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MULTIPLICITY RESULT FOR A CLASS OF NONHOMOGENOUS P-KIRCHHOFF SYSTEM IN $\mathbb{R}^N$

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Abstract *In this paper we study the existence of solutions to the following nonhomogenous p-Kirchhoff elliptic systems in $\mathbb{R}^N$.*

$$\begin{cases} -M\left(\int_{\mathbb{R}^N}\left(|\nabla u|^p+V|u|^p\right)dx\right)\left(\Delta_p u+V|u|^{p-2}u\right)=f_1(x,u,v) & in\ \mathbb{R}^N,\\ -M\left(\int_{\mathbb{R}^N}\left(|\nabla v|^p+V|v|^p\right)dx\right)\left(\Delta_p v+V|v|^{p-2}v\right)=f_2(x,u,v) & in\ \mathbb{R}^N,\\ u(x)\to 0, v(x)\to 0\ as\ |x|\to\infty. \end{cases}$$

Under more relaxed assumptions on $V(x)$ and f_1, f_2. The solutions will be obtained by the Mountain Pass Theorem, Eklend's variational principle and Nehari manifold.

1 Introduction

In this paper we examine the multiplicity results of nontrivial solutions to the following nonhomogenous p-Kirchhoff system

$$\begin{cases} -M\left(\int_{\mathbb{R}^N}\left(|\nabla u|^p+V(x)|u|^p\right)dx\right)\left(\Delta_p u+V(x)|u|^{p-2}u\right)=f_1(x,u,v) & in\ \mathbb{R}^N,\\ -M\left(\int_{\mathbb{R}^N}\left(|\nabla v|^p+V(x)|v|^p\right)dx\right)\left(\Delta_p v+V(x)|v|^{p-2}v\right)=f_2(x,u,v) & in\ \mathbb{R}^N,\\ u(x)\to 0, v(x)\to 0\ \ as\ |x|\to\infty. \end{cases} \tag{1.1}$$

where $f_1(x,u,v)=\frac{1}{r}F_u(u,v)+\lambda|u|^q+g(x)$, $f_2(x,u,v)=\frac{1}{r}F_v(u,v)+\mu|v|^q+h(x)$.

The function F is assumed to be a class C^1 in $\mathbb{R}^2$, and $\Delta_p u=\text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian operator with $1<p<N$, and the functions $g(x), h(x)$ can be seen as a perturbations terms.

Recently, many authors consider the following Kirchhoff-type problem:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u), \tag{1.2}$$

where $a>0, b>0$ are constantants. problem (1.2) is an important nonlocal quasilinear problem because of the appearance of the term $(\int_{\mathbb{R}^N}|\nabla u|^2dx)\Delta u$, which provokes some mathematical difficulties and also makes the study of such a class of problem particularly interesting.

In [18] using Ekeland's variational principle, Corra and Nascimento proved the existence of a weak solution for the boundary problem associated with the nonlocal elliptic system of p-Kirchhoff type.

$$\begin{cases} -\left(M_1 \int\limits_{\Omega} |\nabla u|^p \, dx\right)^{p-1} \Delta_p u = f(u,v) + \rho_1(x) & in\ \Omega, \\ -\left(M_2 \int\limits_{\Omega} |\nabla v|^p \, dx\right)^{p-1} \Delta_p v = g(u,v) + \rho_2(x) & in\ \Omega, \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial u}{\partial \eta} = 0 & on\ \partial\Omega. \end{cases}$$

Wu in [19] obtained five new critical point theorems on the product space and three existence theorems for a sequence of high energy solutions for the following system of Kirchhoff-type:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + V(x)u = F_u(x,u) & in\ \mathbb{R}^N, \\ -\left(a + b\int_{\mathbb{R}^N} |\nabla v|^2 \, dx\right) \Delta u + V(x)v = F_v(x,u) & in\ \mathbb{R}^N, \\ u(x) \to 0, v(x) \to 0 \ \ as\ |x| \to \infty. \end{cases} \tag{1.3}$$

The purpose of this paper is to study the existence and multiplicity results for a coupled system of Kirchhoff type equations in $\mathbb{R}^N$ under some natural assumptions. We will get the existence and multiplicity results of nontrivial solutions by exploiting the Nehari manifold method and the mountain-pass theorem, Ekeland's variational principle.

To state our main theorems, let us introduce the following hypotheses.

We assue that $M(t) = t^k, k > 0, t \geq 0$ and V is a continuous santiying

(H_1) *there exist $b_0 > 0$ such that $V(x) \geq b_0$ in $\mathbb{R}^N$. Morever $V(x) \to +\infty$ as $|x| \to +\infty$.*

(H_2) *Let $F(u,v) \in C^1(\mathbb{R}^2)$ be positively homogeneous of degree $r \in (p, p^*)$, that is, $F(tu, tv) = t^r F(u,v), (t > 0)$ for any $(u,v) \in \mathbb{R}^2$. Also, assume $F(u,0) = F(0,v) = F_u(u,0) = F_v(0,v) = 0$ and $F(u,v) > 0$ for any $(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Furthermore, there exists a consatant $k_1 > 0$ such that*

$$0 \leq F(u,v) \leq k_1 \left(|u|^r + |v|^r\right), \quad \forall (u,v) \in \mathbb{R}^2, \tag{1.4}$$

and for all $(u,v) \in \mathbb{R}^2$,

$$\begin{aligned} |F_u(u,v)| &\leq k_1 \left(|u|^{r-1} + |v|^{r-1}\right), \\ |F_v(u,v)| &\leq k_1 \left(|u|^{r-1} + |v|^{r-1}\right), \end{aligned} \tag{1.5}$$

with $p(k+1) < r < p^*$.

By hypothesis (H_2), we have the so-called Euler identity

$$F_u(u,v)u + F_v(u,v)v = rF(u,v), \quad \forall (u,v) \in \mathbb{R}^2. \tag{1.6}$$

Clearly, the function $F(u,v) = |u|^\alpha |v|^\beta$ with $\alpha + \beta = r$ and $F(u,v) = (u^2+v^2)^{r/2}$ satisfy (H_3).

This work is organized as follows: in section 2 we present some preliminary results and in sectin 3 and 4, we prove the main results.

Theorem 1.1. *Let $g, h \in L^{p'}(\mathbb{R}^N)$ and $g, h \neq 0$ in $\mathbb{R}^N$. Assume that $(H_1), (H_2)$ holds and $1 < p < q < p(k+1) < r < p^*$.*

Then there exist $C_0, C_1 > 0$ such that the problem (1.1) has at least two nontrivial weaks solutions provided

$\left(|\lambda|^{m/m-q} + |\mu|^{m/m-q}\right) < C_0$, *and* $\|g\|_{p'}^{m/m-1} + \|h\|_{p'}^{m/m-1} < C_1 \left(|\lambda|^{m/m-q} + |\mu|^{m/m-q}\right)^{\frac{m-q}{r-q}}$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $m = p(k+1)$.

Theorem 1.2. *Assume (H_1), (H_2) and $1 < p(k+1) \leq q < r < p^*$ hold. Then for any $\lambda, \mu \in \mathbb{R}$, the system (1.1) with $g = h = 0$ admits at least one a pair of solution.*

2 Preliminaries

We introduce some Sobolev space $X = W^{1,p}(\mathbb{R}^N)$ endowed with the norm,

$$\|u\|_X^p = \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx\,, \quad 1 \leq p < \infty$$

The norm in $L^p(\Omega)$ will be denoted by,

$$\|u\|_p^p = \int_{\mathbb{R}^N} |u|^p\, dx.$$

We now consider the following subspace

$$E = \left\{ u \in X \mid \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)\, |u|^p) dx < \infty \right\}. \tag{2.1}$$

E is a Banch space with the norme

$$\|u\|_E^p = \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)\, |u|^p) dx. \tag{2.2}$$

Obviously, we have

$$\|u\|_X \leq \|u\|_E\,, \quad \forall u \in X.$$

The continous embeddings

$$E \hookrightarrow X \hookrightarrow L^q(\mathbb{R}^N) \;\; \text{and} \;\; \|u\|_q \leq S_q \|u\|_X \leq S_q \|u\|_E \quad \forall u \in X \tag{2.3}$$

where $p \leq q \leq p^*$ and $S_q > 0$, see [21, 22]

The following Sobolev inequality [21] is well known. There is a constant $S > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^N)$,

$$S \left(\int_{\mathbb{R}^N} |u|^{p^*}\, dx \right)^{p/p^*} \leq \int_{\mathbb{R}^N} |\nabla u|^p\, dx.$$

Lemma 2.1. *Let (H_1) hold true. Then embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact.*

The proof for Lemma 2.1 in [20]

For the product space $Y = E \times E$, the norme of $(u, v) \in Y$, is defined by

$$\|(u, v)\|^p = \|u\|_E^p + \|v\|_E^p\,.$$

Lemma 2.2. *Under assuption (H_1), the embedding $Y \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ is continuous for $p \leq r \leq p^*$ and $Y \hookrightarrow L^r_{loc}(\mathbb{R}^N) \times L^r_{loc}(\mathbb{R}^N)$ is compact for $p \leq r < p^*$.*

Proof. By [23], we know under the assuption (H_1) the embeding $E \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [p, p^*]$,and $E \hookrightarrow L^r_{loc}(\mathbb{R}^N)$ is compact for $r \in [p, p^*)$,that is, there exist constante $S_r > 0$ such that $\|u\|_r \leq S_r \|u\|_E$, $\forall u \in E$ and for any bounded sequence $\{u_n\} \subset E$, there exists a subsequence of $\{u_n\}$ such that $u_n \rightharpoonup u_0$ in E ans $u_n \to u_0$ in $L^r_{loc}(\mathbb{R}^N)$, $r \in [p, p^*)$. Then for any $(u, v) \in Y$, there exist $C > 0$ such that

$$\|(u, v)\|_r^r \leq S_r^r \left(\|u\|_E^r + \|v\|_E^r\right) \leq S_r^r \|(u, v)\|^r , \tag{2.4}$$

that is, $\|(u, v)\|_r^r \leq S_r^r \|(u, v)\|^r$, that is $Y \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$ is continuous for $r \in [p, p^*]$ .On the other hand, suppose $\{(u_n, v_n)\} \subset Y$ are bouneded, that is, $\{u_n\}$ and $\{v_n\}$ are bounded in E, then there exist $\{u_n\}$ and $\{v_n\}$ such that

$$u_n \rightharpoonup u_0, v_n \rightharpoonup v_0 \text{ in } L^r_{loc}(\mathbb{R}^N),\ r \in [p, p^*) .$$

Therefor,

$$\|(u_n, v_n) - (u_0, v_0)\|_r^r \leq S_r^r \left(\|u_n - u_0\|_r^r + \|v_n - v_0\|_r^r\right) \to 0, \text{ as } n \to \infty,$$

that is

$$(u_n, v_n) \to (u_0, v_0), \text{ in } L^r_{loc}(\mathbb{R}^N) \times L^r_{loc}(\mathbb{R}^N),\ r \in [p, p^*) ,$$

that is, $Y \hookrightarrow L^r_{loc}(\mathbb{R}^N) \times L^r_{loc}(\mathbb{R}^N)$ is compact for $p \leq r < p^*$. The proof is completed. □

Definition 2.3. We say that (u, v) is a weak solution to (1.1) if
for all $(\varphi_1, \varphi_2) \in Y$, we have

$$\begin{aligned}
&\|u\|_E^{pk} \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla \varphi_1 + V |u|^{p-2} u\varphi_1) dx \\
&+ \|v\|_E^{pk} \int_{\mathbb{R}^N} (|\nabla v|^{p-2} \nabla v \nabla \varphi_2 + V |v|^{p-2} v\varphi_2) dx \\
&- \frac{1}{r} \int_{\mathbb{R}^N} \left(F_u(u, v)\varphi_1 + F_v(u, v)\varphi_2\right) dx \\
&- \int_{\mathbb{R}^N} \left(\lambda |u|^{q-2} u\varphi_1 + \mu |v|^{q-2} \varphi_2\right) dx \\
&- \int_{\mathbb{R}^N} (g\varphi_1 + h\varphi_2) dx \\
&= 0.
\end{aligned}$$

We see that weak solutions of system (1.1) are critical points of the functional $I : Y \to \mathbb{R}$ given by,

$$\begin{aligned}
I(u, v) = &\frac{1}{m} \left(\|u\|_E^m + \|v\|_E^m\right) - \frac{1}{r} \int_{\mathbb{R}^N} F(u, v) dx \\
&- \frac{1}{q} \int_{\mathbb{R}^N} \left(\lambda |u|^q + \mu |v|^q\right) dx - \int_{\mathbb{R}^N} (gu + hv) dx.
\end{aligned}$$

Definition 2.4. Let $c \in \mathbb{R}$, X be a Banach space and $I \in C^1(X, \mathbb{R})$

(i) $\{z_n\}$ is a $(PS)_c-$sequence in X for I if $I(z_n) = c + o(1)$ and $I'(z_n) = o(1)$ strongly in X^{-1} as $n \to \infty$.

(ii) We say that I satisfies the (PS) condition if any $(PS)_c$-sequence $\{z_n\}$ in X for I has a convergent subsequence.

Lemma 2.5. *[14](Mountain Pass Theorem)*
Suppose X is a Banach space, $I \in C^1(X,\mathbb{R})$ with $I(0)=0$. If I satisfies (PS) condition and
(A_1) there are $\rho, \alpha_0 > 0$, such that $I(u) \geq \alpha_0$ when $\|u\|_X = \rho$
(A_2) there is $e \in X$, $\|e\|_X > \rho$ such that $I(e) < 0$.
Define

$$\Gamma = \left\{\gamma \in C^1([0,1], X,) : \gamma(0) = o,\ \gamma(1) = e\right\}.$$

Then

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)) \geq \alpha_0$$

is a critical value of I.

3 Proof of Theorem 1.1

Lemma 3.1. *Assume $(H_1),(H_2)$ and (H_3) holds.Then there exist $C_0 > 0$ such that $I(u,v)$ satisfies the assuptions $(A_1)-(A_2)$ in lemma 2.5 provided*
$\left(|\lambda|^{m/m-q} + |\mu|^{m/m-q}\right) < C_0$, *and* $\|g\|_{p'}^{m/m-1} + \|h\|_{p'}^{m/m-1} < C_1\left(|\lambda|^{m/m-q} + |\mu|^{m/m-q}\right)^{\frac{m-q}{r-q}}$
where $\frac{1}{p} + \frac{1}{p'} = 1$ and $m = p(k+1)$.

Proof. In fact, it follows from (H_3) and (2.4), we have

$$I(u,v) \geq \frac{1}{2^k m}\left((\|u\|_E^p + \|v\|_E^p)\right)^{k+1} - \frac{1}{r}\int_{\mathbb{R}^N} F(u,v)dx \tag{3.1}$$
$$- \frac{1}{q}\int_{\mathbb{R}^N}(\lambda|u|^q + \mu|v|^q)\,dx - \int_{\mathbb{R}^N}(gu+hv)\,dx,$$

$$I(u,v) \geq \frac{1}{2^k m}\|(u,v)\|^m - \frac{k_1}{r}\int_{\mathbb{R}^N}(|u|^r + |v|^r)\,dx \tag{3.2}$$
$$- \frac{1}{q}\int_{\mathbb{R}^N}(\lambda|u|^q + \mu|v|^q)\,dx - \|g\|_{p'}\|u\|_E - \|h\|_{p'}\|v\|_E.$$

Using Young inequality, we get

$$\|g\|_{p'}\|u\|_E \leq \frac{\varepsilon_0^m}{m}\|u\|_E^m + C_2\|g\|_E^{m/m-1}, \tag{3.3}$$

where $C_2 = C_2(m,\varepsilon_0) = (m-1)\varepsilon_0^{m/m-1}/m$ and $\varepsilon_0 \in (0, e^{-(k+1)\ln 2/m})$.

$$I(u,v) \geq \frac{1}{2^k m}\|(u,v)\|^m - \frac{k_1}{r}S_r^r\|(u,v)\|^r$$
$$- \frac{S_q^q}{q}(|\lambda|^\theta + |\mu|^\theta)^{1/\theta}\|(u,v)\|^q - \frac{2\varepsilon_0^m}{m}\|(u,v)\|^m$$
$$- C_2\left(\|g\|_{p'}^{m/m-1} + \|h\|_{p'}^{m/m-1}\right)$$

with $\theta = \frac{m}{m-q}$.

$$I(u,v) \geq \left(\frac{1-2^{k+1}\varepsilon_0^m}{2^k m}\right)\|(u,v)\|^m - \frac{k_1}{r}S_r^r\|(u,v)\|^r \tag{3.4}$$
$$- \frac{S_q^q}{q}(|\lambda|^\theta + |\mu|^\theta)^{1/\theta}\|(u,v)\|^q$$
$$- C_2\left(\|g\|_{p'}^{m/m-1} + \|h\|_{p'}^{m/m-1}\right).$$

Let $\alpha(t) = t^m (a_1 - \beta(t))$, with $\beta(t) = a_2 t^{r-m} + a_3 t^{q-m}$ for $t \geq 0$, where

$$a_1 = \frac{1 - 2^{k+1}\varepsilon_0^m}{2^k m}, \ a_2 = k_1 S_r^r \text{ and } a_3 = \frac{S_q^q}{q}(|\lambda|^\theta + |\mu|^\theta)^{1/\theta}$$

Note that $\beta(t) \to +\infty$ as $t \to +\infty$.The function β has a minimum at:

$$t_{\min} = \left(\frac{(m-q)\, a_3}{(r-m)\, a_2}\right)^{1/r-q} > 0.$$

Moveover $\beta(t) < a_1$ implies that

$$a_4(|\lambda|^{m/m-q} + |\mu|^{m/m-q})^{\frac{(m-q)(r-m)}{m(r-q)}} < a_1$$

with $a_4 = a_2 \left(\frac{(m-q)S_q^q}{q(r-m)a_2}\right)^{1/r-q} + \frac{1}{q}S_q^q$
and consequently there existe some $C_0 > 0$ such that

$$(|\lambda|^{m/m-q} + |\mu|^{m/m-q}) < C_0$$

To verify (A_1) in Lemma 2.5 it suffice to show that

$$\alpha(t_{\min}) - C_2\left(\|g\|_{p'}^{m/m-1} + \|h\|_{p'}^{m/m-1}\right) > 0 \tag{3.5}$$

We deduce that for some constant $C_1 > 0$

$$\|g\|_{p'}^{m/m-1} + \|h\|_{p'}^{m/m-1} < C_1\left(|\lambda|^{m/m-q} + |\mu|^{m/m-q}\right)^{m-q/r-q}. \tag{3.6}$$

Then from (3.4) and (3.6), that there exist C_0, C_1 and $\alpha_0 > 0$ such that
$I(u,v) \geq \alpha_0$ with $\|g\|_{p'}^{m/m-1} + \|h\|_{p'}^{m/m-1} < C_0$ and
$\|g\|_{p'}^{m/m-1} + \|h\|_{p'}^{m/m-1} < C_1\left(|\lambda|^{m/m-q} + |\mu|^{m/m-q}\right)^{m-q/r-q}$.
Thus (A_1) in Lemma 2.5 is true.

We now verify (A_2) in Lemma 2.5.
Choose $(\varphi_1, \varphi_2) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ such that $F(\varphi_1, \varphi_2) > 0$. Then

$$\begin{aligned} I(t\varphi_1, t\varphi_2) &= \frac{t^m}{m}(\|\varphi_1\|_E^m + \|\varphi_2\|_E^m) - \frac{t^r}{r}\int_{\mathbb{R}^N} F(\varphi_1,\varphi_2)dx \\ &\quad - \frac{t^q}{q}\int_{\mathbb{R}^N}(\lambda|\varphi_1|^q + \mu|\varphi_2|^q)\,dx - t\int_{\mathbb{R}^N}(g\varphi_1 + h\varphi_2)\,dx. \end{aligned}$$

and $I(t\varphi_1, t\varphi_2) \to -\infty$ as $t \to +\infty$ since $m < r$. Therefore, there there exists t large enough, such that $I(t\varphi_1, t\varphi_2) < 0$. Then , we take $e = (t\varphi_1, t\varphi_2) \in Y$ and $I(e) < 0$ and (A_2) in Lemma 2.5 is true. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $c \in \mathbb{R}$. Then each $(PS)_c$ sequence for I is bounded in Y.*

Proof. Let $\{(u_n, v_n)\}$ be an arbitrary $(PS)_c$ sequence of I in Y, that is

$$I(u_n, v_n) \to c, \quad I'(u,v) \to 0 \text{ in } Y^{-1} \tag{3.7}$$

$$
\begin{aligned}
c+1+\|(u_n,v_n)\| &\geq I(u_n,v_n)-\frac{1}{r}\langle I(u_n,v_n),(u_n,v_n)\rangle\\
&=(m^{-1}-r^{-1})\left(\|u\|_E^m+\|u\|_E^m\right)\\
&-r^{-1}\int_{\mathbb{R}^N}(rF(u_n,v_n)-F_u(u_n,v_n)u_n-F_v(u_n,v_n)v_n)\,dx\\
&-(q^{-1}-r^{-1})\int_{\mathbb{R}^N}(\lambda|u|^q+\mu|v|^q)\,dx+(r^{-1}-1)\int_{\mathbb{R}^N}(gu_n+hv_n)\,dx\\
&\geq 2^{-k}(m^{-1}-r^{-1})\|(u,v)\|^m-S_q^q(q^{-1}-r^{-1})(|\lambda|^\theta+|\mu|^\theta)^{1/\theta}\|(u,v)\|^m\\
&+(r^{-1}-1)\max(\|g\|_{p'},\|h\|_{p'})\|(u_n,v_n)\|.
\end{aligned}
$$

Since $1<p<q<m$, we conclude that $\{(u_n,v_n)\}$ is bounded in Y. □

Lemma 3.3. *The functional I satisfies (PS) condition on Y.*

Proof. Let $\{(u_n,v_n)\}$ be an arbitrary $(PS)_c$ sequence of I in Y

By Lemma 3.2 $\{(u_n,v_n)\}$ is bounded in Y. Then there exist subsequence (still denote by $\{(u_n,v_n)\}$) and $(u,v)\in X$ such that $\|(u_n,v_n)\|\to t_0\geq 0$.

If $t_0=0$, then the proof is finished. In the following, we assume $t_0>0$. Then for n sufficiently large, $\|(u_n,v_n)\|\geq \frac{1}{2}t_0>0$.

We now show that $\{(u_n,v_n)\}$ has convergent subsequence in Y.

$$u_n\rightharpoonup u,\ v_n\rightharpoonup v \quad \text{weakly in } E. \tag{3.8}$$

$$u_n\rightharpoonup u,\ v_n\rightharpoonup v \quad \text{a.e in } \mathbb{R}^N, \tag{3.9}$$

and by Lemma 2.1

$$u_n\to u,\ v_n\to v \quad \text{strongly in } L^p(\mathbb{R}^N), \tag{3.10}$$

we assume $\|(u_n,v_n)\|\leq M$ for some constant $M>0$, and all $n\in\mathbb{N}$.

Let

$$
\begin{aligned}
P_n &= \langle I'(u_n,v_n),(u_n-u,v_n-v)\rangle\\
&=\|u_n\|_E^{pk}\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)+V|u_n|^{p-2}u_n(u_n-u)dx\\
&+\|v_n\|_E^{pk}\int_{\mathbb{R}^N}|\nabla v_n|^{p-2}\nabla v_n\nabla(v_n-v)+V|v_n|^{p-2}v_n(v_n-v)dx \qquad (3.11)\\
&-\frac{1}{r}\int_{\mathbb{R}^N}(F_u(u_n,v_n)(u_n-u)+F_v(u_n,v_n)(v_n-v))\,dx\\
&-\int_{\mathbb{R}^N}\lambda|u_n|^{q-2}u_n(u_n-u)dx-\int_{\mathbb{R}^N}\mu|v_n|^{q-2}v_n(v_n-v)dx\\
&-\int_{\mathbb{R}^N}g(u_n-u)\,dx-\int_{\mathbb{R}^N}h(v_n-v)\,dx. \qquad (3.12)
\end{aligned}
$$

The fact $I'(u_n,v_n)\to 0$ in Y^* implies that $P_n\to 0$ as $n\to\infty$. Similarly, the fact $u_n\rightharpoonup u$, $v_n\rightharpoonup v$ in Y implies that $Q_n\to 0$ as $n\to\infty$, where

$$
\begin{aligned}
Q_n &= \|u_n\|_E^{pk}\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla(u_n-u)+V|u|^{p-2}u(u_n-u)dx \qquad (3.13)\\
&+\|v_n\|_E^{pk}\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla(v_n-v)+V|v|^{p-2}v(v_n-v)dx.
\end{aligned}
$$

We define,

$$R_n = \int_{\mathbb{R}^N} F_u(u_n, v_n)\,(u_n - u)\,dx \tag{3.14}$$

$$S_n = \int_{\mathbb{R}^N} F_v(u_n, v_n)\,(v_n - v)\,dx \tag{3.15}$$

$$T_n = \int_{\mathbb{R}^N} g\,(u_n - u)\,dx \tag{3.16}$$

$$K_n = \int_{\mathbb{R}^N} h\,(v_n - v)\,dx \tag{3.17}$$

$$D_n = \int_{\mathbb{R}^N} \lambda\,|u_n|^{q-2}\,u_n(u_n - u)dx \tag{3.18}$$

$$L_n = \int_{\mathbb{R}^N} \mu\,|v_n|^{q-2}\,v_n(v_n - v)dx \tag{3.19}$$

We now prove $R_n \to 0, S_n \to 0, T_n \to 0$ and $k_n \to 0$ as $n \to \infty$.
It follows from the assuption (H_3) and by Hleder's ineqaulity, we have

$$\begin{aligned} |R_n| &\le \int_{\mathbb{R}^N} F_u(u_n, v_n)\,(u_n - u)\,dx \\ &\le k_1 \int \left(|u|^{r-1} + |v|^{r-1}\right)|u_n - u_n|\,dx \\ &\le k_1 \left(\|u\|_r^{r-1} + \|u\|_r^{r-1}\right)\|u_n - u_n\|_r \\ &\le k_1 S_r \left(\|u\|_E^{r-1} + \|u\|_E^{r-1}\right)\|u_n - u_n\|_r \\ &\le 2k_1 S_r M^{r-1}\,\|u_n - u\|_r\,. \end{aligned} \tag{3.20}$$

By interpolation inequality we have

$$\|u_n - u\|_r \le \|u_n - u\|_p^{\theta}\,\|u_n - u\|_{p^*}^{1-\theta}, \tag{3.21}$$

withe $\theta \in (0,1)$ and $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$

Since $\{u_n\}$ bounded in E, then $\{u_n\}$ is bounded in $L^{p^*}(\mathbb{R}^N)$. Morover, il follows from (3.21) that $\|u_n - u_n\|_r \to 0$, as $n \to \infty$ and thus, $R_n \to 0$ as.$n \to \infty$
Similarly, we have

$$|S_n| \le 2k_1 S_r M^{r-1}\,\|v_n - v\|_r\,.$$

And $S_n \to 0$ as $n \to \infty$,
we have

$$\begin{aligned} |T_n| &\le \int_{\mathbb{R}^N} |g|\,|u_n - u|\,dx \\ &\le \|g\|_{p'}\,\|u_n - u\|_p \to 0 \text{ as } n \to \infty \end{aligned} \tag{3.22}$$

We prove $\|(u_n - u, v_n - v)\| \to 0$ in Y.
Notice that

$$P_n - Q_n = \|u_n\|_E^{pk}\,U_n + \|v_n\|_E^{pk}\,V_n - R_n - S_n - T_n - K_n. \tag{3.23}$$

Where

$$U_n = \int_{\mathbb{R}^N} \left(\left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \nabla (u_n - u) \right) dx + \int_{\mathbb{R}^N} \left(V \left(|u_n|^{p-2} u_n - |u|^{p-2} u \right) (u_n - u) \right) dx, \tag{3.24}$$

and

$$V_n = \int_{\mathbb{R}^N} \left(\left(|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v \right) \nabla (v_n - v) \right) + \int_{\mathbb{R}^N} \left(V \left(|v_n|^{p-2} v_n - |v|^{p-2} v \right) (v_n - v) \right) dx. \tag{3.25}$$

Using the standard inequality in $\mathbb{R}^N$.
For any $x, y \in \mathbb{R}^N$

$$\left\langle |x|^{p-2} x - |y|^{p-2} y, x - y \right\rangle \geq C_p |x - y|^p, \qquad p \geq 2, \tag{3.26}$$

and

$$\left\langle |x|^{p-2} x - |y|^{p-2} y, x - y \right\rangle \geq \frac{C_p |x - y|^2}{(|x| + |y|)^{2-p}}, \qquad 1 < p < 2. \tag{3.27}$$

Then, $P_n - Q_n \to 0$ as $n \to \infty$, that $U_n \to 0$, $V_n \to 0$ as $n \to \infty$.
We obtain $\|(u_n - u, v_n - v)\| \to 0$ in Y. Thus $I(u, v)$ satisfies (PS) on Y and we finish the proof of Lemma 3.3. □

By Lemma 3.1 and Lemma 3.3 I satisfies all assumptions in Lemma 2.5. Then there exists $(u_1, v_1) \in Y$ such that (u_1, v_1) is a solution of problem (1.1) Furthermore $I(u_1, v_1) \geq \alpha_0 > 0$.

We now seek a solution (u_2, v_2) of problem (1.1).
Choose $(\varphi_1, \varphi_2) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} (g\varphi_1 dx + h\varphi_2)\, dx > 0$ and

$$I(t\varphi_1, t\varphi_2) = \frac{t^m}{m} \left(\|\varphi_1\|_E^m + \|\varphi_2\|_E^m \right) - \frac{t^r}{r} \int_{\mathbb{R}^N} F(\varphi_1, \varphi_2) dx - t \int_{\mathbb{R}^N} (gu + hv)\, dx. \tag{3.28}$$

Since $r > m$, we have from (3.28) that $I(t\varphi_1, t\varphi_2) < 0$ for $t > 0$ samll. Thus

$$-\infty < c_\rho = \inf_{\overline{B}_\rho} I(u, v) < 0 \text{ and } \inf_{\partial B_\rho} I(u, v) > 0, \tag{3.29}$$

where ρ given in Lemma 2.5 and B_ρ is open ball in Y centered at the origin with radius ρ
Let $\varepsilon_n \to 0$ such that

$$c_\rho < \varepsilon_n < \inf_{B_\rho} I(u, v) - \inf_{\partial B_\rho} I(u, v) \tag{3.30}$$

Then, by Ekeland's variational principle in [15]

$$c_\rho \leq I(u_n, v_n) < c_\rho + \varepsilon_n \tag{3.31}$$

and

$$I(u_n, v_n) < I(u, v) + \varepsilon_n \|(u_n - u, v_n - v)\| \tag{3.32}$$

Then it follows from (3.29) and (3.31) that

$$I(u_n, v_n) < c_\rho + \varepsilon_n \leq \inf_{B_\rho} I(u, v) + \varepsilon_n < \inf_{\partial B_\rho} I(u, v) \tag{3.33}$$

so that $(u_n, v_n) \in B_\rho$.
We now consider the functional $F : \overline{B}_\rho \to \mathbb{R}$ *given by*

$$F(u, v) = I_{\lambda,\mu}(u_n, v_n) + \varepsilon_n \|(u_n - u, v_n - v)\| \tag{3.34}$$

Then (3.32)*shows that* $F(u_n, v_n) < F(u, v)$ *for* $(u, v) \in \overline{B}_\rho, (u_n, v_n) \neq (u, v)$ *and thus* (u, v) *is a strict local minimum of* F.
Moreover

$$t^{-1} \left(F(u_n + t\varphi_1, v_n + t\varphi_2) - F(u_n, v_n)\right) \geq 0 \tag{3.35}$$

for small $t > 0$ *and* $\|(\varphi_1, \varphi_2)\| \leq 1$. *Hence*

$$t^{-1} \left(I(u_n + t\varphi_1, v_n + t\varphi_2) - I(u_n, v_n)\right) + \varepsilon_n \|(\varphi_1, \varphi_2)\| \geq 0. \tag{3.36}$$

Let $t \to 0^+$

$$\langle I'(u_n, v_n), (\varphi_1, \varphi_2)\rangle + \varepsilon_n \|(\varphi_1, \varphi_2)\| \geq 0, \ \ \forall(\varphi_1, \varphi_2) \in B_1 \tag{3.37}$$

Replacing (φ_1, φ_2) *in* (3.37) *by* $(-\varphi_1, -\varphi_2)$, *we get*

$$-\langle I'(u_n, v_n), (\varphi_1, \varphi_2)\rangle + \varepsilon_n \|(\varphi_1, \varphi_2)\| \geq 0, \ \ \forall(\varphi_1, \varphi_2) \in B_1. \tag{3.38}$$

So that

$$\|I'(u_n, v_n)\| \leq \varepsilon. \tag{3.39}$$

Therfore, there is a sequence $\{(u_n, v_n)\} \subset B_\rho$ *such that* $I'_{\lambda,\mu}(u_n, v_n) \to c_\rho < 0$ *and* $I'_{\lambda,\mu}(u_n, v_n) \to 0$ *in* Y^{-1}*as* $n \to \infty$.

By Lemma 3.3 $\{(u_n, v_n)\}$ *has a convergent subsequence in* X, *still denoted by* $\{(u_n, v_n)\}$, *such that* $(u_n, v_n) \to (u_2, v_2)$ *in* X.

Thus (u_2, v_2) *is a solution of* (1.1) *with* $I'(u_2, v_2) < 0$. *Then the proof of theorem 1.1 is complete.*

4 Proof of Theorem 1.2

To prove of solution for the system (1.1), we introduce the Nehari manifold.

$$\mathcal{N} = \{(u, v) \in Y \setminus (0, 0) : I'(u, v)(u, v) = 0\} \tag{4.1}$$

that is, $(u, v) \in \mathcal{N}$ *if and only if* $(u, v) \neq 0$ *and*

$$\|u\|_E^m + \|v\|_E^m = \int_{\mathbb{R}^N} (F(u, v) + \lambda |u|^q + \mu |v|^q) \, dx. \tag{4.2}$$

Furthermore, we define the fibering maps $\phi(t) = I(u, v)$ *for* $t > 0$. *Clearly,* $(u, v) \in \mathcal{N}$ *if and only if* $\phi'(1) = 0$ *and, more generally,* $(tu, tv) \in \mathcal{N}$ *if and only if* $\phi'(t) = 0$, *that is, elements in* $\mathcal{N}$ *correspond to stationary points of fibring maps* $\phi(t)$. *By definition, we have*

$$\begin{aligned} \phi(t) = \frac{t^m}{m} \left(\|u\|_E^m + \|v\|_E^m\right) - \frac{t^r}{r} \int_{\mathbb{R}^N} F(u, v) dx \\ - \frac{t^q}{q} \int_{\mathbb{R}^N} (\lambda |u|^q + \mu |v|^q) \, dx, \end{aligned}$$

and

$$\phi'(t) = t^{m-1}\left(\|u\|_E^m + \|v\|_E^m\right) - t^{r-1}\int_{\mathbb{R}^N} F(u,v)dx$$
$$- t^{q-1}\int_{\mathbb{R}^N}\left(\lambda |u|^q + \mu |v|^q\right) dx.$$

Notice that, if $(u, v) \in \mathcal{N}$, then

$$I(u,v) = (\frac{1}{m} - \frac{1}{q})\left(\|u\|_E^m + \|v\|_E^m\right) + (\frac{1}{q} - \frac{1}{r})\int_{\mathbb{R}^N} F(u,v)dx$$
$$= (\frac{1}{m} - \frac{1}{r})\left(\|u\|_E^m + \|v\|_E^m\right) + (\frac{1}{r} - \frac{1}{q})\int_{\mathbb{R}^N}\left(\lambda |u|^q + \mu |v|^q\right) dx. \tag{4.3}$$

In the following, we derive some properties for the Nehari manifold $\mathcal{N}$.

Lemma 4.1. *Let $p < q < r$ and (H_2). then the Nehari manifold $\mathcal{N} \neq \emptyset$.*

Proof. Let $(u, v) \in Y$, $(u, v) \neq 0$. consider the following function for $t > 0$,
$\gamma(t) = I'(tu, tv)(tu, tv) = t^m\left(\|u\|_E^m + \|v\|_E^m\right) - t^r\int_{\mathbb{R}^N} F(u,v)dx - t^q\int_{\mathbb{R}^N}\left(\lambda |u|^q + \mu |v|^q\right) dx.$
Since $p < q < r$, it follows that $\gamma(t) > 0$ for small $t > 0$, and $\gamma(t) \to -\infty$ as $t \to +\infty$.
Then there exists $t_1 > 0$ such that $\gamma(t_1) = 0$. Obviously, $(t_1 u, t_1 v) \neq (0, 0)$.
We conclude that $(t_1 u, t_1 v) \in \mathcal{N}$ and $\mathcal{N} \neq \emptyset$. □

Lemma 4.2. *Let the conditions in theorem 1.2 hold. then, the functional I is coercive and bounded from below on $\mathcal{N}$. Moreover*

$$d = \inf_{(u,v)\in\mathcal{N}} I(u, v).$$

Proof. Let $(u, v) \in \mathcal{N}$. Then it follows from (2.3) and (4.2) that

$$\|u\|_E^m + \|v\|_E^m = \int_{\mathbb{R}^N}\left(F(u,v) + \lambda |u|^q + \mu |v|^q\right) dx$$
$$\leq C_0\left(\|u\|_E^r + \|v\|_E^r + \|u\|_E^q + \|v\|_E^q\right) \tag{4.4}$$

where $C_0 = \max\left\{S_r k_1, S_q, \max(|\lambda|, |\mu|)\right\}$
Inequality (4.4) implies

$$2^{-k} \leq C_0(\|(u, v)\|^{r-m} + \|(u, v)\|^{q-m}). \tag{4.5}$$

If $\|(u, v)\| \leq 1$, (4.5) gives $2^{-k} \leq 2C_0 \|(u, v)\|^{q-m}$.
So we have

$$\|(u, v)\| \geq \min\left\{1, \left(2^{k+1}C_0\right)^{\frac{1}{q-m}}\right\} := C_1, \quad \forall (u, v) \in \mathcal{N}. \tag{4.6}$$

Therfore, if $(u, v) \in \mathcal{N}$, we have from (4.3) that

$$I(u, v) \geq (\frac{1}{m} - \frac{1}{q})\left(\|u\|_E^m + \|v\|_E^m\right) \geq C_2$$

where $C_2 = 2^{-k}(\frac{1}{m} - \frac{1}{q})C_1^p > 0$.
Thus, the proof of Lemma 4.2 is finished. □

Lemma 4.3. *Let all conditions in Theorem 1.2 hold. Then, there exists a nonnegative function $(u_0, v_0) \in \mathcal{N}$ such that $d = \inf_{(u,v)\in\mathcal{N}} I(u, v) = I(u_0, v_0)$.*

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence for d in $\mathcal{N}$. The fact $I(u_n, v_n) = I(|u_n|, |v_n|)$ implies that $\{(|u_n|, |v_n|)\}$ is also a minimizing sequence, so that you can assume from beginning $u_n, v_n > 0$ a.e in $\mathbb{R}^N$. Since I is coercive and bounded from below on $\mathcal{N}$, the sequence $\{(u_n, v_n)\}$ is bounded in Y. We can assume that, up to subsequence, $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in Y. By Lemma**??**, we have

$$u_n \to u_0, v_n \to v_0 \text{ in } L^r(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$$

and

$$u_n(x) \to u_0(x),\ v_n(x) \to v_0(x) \text{ a.e. in } \mathbb{R}^N.$$

We now prove that $(u_0, v_0) \in \mathcal{N}$ and $d = I(u_0, v_0)$.
Since $(u_n, v_n) \in \mathcal{N}$, then

$$\|u_n\|_E^m + \|v_n\|_E^m = \int_{\mathbb{R}^N} (F(u_n, v_n) + \lambda |u_n|^q + \mu |v_n|^q)\, dx. \tag{4.7}$$

By the weakly lower semi-continuity of norms, we have from (4.7).

$$\begin{aligned} \|u_0\|_E^m + \|v_0\|_E^m &\le \lim \inf_{n\to\infty} (\|u_n\|_E^m + \|v_n\|_E^m) \\ &\le \int_{\mathbb{R}^N} (F(u_0, v_0) + \lambda |u_0|^q + \mu |v_0|^q)\, dx. \end{aligned} \tag{4.8}$$

If the equality in (4.8) holds, then $(u_0, v_0) \in \mathcal{N}$.
So, arguing by contradiction, we assume that

$$\|u_0\|_E^m + \|v_0\|_E^m < \int_{\mathbb{R}^N} (F(u_0, v_0) + \lambda |u_0|^q + \mu |v_0|^q)\, dx.$$

Let $\phi(t) = I'(tu_0, tv_0)(tu_0, tv_0)$. Clearly, $\phi(t) > 0$ for small $t > 0$ and $\phi(1) < 0$. So that there exists $t \in (0, 1)$ such that $\phi(t) = 0$ and $(u_0, v_0) \in \mathcal{N}$. Then we have

$$\begin{aligned} d \le I(tu_0, tv_0) &= t^m(\frac{1}{m} - \frac{1}{q})\, (\|u_0\|_E^m + \|v_0\|_E^m) + t^r(\frac{1}{q} - \frac{1}{r}) \int_{\mathbb{R}^N} F(u_0, v_0) dx \\ &< (\frac{1}{m} - \frac{1}{q})\, (\|u_0\|_E^m + \|v_0\|_E^m) + (\frac{1}{q} - \frac{1}{r}) \int_{\mathbb{R}^N} F(u_0, v_0) dx \\ &\le \lim \inf_{n\to\infty} \left((\frac{1}{m} - \frac{1}{q})\, (\|u_n\|_E^m + \|v_n\|_E^m) + (\frac{1}{q} - \frac{1}{r}) \int_{\mathbb{R}^N} F(u_n, v_n) dx \right) \\ &\le \lim \inf_{n\to\infty} I(u_n, v_n). \end{aligned}$$

This contradiction prove that the equality in (4.8) holds and then $(u_0, v_0) \in \mathcal{N}$ and the proof of Lemma (4.3) is completed. □

Clearly, it is enough to prove that (u_0, v_0) *is a critical point for I in Y, that is,* $I'(u_0, v_0)(\varphi, \psi) = 0$ *for all* $(\varphi, \psi) \in Y$ *and* $I'(u_0, v_0) = 0$ *in* Y^*, *where* (u_0, v_0) *is in the position of Lemma (4.3)*
For every $(\varphi, \psi) \in Y$, *we choose* $\varepsilon > 0$ *such that* $(u_0 + s\varphi, v_0 + s\psi) \neq 0$ *for all* $s \in (-\varepsilon, \varepsilon)$. *Define a function*

$$\begin{aligned} w(s, t) &= I'(t\,(u_0 + s\varphi)\,, t\,(v_0 + s\psi))(t\,(u_0 + s\varphi)\,, t\,(v_0 + s\psi)) \\ &= \|t\,(u_0 + s\varphi)\|_E^m + \|t\,(v_0 + s\psi)\|_E^m - t^r \int_{\mathbb{R}^N} F(u_0 + s\varphi, v_0 + s\psi) dx \\ &\quad t^q \left(\lambda \|u_0 + s\varphi\|_q^q + \mu \|v_0 + s\psi\|_q^q \right). \end{aligned} \tag{4.9}$$

Then

$$w(0,1) = \|u_0\|_E^m + \|v_0\|_E^m - \int_{\mathbb{R}^N} F(u_0,v_0)dx - \left(\lambda \|u_0\|_q^q + \mu \|v_0\|_q^q\right) \tag{4.10}$$

and

$$\frac{\partial w}{\partial t} = m\left(\|u_0\|_E^m + \|v_0\|_E^m\right) - r\int_{\mathbb{R}^N} F(u_0,v_0)dx - q\left(\lambda \|u_0\|_q^q + \mu \|v_0\|_q^q\right) = (m-q)\left(\|u_0\|_E^m + \|v_0\|_E^m\right) + (q-r)\int_{\mathbb{R}^N} F(u_0,v_0)dx < 0. \tag{4.11}$$

So, by the implicit function Theorem, there exists a C^1 function $t(s)$ such that $t(0) = 1$ and $w(s,t(s)) = 0$ for every $s \in (-\varepsilon_0, \varepsilon_0) \subset (-\varepsilon, \varepsilon)$.

This also shows that $t(s) \neq 0$, at least for ε_0 small enough.

Therfore, $t(s)(u_0 + s\varphi, v_0 + s\psi) \in \mathcal{N}$. Denote $t = t(s)$ and

$$\chi(s) = I(t(u_0+s\varphi), t(v_0+s\psi)) \quad \frac{t^m}{m}\left(\|t(u_0+s\varphi)\|_E^m + \|t(v_0+s\psi)\|_E^m\right) - \frac{t^r}{r}\int_{\mathbb{R}^N} F(u_0+s\varphi, v_0+s\psi)dx - \frac{t^q}{q}\left(\lambda\|u_0+s\varphi\|_q^q + \mu\|v_0+s\psi\|_q^q\right). \tag{4.12}$$

We see that the function $\chi(s)$ is differentiable and has a minimum point at $s = 0$. Therefore,

$$0 = \chi'(s) = t'(0)w(0,1) + I'(u_0,v_0)(\varphi,\psi).$$

Since $(u_0,v_0) \in \mathcal{N}$, it follows from (4.10) that $I'(u_0,v_0)(\varphi,\psi) = 0$ for all $(\varphi,\psi) \in Y$ and thus $I'(u_0,v_0) = 0$ in Y^*. So, (u_0,v_0) is a critical point for I and then (u_0,v_0) is a weak solution of the problem (1.1). Thus the proof of Theorem1.2 is completed.

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