** Consider the code $\mathfrak{C} = \langle g_0(\mathfrak{z}), vg_1(\mathfrak{z}) \rangle$ where $g_0(\mathfrak{z}) = \mathfrak{z}^5 + t\mathfrak{z}^4 + t^2\mathfrak{z}^3 + t^2\mathfrak{z}^2 + t\mathfrak{z} + 1$ and $g_1(\mathfrak{z}) = \mathfrak{z}^3 + t\mathfrak{z}^2 + t\mathfrak{z} + 1$. Then $\mathfrak{C}$ is a skew cyclic of length $n = 12$ over $\mathcal{R}$. As polynomials $g_0(\mathfrak{z})$ and $g_1(\mathfrak{z})$ are palindromic, implies that the code $\mathfrak{C}$ is reversible.

| Length | $g(\mathfrak{z})$; $\mathfrak{C} = \langle g(\mathfrak{z}), vg(\mathfrak{z})\rangle$ | Gray image |
|---|---|---|
| 10 | $\mathfrak{z}^5 + \mathfrak{z}^4 + \omega x^3 + \omega\mathfrak{z}^2 + \mathfrak{z} + 1$ | $(20, 4^{10}, 4)$ |
| 12 | $\mathfrak{z}^5 + \omega^2\mathfrak{z}^3 + \omega^2\mathfrak{z}^2 + 1$ | $(24, 4^{14}, 4)$ |
| 12 | $\mathfrak{z}^7 + \omega\mathfrak{z}^5 + \omega^2\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \omega\mathfrak{z}^2 + 1$ | $(24, 4^{10}, 6)$ |
| 14 | $\mathfrak{z}^7 + \mathfrak{z}^5 + \omega^2\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \mathfrak{z}^2 + 1$ | $(28, 4^{14}, 6)$ |
| 18 | $\mathfrak{z}^7 + \mathfrak{z}^6 + \omega\mathfrak{z}^4 + \omega\mathfrak{z}^3 + \mathfrak{z} + 1$ | $(36, 4^{22}, 4)$ |
| 20 | $\mathfrak{z}^7 + \omega^2\mathfrak{z}^6 + \omega\mathfrak{z}^5 + \omega\mathfrak{z}^2 + \omega^2\mathfrak{z} + 1$ | $(40, 4^{26}, 4)$ |
| 24 | $\mathfrak{z}^7 + \mathfrak{z}^6 + \omega\mathfrak{z}^5 + \omega\mathfrak{z}^2 + \mathfrak{z} + 1$ | $(48, 4^{34}, 4)$ |

**Table 3.** DNA codes and their Gray image

## 3 Reversible skew cyclic code over $\mathbb{F}_4\mathcal{R}$

This section extends our study to the mixed alphabet $\mathbb{F}_4\mathcal{R}$. Here, we begin with the following definition.

**Definition 3.1.** Let $\Theta \in Aut(\mathcal{R})$. Then a non-empty subset $\mathfrak{C}$ of $\mathbb{F}_4^n\mathcal{R}^m$ is said to be skew cyclic code of length $n+m$ if

- $(i)$ the set $\mathfrak{C}$ is a left $\mathcal{R}$ submodule of module $\mathbb{F}_4^n\mathcal{R}^m$;
- $(ii)$ there exists a skew cyclic shift operator $\Upsilon : \mathbb{F}_4^n\mathcal{R}^m \longrightarrow \mathbb{F}_4^n\mathcal{R}^m$ such that the set $\mathfrak{C}$ is closed under $\Upsilon$, i.e., for any codeword $(\boldsymbol{s},\boldsymbol{t}) = (s_0, s_1, \ldots, s_{n-1} \mid t_0, t_1, \ldots, t_{m-1}) \in \mathfrak{C}$, we have

  $$\Upsilon((\boldsymbol{s},\boldsymbol{t})) = (\Theta(s_{n-1}), \Theta(s_0), \ldots, \Theta(s_{n-2}) \mid \Theta(t_{n-1}), \Theta(t_0), \ldots, \Theta(t_{m-2})) \in \mathfrak{C},$$

  where $\boldsymbol{s} \in \mathbb{F}_4^n$ and $\boldsymbol{t} \in \mathcal{R}^m$.

Let $\mathcal{R}_{(n,m)} = \frac{\mathbb{F}_4[\mathfrak{z},\Theta]}{\langle \mathfrak{z}^n-1\rangle} \times \frac{\mathcal{R}[\mathfrak{z};\Theta]}{\langle \mathfrak{z}^m-1\rangle}$ where $\Theta \in Aut(\mathcal{R})$. Here, we can associate each element $(\boldsymbol{s},\boldsymbol{t}) \in \mathbb{F}_4^n\mathcal{R}^m$ by a polynomial $(s(\mathfrak{z}), t(\mathfrak{z})) \in \mathcal{R}_{(n,m)}$ under the correspondence $(\boldsymbol{s},\boldsymbol{t}) \longmapsto (s(\mathfrak{z}), t(\mathfrak{z}))$, where $s(\mathfrak{z}) = s_0 + s_1\mathfrak{z} + \cdots + s_{n-1}\mathfrak{z}^{m-1} \in \frac{\mathbb{F}_4[\mathfrak{z}]}{\langle \mathfrak{z}^n-1\rangle}$ and $t(\mathfrak{z}) = t_0 + t_1\mathfrak{z} + \cdots + t_{m-1}\mathfrak{z}^{m-1} \in \frac{\mathcal{R}[\mathfrak{z};\Theta]}{\langle \mathfrak{z}^m-1\rangle}$ are used for $\boldsymbol{s} = (s_0, s_1, \ldots, s_{n-1}) \in \mathbb{F}_4^n$ and $\boldsymbol{t} = (t_0, t_1, \ldots, t_{m-1}) \in \mathcal{R}^m$, respectively. Moreover, $\mathcal{R}_{(n,m)}$ is a left $\mathcal{R}[\mathfrak{z};\Theta]$-module with respect to left multiplication defined by $r(\mathfrak{z}) \star (s(\mathfrak{z}), t(\mathfrak{z})) = (\Gamma(r(\mathfrak{z}))s(\mathfrak{z}), r(\mathfrak{z})t(\mathfrak{z}))$ for $s(\mathfrak{z}) \in \frac{\mathbb{F}_4[\mathfrak{z},\Theta]}{\langle \mathfrak{z}^n-1\rangle}$, $t(\mathfrak{z}) \in \frac{\mathcal{R}[\mathfrak{z},\Theta]}{\langle \mathfrak{z}^m-1\rangle}$ and $r(\mathfrak{z}) = r_0 + r_1\mathfrak{z} + \cdots + r_\ell\mathfrak{z}^\ell \in \mathcal{R}[\mathfrak{z};\Theta]$, where $\Gamma(r(\mathfrak{z})) = \Gamma(r_0) + \Gamma(r_1)\mathfrak{z} + \cdots + \Gamma(r_\ell)\mathfrak{z}^\ell$. Since $\Gamma$ is a projection and $\frac{\mathcal{R}[\mathfrak{z};\Theta]}{\langle \mathfrak{z}^m-1\rangle}$ is left $\mathcal{R}[\mathfrak{z};\Theta]$-module, thus the multiplication is well defined.
From the above discussion, the following lemma holds.

**Lemma 3.2.** *Let $\mathfrak{C}$ be a linear code of length $(n,m)$ over $\mathbb{F}_4^n\mathcal{R}^m$. Then $\mathfrak{C}$ is a skew cyclic code if and only if $\mathfrak{C}$ is a left $\mathcal{R}[\mathfrak{z};\Theta]$-submodule of $\mathcal{R}_{(n,m)}$.*

Let $\Pi_n : \mathbb{F}_4^n\mathcal{R}^m \longrightarrow \mathbb{F}_4^n$ and $\Pi_m : \mathbb{F}_4^n\mathcal{R}^m \longrightarrow \mathcal{R}^m$ be canonical projection maps defined by $\Pi_n(\boldsymbol{s},\boldsymbol{t}) = \boldsymbol{s}$ and $\Pi_m(\boldsymbol{s},\boldsymbol{t}) = \boldsymbol{t}$, respectively. Since these maps are linear, for any $\mathbb{F}_4\mathcal{R}^m$ linear code $\mathfrak{C}$ of length $(n,m)$, its canonical projection $\Pi_n(\mathfrak{C}) = \mathfrak{C}_n$ and $\Pi_m(\mathfrak{C}) = \mathfrak{C}_m$ are linear codes over $\mathbb{F}_4$ and $\mathcal{R}$ of length $n$ and $m$, respectively. Moreover, $\mathfrak{C}$ of block length $(n,m)$ over $\mathbb{F}_4\mathcal{R}$ is said to be separable code if and only if $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$.

Let $\mathbf{c} = (\mathbf{s},\mathbf{t}) = (s_0, s_1, \ldots, s_{n-1}, t_0, t_1, \ldots, t_{m-1}) \in \mathfrak{C}$ then reversible, complement and reversible-complement of $\mathbf{c}$ is defined by $\mathbf{c}^r = (s_{n-1}, s_{n-2}, \ldots, s_1, s_0, t_{m-1}, t_{m-2}, \ldots, t_1, t_0)$, $\mathbf{c}^c = (s_0^c, s_1^c, \ldots, s_{n-1}^c, t_0^c, t_1^c, \ldots, t_{m-1}^c)$ and $\mathbf{c}^{rc} = (s_{n-1}^c, s_{n-2}^c, \ldots\ldots, s_1^c, s_0^c, t_{m-1}^c, t_{m-2}^c, \ldots, t_1^c, t_0^c)$, respectively.

**Definition 3.3.** Let $\mathfrak{C}$ be a $\mathbb{F}_4\mathcal{R}$-linear code of block length $(n,m)$. Then $\mathfrak{C}$ is said to be reversible if for any $\mathbf{c} = (\mathbf{s},\mathbf{t}) \in \mathfrak{C}$, $\mathbf{c}^r = (\mathbf{s}^r,\mathbf{t}^r) \in \mathfrak{C}$, and complement if for any $\mathbf{c} \in \mathfrak{C}$, $\mathbf{c}^c = (\mathbf{s}^c,\mathbf{t}^c) \in \mathfrak{C}$ and reversible-complement if for any $\mathbf{c} = (\mathbf{s},\mathbf{t}) \in \mathfrak{C}$, $\mathbf{c}^{rc} = (\mathbf{s}^{rc},\mathbf{t}^{rc}) \in \mathfrak{C}$.

Now, we give the structure of separable skew cyclic codes over the ring $\mathbb{F}_4\mathcal{R}$.

**Theorem 3.4.** *Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a linear code of length $(n, m)$ over $\mathbb{F}_4\mathcal{R}$. The image of $\mathfrak{C}$ is a skew cyclic code under automorphism $\Theta$ if and only if $\mathfrak{C}_n$ is a cyclic code of length $n$ over $\mathbb{F}_4$ and $\mathfrak{C}_m$ is a skew cyclic code of length $m$ over $\mathcal{R}$.*

The following theorem gives the structure of cyclic codes of length $n$ and their dimension over $\mathbb{F}_4$. The proof follows the similar approach given in [13].

**Theorem 3.5.** *Let $\mathfrak{C}_n$ be a cyclic code of length $n$ over the field $\mathbb{F}_4$. Then there exists a unique monic polynomial $f(\mathfrak{z}) \in \mathbb{F}_4[\mathfrak{z}]/\langle \mathfrak{z}^n - 1\rangle$ such that $\mathfrak{C}_n = \langle f(\mathfrak{z})\rangle$ where $f(\mathfrak{z})|(\mathfrak{z}^n - 1)$. The dimension of the code $\mathfrak{C}$ is $n - deg(f(\mathfrak{z}))$.*

In our next attempt, we provide the condition for a cyclic code to be reversible and a reversible-complement code over $\mathbb{F}_4$. Part of the proof is given in [14] and reverse-complement condition is straightforward by the codons correspondence.

**Theorem 3.6.** *Let $\mathfrak{C}_n$ be a cyclic code of length $n$ over field $\mathbb{F}_4$ generated by the polynomial $f(\mathfrak{z})$. Then the code $\mathfrak{C}_n$ is reversible if and only if polynomial $f(\mathfrak{z})$ is a self-reciprocal polynomial. Furthermore, $\mathfrak{C}_n$ is a reversible-complement code if and only if $f(\mathfrak{z})$ is a self-reciprocal polynomial and $\frac{\mathfrak{z}^n - 1}{\mathfrak{z} - 1} \in \mathfrak{C}_n$.*

Now, we give the structure of additive DNA codes. Here, we study separable codes to find the DNA codes.

**Theorem 3.7.** *Let $\mathfrak{C} = \mathfrak{C}_n \times \mathfrak{C}_m$ be a linear code of block length $(n, m)$ over $\mathbb{F}_4\mathcal{R}$. Then $\mathfrak{C}$ is a DNA codes if and only if $\mathfrak{C}_n$ and $\mathfrak{C}_m$ are DNA codes over $\mathbb{F}_4$ and $\mathcal{R}$, respectively.*

| Length | $f(\mathfrak{z})$ | $g(\mathfrak{z});\ \mathfrak{C}=\langle g(\mathfrak{z}), vg(\mathfrak{z})\rangle$ | Gray image |
|---|---|---|---|
| $(13,12)$ | $\mathfrak{z}^7+\omega\mathfrak{z}^6+\omega^2\mathfrak{z}^5+\omega\mathfrak{z}^4+\omega\mathfrak{z}^3+\omega^2\mathfrak{z}^2+\omega\mathfrak{z}+1$ | $\mathfrak{z}^7+\omega\mathfrak{z}^5+\omega^2\mathfrak{z}^4+\omega^2\mathfrak{z}^3+\omega\mathfrak{z}^2+1$ | $(37,4^{16},6)$ |
| $(15,14)$ | $\mathfrak{z}^4+\omega^2\mathfrak{z}^3+\omega\mathfrak{z}^2+\omega^2\mathfrak{z}+1$ | $\mathfrak{z}^5+\omega^2\mathfrak{z}^4+\omega^2\mathfrak{z}^3+\omega\mathfrak{z}+\omega^2$ | $(43,4^{29},4)$ |
| $(15,14)$ | $\mathfrak{z}^7+\omega^2\mathfrak{z}^6+\omega^2\mathfrak{z}^5+\omega\mathfrak{z}^3+\omega\mathfrak{z}+1$ | $\mathfrak{z}^7+\omega\mathfrak{z}^6+\omega^2\mathfrak{z}^5+\omega^2\mathfrak{z}^4+\omega\mathfrak{z}^2+\omega^2\mathfrak{z}+1$ | $(43,4^{22},6)$ |
| $(20,14)$ | $\mathfrak{z}^4+\mathfrak{z}^3+\mathfrak{z}^2+\omega\mathfrak{z}+\omega^2$ | $\mathfrak{z}^5+\omega^2\mathfrak{z}^4+\omega^2\mathfrak{z}^3+\omega\mathfrak{z}+\omega^2$ | $(48,4^{33},4)$ |
| $(17,20)$ | $\mathfrak{z}^4+\omega\mathfrak{z}^3+\mathfrak{z}^2+\omega\mathfrak{z}+1$ | $\mathfrak{z}^6+\omega^2\mathfrak{z}^4+\mathfrak{z}^3+\omega^2\mathfrak{z}^2+\omega\mathfrak{z}+\omega^2$ | $(57,4^{41},4)$ |
| $(17,24)$ | $\mathfrak{z}^4+\omega^2\mathfrak{z}^3+\mathfrak{z}^2+\omega^2\mathfrak{z}+1$ | $\mathfrak{z}^6+\omega^2\mathfrak{z}^5+\mathfrak{z}^4+\mathfrak{z}+\omega^2$ | $(65,4^{49},4)$ |
| $(43,16)$ | $\mathfrak{z}^7+\omega^2\mathfrak{z}^5+\mathfrak{z}^4+\mathfrak{z}^3+\omega\mathfrak{z}^2+1$ | $\mathfrak{z}^7+\mathfrak{z}^6+\mathfrak{z}^5+\mathfrak{z}^3+\omega^2\mathfrak{z}+\omega$ | $(75,4^{54},5)$ |
| $(43,18)$ | $\mathfrak{z}^7+\omega^2\mathfrak{z}^5+\mathfrak{z}^4+\mathfrak{z}^3+\omega\mathfrak{z}^2+1$ | $\mathfrak{z}^7+\omega^2\mathfrak{z}^6+\mathfrak{z}^4+\omega^2\mathfrak{z}^3+\omega\mathfrak{z}^2+\omega^2\mathfrak{z}+\omega$ | $(79,4^{58},5)$ |
| $(30,28)$ | $\mathfrak{z}^5+\omega\mathfrak{z}^2+\omega^2\mathfrak{z}+\omega^2$ | $\mathfrak{z}^6+\omega\mathfrak{z}^5+\omega^2\mathfrak{z}^4+\omega^2\mathfrak{z}^3+\mathfrak{z}+\omega$ | $(86,4^{61},4)$ |
| $(35,30)$ | $\mathfrak{z}^5+\omega\mathfrak{z}^4+\omega^2\mathfrak{z}^2+\omega^2\mathfrak{z}+1$ | $\mathfrak{z}^6+\omega\mathfrak{z}^2+\omega\mathfrak{z}+\omega^2$ | $(95,4^{78},4)$ |
| $(17,42)$ | $\mathfrak{z}^4+\omega^2\mathfrak{z}^3+\mathfrak{z}^2+\omega^2\mathfrak{z}+1$ | $\mathfrak{z}^6+\omega^2\mathfrak{z}^5+\omega^2\mathfrak{z}^4+\omega\mathfrak{z}^3+\omega\mathfrak{z}^2+1$ | $(101,4^{85},4)$ |
| $(35,42)$ | $\mathfrak{z}^5+\omega\mathfrak{z}^4+\omega^2\mathfrak{z}^2+\omega^2\mathfrak{z}+1$ | $\mathfrak{z}^6+\omega^2\mathfrak{z}^5+\omega^2\mathfrak{z}^4+\omega\mathfrak{z}^3+\omega\mathfrak{z}^2+1$ | $(119,4^{102},4)$ |

**Table 4.** Additive codes and their Gray image

*Proof.* Suppose $\mathfrak{C}$ is a DNA code, then for any $\mathbf{c} = (\mathbf{s}, \mathbf{t}) = (s_0, s_1, \ldots\ldots, s_{n-1}, t_0, t_1, \ldots, t_{m-1}) \in \mathfrak{C}$ we get $\mathbf{c}^{rc} = (s^c_{n-1}, s^c_{n-2}, \ldots, s^c_1, s^c_0, t^c_{m-1}, t^c_{m-2}, \ldots, t^c_1, t^c_0) \in \mathfrak{C}$. Now $\mathbf{c}^{rc} \in \mathfrak{C}$ implies $\mathbf{s}^{rc} \in \mathfrak{C}_n$ and $\mathbf{t}^{rc} \in \mathfrak{C}_m$. Therefore $\mathfrak{C}_n$ and $\mathfrak{C}_m$ are DNA codes.
Conversely, suppose $\mathfrak{C}_n$ and $\mathfrak{C}_m$ are DNA codes over $\mathbb{F}_4$ and $\mathcal{R}$ respectively. Then for $\mathbf{s} \in \mathfrak{C}_n$ and $\mathbf{t} \in \mathfrak{C}_m$ we get $\mathbf{s}^{rc} \in \mathfrak{C}_n$ and $\mathbf{t}^{rc} \in \mathfrak{C}_m$. Now for $\mathbf{c} = (\mathbf{s}, \mathbf{t}) \in \mathfrak{C}_n \times \mathfrak{C}_m = \mathfrak{C}$ we have $(\mathbf{s}^{rc}, \mathbf{t}^{rc}) \in \mathfrak{C}_n \times \mathfrak{C}_m = \mathfrak{C}$. Thus, $\mathfrak{C}$ is a DNA code. □

## 4 DNA codes over $\mathcal{R}$ and $\mathbb{F}_4\mathcal{R}$

In this section, we obtain the complementary condition for the reversible codes. A code is a DNA code if it satisfies both reversible and complementary conditions. In the following lemma, we establish a relation between the code alphabets and their complement using considered Gray map (1.1).

**Lemma 4.1.** *For the given skew cyclic codes in Section 2 and 3, the following conditions hold:*

*(1) For any* $r \in \mathcal{R},\ r^c = r + 1$.

*(2) For any* $r_1, r_2 \in \mathcal{R},\ r_1^c + r_2^c = (r_1 + r_2)^c + 1$.

*(3) For* $(a, r) \in \mathbb{F}_4\mathcal{R},\ (a, r)^c = (a, r) + (1, 1)$.

**Remark 4.2.** We identify $\mathfrak{i}_n(\mathfrak{z})$ by the polynomial $1 + \mathfrak{z} + \mathfrak{z}^2 + \cdots + \mathfrak{z}^n$.

**Theorem 4.3.** *Given a polynomial* $a(\mathfrak{z})$ *of degree* $n$ *in* $\mathcal{R}[\mathfrak{z}]$, *we have*

$$a(\mathfrak{z})^{rc} = a(\mathfrak{z})^r + \mathfrak{i}_n(\mathfrak{z}).$$

*Proof: Let* $a(\mathfrak{z}) = a_0 + a_1\mathfrak{z} + \ldots + a_{n-1}\mathfrak{z}^{n-1} + \mathfrak{z}^n$ *be a polynomial of degree* $n$ *in* $\mathcal{R}[\mathfrak{z}]$ *where* $a_0$ *is a non-zero element of* $\mathcal{R}$. *Then*

$$\begin{aligned} a(\mathfrak{z})^{rc} =& a_n^c + a_{n-1}^c\mathfrak{z} + \ldots + a_1^c\mathfrak{z}^{n-1} + a_0^c\mathfrak{z}^n \\ =& a_n + 1 + (a_{n-1} + 1)\mathfrak{z} + (a_{n-2} + 1)\mathfrak{z}^2 + \ldots + \\ & (a_1 + 1)\mathfrak{z}^{n-1} + (a_0 + 1)\mathfrak{z}^n \\ =& \mathfrak{i}_n(\mathfrak{z}) + a(\mathfrak{z})^r. \end{aligned}$$

The following corollaries are obvious from the above theorems.

**Corollary 4.4.** *Let* $\mathfrak{C}$ *be a cyclic code of length* $m$ *over* $\mathcal{R}$. *If code* $\mathfrak{C}$ *is reversible and all-ones vector ( that is, the value of each entry is* 1*) is in* $\mathfrak{C}$, *then* $\mathfrak{C}$ *is a DNA code.*

**Corollary 4.5.** *Let* $\mathfrak{C}$ *be an additive code of block length* $(n, m)$ *over* $\mathbb{F}_4\mathcal{R}$. *If code* $\mathfrak{C}$ *is a reversible and all-ones vector is in* $\mathfrak{C}$, *then* $\mathfrak{C}$ *is a DNA code.*

| Length | $f(\mathfrak{z})$ | $g(\mathfrak{z})$; $\mathfrak{C} = \langle g(\mathfrak{z}), vg(\mathfrak{z})\rangle$ | Gray image |
|---|---|---|---|
| $(17,10)$ | $\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \mathfrak{z}^2 + \omega^2\mathfrak{z} + 1$ | $\mathfrak{z}^5 + \mathfrak{z}^4 + \omega\mathfrak{z}^3 + \omega\mathfrak{z}^2 + \mathfrak{z} + 1$ | $(37, 4^{23}, 4)$ |
| $(13,12)$ | $\mathfrak{z}^7 + \omega\mathfrak{z}^6 + \omega^2\mathfrak{z}^5 + \omega\mathfrak{z}^4 + \omega\mathfrak{z}^3 + \omega^2\mathfrak{z}^2 + \omega\mathfrak{z} + 1$ | $\mathfrak{z}^7 + \omega\mathfrak{z}^5 + \omega^2\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \omega\mathfrak{z}^2 + 1$ | $(37, 4^{16}, 6)$ |
| $(15,12)$ | $\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \omega\mathfrak{z}^2 + \omega^2\mathfrak{z} + 1$ | $\mathfrak{z}^5 + \omega^2\mathfrak{z}^3 + \omega^2\mathfrak{z}^2 + 1$ | $(39, 4^{25}, 4)$ |
| $(15,14)$ | $\mathfrak{z}^7 + \omega\mathfrak{z}^5 + \omega\mathfrak{z}^4 + \omega\mathfrak{z}^3 + \omega\mathfrak{z}^2 + 1$ | $\mathfrak{z}^7 + \mathfrak{z}^5 + \omega^2\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \mathfrak{z}^2 + 1$ | $(43, 4^{22}, 6)$ |
| $(20,12)$ | $\mathfrak{z}^5 + \omega\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \omega^2\mathfrak{z}^2 + \omega\mathfrak{z} + 1$ | $\mathfrak{z}^5 + \omega^2\mathfrak{z}^3 + \omega^2\mathfrak{z}^2 + 1$ | $(44, 4^{28}, 4)$ |
| $(21,14)$ | $\mathfrak{z}^7 + \omega\mathfrak{z}^6 + \mathfrak{z}^4 + \mathfrak{z}^3 + \omega\mathfrak{z} + 1$ | $\mathfrak{z}^7 + \mathfrak{z}^5 + \omega^2\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \mathfrak{z}^2 + 1$ | $(49, 4^{28}, 5)$ |
| $(17,18)$ | $\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \mathfrak{z}^2 + \omega^2\mathfrak{z} + 1$ | $\mathfrak{z}^7 + \mathfrak{z}^6 + \omega\mathfrak{z}^4 + \omega\mathfrak{z}^3 + \mathfrak{z} + 1$ | $(53, 4^{35}, 4)$ |
| $(20,20)$ | $\mathfrak{z}^5 + \omega\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \omega^2\mathfrak{z}^2 + \omega\mathfrak{z} + 1$ | $\mathfrak{z}^7 + \omega^2\mathfrak{z}^6 + \omega\mathfrak{z}^5 + \omega\mathfrak{z}^2 + \omega^2\mathfrak{z} + 1$ | $(60, 4^{41}, 4)$ |
| $(20,24)$ | $\mathfrak{z}^5 + \omega\mathfrak{z}^4 + \omega^2\mathfrak{z}^3 + \omega^2\mathfrak{z}^2 + \omega\mathfrak{z} + 1$ | $\mathfrak{z}^7 + \mathfrak{z}^6 + \omega\mathfrak{z}^5 + \omega\mathfrak{z}^2 + \mathfrak{z} + 1$ | $(68, 4^{49}, 4)$ |

**Table 5.** DNA codes from $\mathbb{F}_4\mathcal{R}-$skew cyclic codes

## 5 Conclusion

In this work, we have studied the reversible and the DNA codes over a chain-ring $\mathcal{R} = \mathbb{F}_4[v]/\langle v^2\rangle$ and $\mathbb{F}_4\mathcal{R}$. First, we gave a table for a string of oligonucleotides corresponding to the elements of the respective structure using a Gray map. Then, we have discussed the structure of reversible and DNA codes over these structures. Interestingly, we found some better DNA codes using this method.

## Declarations

**Data Availability Statement**: The authors declare that [the/all other] data supporting the findings of this study are available within the article. Any clarification may be requested from the corresponding author, provided it is essential.

**Competing interests**: The authors declare that there is no conflict of interest regarding the publication of this manuscript.

**Use of AI tools declaration** The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this manuscript.

## References


[1] T. Abualrub and I. Siap, *Cyclic codes over the rings $Z_2 + uZ_2$ and $Z_2 + uZ_2 + u^2Z_2$*, Des. Codes, Cryptogr., **42**(3), 273-287, (2007).

[2] L. M. Adleman, *Molecular computation of solutions to combinatorial problems*, Science, **266**(5187), 1021-1024, (1994).

[3] M. R. Alimoradi, *Decoding of cyclic codes over a class of finite rings*, Palest. J. Math., **12**(2), 321–328, (2023).

[4] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system I: The user language*, J. Symb. Comput., **24**(3-4), 235-265, (1997).

[5] D. Boucher, W. Geiselmann, and F. Ulmer, *Skew-cyclic codes*, Appl. Algebra Eng. Commun. Comput., **18**(4), 379-389, (2007).

[6] D. Boucher, P. Solé and F. Ulmer, *Skew constacyclic codes over Galois rings*, Adv. Math. Commun., **2**(3), 273, (2008).

[7] D. Boucher, and F. Ulmer, *Coding with skew polynomial rings*, J. Symb. Comput., **44**(12), 1644-1656, (2009).

[8] H. Q. Dinh, S. Pathak, A. K. Upadhyay and W. Yamaka, *New DNA codes from cyclic codes over mixed alphabets*, Mathematics, **8**(11), 1977, (2020).

[9] K. Guenda and T. A. Gulliver, *Construction of cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$ for DNA computing*, Appl. Algebra Eng. Commun. Comput., **24**(6), 445-459, (2013).

[10] F. Gursoy, E. S. Oztas, and I. Siap, *Reversible DNA codes using skew polynomial rings*, Appl. Algebra Eng. Commun. Comput., **28**(4), 311-320, (2017).

[11] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. Sloane, and P. Solé, *The $\mathbb{Z}_4-$linearity of Kerdock, Preparata, Goethals, and related codes*, IEEE Trans. Inf. Theory, **40**(2), 301-319, (1994).

[12] S. Jitman, S. Ling, and P. Udomkavanich, *Skew constacyclic codes over finite chain rings*, Adv. Math. Commun., **6**(1), 39, (2012).

[13] S. Ling, and C. Xing, *Coding theory: a first course*, Cambridge University Press (2004).

[14] J. L. Massey, *Reversible codes*. Inf. Control., **7**(3), 369-380, (1964).

[15] O. Prakash, A. Singh, R. K. Verma, P. Solé, and W. Cheng, *DNA Code from Cyclic and Skew Cyclic Codes over $F_4[v]/\langle v^3\rangle$*, Entropy, **25**(2), Paper No. 239, 12 pp, (2023).

[16] O. Prakash, R.K. Verma, and A. Singh, *Quantum and LCD codes from skew constacyclic codes over a finite non-chain ring*, Quantum Inf. Process., **22**(5) (2023), Paper No. 200, 25 pp. https://doi.org/10.1007/s11128-023-03951-0

[17] O. Prakash, S. Yadav, and P. Sharma, *Reversible cyclic codes over a class of chain rings and their application to DNA codes*, Int. J. Inf. Coding Theory, **6**(1), 52-70, (2022).

[18] N. Rehman, M. Azmi and G. Mohammad, *Reversible cyclic codes over $F_2 + uF_2 + u^2F_2 + \cdots + u^{k-1}F_2$*, Palest. J. Math., **12**(2), 395–407, (2023).

[19] I. Siap, T. Abualrub, N. Aydin, and P. Seneviratne, *Skew cyclic codes of arbitrary length*, Int. J. Inf. Coding Theory, **2**(1), 10-20, (2011).

[20] R. K. Verma, O. Prakash, H. Islam, and A. Singh, *New non-binary quantum codes from skew constacyclic and additive skew constacyclic codes*, Eur. Phys. J. Plus, **137**(2), 1-13, (2022).

[21] R. K. Verma, O. Prakash, A. Singh and H. Islam, *New quantum codes from skew constacyclic codes*, Adv. Math. Commun., **17**(4), 900-919, (2023).

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