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Packing Chromatic Numbers of Some Mycielski and Power Graphs

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Abstract A *packing coloring* of order k of a given graph G is to assign k different colors from $\{1,\ 2, ...,\ k\}$ to the vertices of G, such that two distinct vertices colored with color i are at a distance greater than i. The *packing chromatic number* of G, denoted by $\chi_\rho\ (G)$, is the minimum order of a packing coloring. In the first part of this paper, we determine the exact values of the packing chromatic numbers of some Mycielski graphs. In the second part, we present the packing chromatic numbers of the second power graph of the Mycielski graph of paths and cycles.

1 Introduction

Throughout this paper, we consider only finite, simple graphs. Let G be a graph and u an arbitrary vertex of G. The *(open) neighborhood* of a vertex u in G, $N_G(u)$, is the set of all vertices adjacent to u in G i.e., $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ (we frequently omit the subscript if the graph G is obvious from the context). The *degree* of u, denoted by $deg_G(u)$, is $|N_G(u)|$. A vertex u with the property that $deg_G(u) = 1$ is called a *leaf* or a *pendant vertex*.

The distance between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$ (or $d(u, v)$ when the graph G is obvious from the context), is the length of the shortest u, v-path. The *diameter* of G, denoted by diam (G), is $max\{d_G(u, v) : u, v \in V(G)\}$.

The *vertex cover number* of G, denoted by $\tau\ (G)$, is the minimum cardinality of a subset S of vertices such that every edge in G has at least one endpoint in S. Note that if a set of vertices is a vertex cover, then its complement is an independent set. It is important to note that the vertex cover problem is NP-complete for graphs of diameter 2. However, for perfect graphs, including bipartite graphs, the problem can be solved in polynomial time (see [18, 22]).

Given a positive integer i, an *i-packing* in G is a subset W of the vertices of G such that the distance between any two distinct vertices from this set is greater than i. This concept encompasses also the notion of an independent set, which is edentical to a 1-packing. The *packing chromatic number* of a graph G is the smallest integer k such that the vertex set of G can be partitioned into disjoint sets $W_1, W_2, \ldots, W_k$, where W_i is an i-packing for each $i \in \{1, 2, \ldots, k\}$. We denote this number by $\chi_\rho(G)$. The corresponding mapping $c : V\ (G) \rightarrow \{1, ...,\ k\}$, satisfying the property that $c(u) = c(v) = i$ implies $d_G(u, v) > i$, is called a *packing coloring* of order k. The packing coloring is optimal, when $k = \chi_\rho(G)$.

The concept of packing chromatic number was introduced in 2008 by Goddard et al. [19]. First, it was presented under the name broadcast chromatic number, and the current name was given by Brešar et al. in [7]. The concept arose from the area of frequency assignment in wireless networks [16, 31] and also has several applications, such as in resource replacement and biological diversity [31]. The packing chromatic number has been investigated in a number of papers. Many contributions to this invariant have been published in the last few years (see [2, 3, 5, 21, 6, 9, 24, 25, 27, 28, 29]). This confirm a wide interest given to this concept. The concept has been investigated to determine the packing chromatic numbers of infinite graphs for different types and classes of graphs, such as infinite grids, lattices, distance graphs, etc. [4, 7, 13, 17, 26, 15]. The determination of the packing chromatic number is (very) hard [16] as its decision version is NP-complete even when restricted to trees (to learn more, see [24]).

In a given graph, operations or editing operations, also known as graph edit operations, are used to create a new graph from an existing one by making a simple local change. Examples of such changes include the addition or deletion of a vertex or of an edge, the merging and splitting of vertices, the contraction and subdivision of edges, etc. Some of these operations have been studied by Brešar et al. in [8]. In this paper, we study the packing chromatic numbers of power graphs. The *$p-th$ power graph* G^p for $p \geq 1$ of a graph G is the graph obtained from G by adding an edge between each pair of vertices that are at distance at most p in G. When $p = 1$, the graph G^1 is isomorphic to G. Note that, if the graph G has a diameter p, then G^p is the complete graph. The power graphs have been investigated in more papers, see [1, 11, 23]. The *Mycielski* graph of

any graph G, denoted by $\mu(G)$, is a connected graph with vertex set $V(\mu(G)) = V(G) \cup V' \cup \{w\}$, where $V = \{v_1, v_2, ..., v_n\}$, $V' = \{x_1, x_2, ..., x_n\}$ and the edge set $E(\mu(G)) = E(G) \cup E' \cup E''$ with $E' = \{v_i x_j, v_j x_i : v_i v_j \in E(G);\ 1 \leq i, j \leq n\}$ and $E'' = \{w x_i;\ 1 \leq i \leq n\}$. Thus, if G has n vertices and m edges, then $\mu(G)$ has $2n + 1$ vertices and $3m + n$ edges. It was developed by Mycielski [30] in search of triangle free graphs with large chromatic number, and then, it caught the interest of a number of authors [9, 12, 14, 20]. In [10], Bidine et al. determined the packing chromatic number of some Mycielski Graphs. They gave the lower and upper bound for packing chromatic numbers of the Mycielskian of paths and cycles. In this paper, we strenghten these results by providing the exact values of packing chromatic numbers of these two graphs.

The paper is organized as follows. In the next section, we present some results of packing chromatic numbers of Mycielskian graphs. We determine the exact values of $\chi_p(\mu(K_{1,n}))$, $\chi_p(\mu(W_n))$ and $\chi_p(\mu(T))$, where T is a tree of diameter 3. We finish this section with determining the exact values of $\chi_p(\mu(P_n))$ and $\chi_p(\mu(C_n))$. In Section 3, we consider the power graph. We provide the exact values for $\chi_p((\mu(P_n))^2)$ and $\chi_p((\mu(C_n))^2)$. We end this paper by giving some remarks and open problems.

It is well known that the packing chromatic number of any graph cannot be smaller than the packing chromatic number of its subgraphs. This concept is given in the following proposition.

Proposition 1.1. *[19] Let H be a subgraph of a given graph G. Then*

$$\chi_\rho(H) \leq \chi_\rho(G).$$

Now, recall two well known propositions, which will be used in the sequel of this paper. While the first proposition provides the values of packing chromatic numbers for cycles, the second proposition presents the upper bound of packing chromatic numbers for any graph using the vertex cover number.

Proposition 1.2. *[19] If C_n is a cycle of order n, then*

$$\chi_\rho(C_n) = \begin{cases} 3; & n = 4k, k \geq 1, \text{ or } n = 3, \\ 4; & \text{otherwise}. \end{cases}$$

Proposition 1.3. *[19] For any graph G, $\chi_p(G) \leq \tau(G) + 1$, with equality if diam $(G) = 2$.*

2 Mycielski graph

In this section, we continue with the consideration of packing chromatic numbers of Mycielski graphs. First, we provide the exact values for $\chi_p(\mu(K_{1,n}))$, $\chi_p(\mu(W_n))$ and $\chi_p(\mu(T))$, where T is a tree of diameter 3. At the end of this section, we provide the exact values for $\chi_p(\mu(P_n))$ and $\chi_p(\mu(C_n))$.

Initially, we begin by considering the following proposition, which provides the value of $\chi_p\,(\mu\,(K_{1,n}))$ and $\chi_p(\mu(W_n))$. It is important to note that graphs $\mu(K_{1,n})$ and $\mu(W_n)$ have a diameter of 2 and by using Proposition 1.3, the results can be determined (see Figures 1 and 2). The proof of the first part of this proposition is also straightforward.

Proposition 2.1. *For every $n \geq 2$, $\chi_p(\mu(K_{1,n})) = 4$ and for every $n \geq 4$, $\chi_p(\mu(W_n)) = n + 2$.*

Proof. Let denote by v_i, $i \in \{1, ..., n\}$ the vertices of the wheel graph W_n such that $d_{\mu(W_n)}\,(v_1) = 2n - 2$ and by $x_{i,}$ $i \in \{1, ..., n\}$ the corresponding vertices to v_i. Let w be the vertex connecting the vertices x_i. Since $diam(\mu(W_n)) = 2$, the result follows from Proposition 1.3. Let $A = \{w,\ v_i;\ 1 \leq i \leq n\}$, where $\tau\,(\mu\,(W_n)) = |A| = n + 1$. □

Figure 1 provides a packing coloring of $\mu(K_{1,8})$ and Figure 2 shows a packing coloring of $\mu(W_9)$.

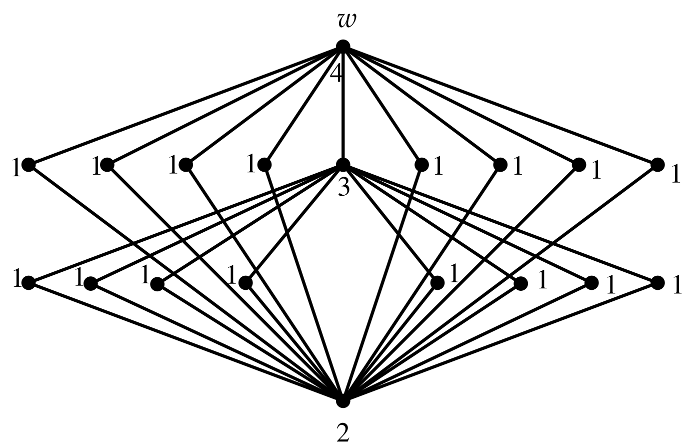

Figure 1. A packing coloring of $\mu(K_{1,8})$

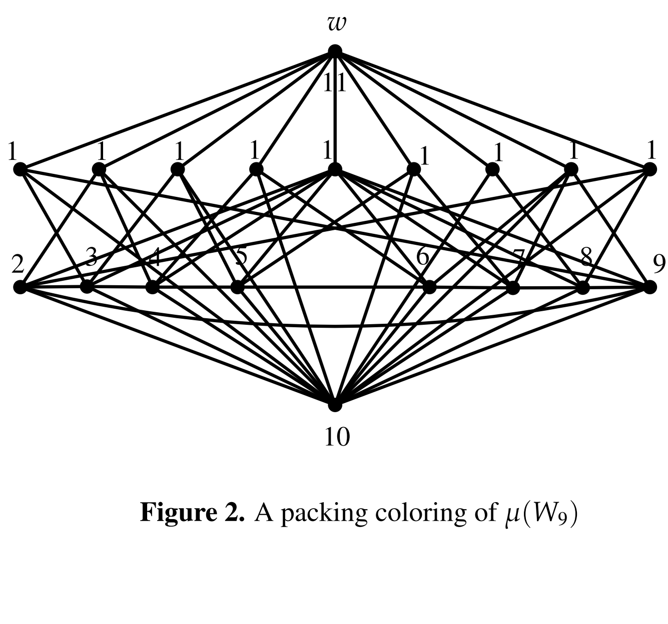

Figure 2. A packing coloring of $\mu(W_9)$

We continue with determining the packing coloring number of the Mycielski graph obtained from a tree of diameter three using the Mycielski construction.

Proposition 2.2. *Let T be a tree of diameter 3 different from P_4. It holds that $\chi_p(\mu(T)) = 6$.*

Proof. Let T be a tree of diameter 3 and of order n. Denote by c_1 and c_2 the two non-pendant vertices of T. Denote the pendant neighbors of c_1 and c_2 by $v_{1,1}, v_{1,2}, ..., v_{1,p}$ and $v_{2,1}, v_{2,2},..., v_{2,q}$, respectively. Denote by u_1 and u_2 respectively, the two corresponding vertices of c_1 and c_2 in $\mu(T)$ and the corresponding vertices of $v_{1,1}, v_{1,2},..., v_{1,p}$ in $\mu(T)$ are $x_{1,1}, x_{1,2}, ..., x_{1,p}$. Similarly, $x_{2,1}, x_{2,2},..., x_{2,q}$ are the corresponding vertices of $v_{2,1}, v_{2,2},..., v_{2,q}$ in $\mu(T)$. Let w be the vertex that connects the vertices $x_{1,i}$, $i \in \{1, ..., p\}$, $x_{2,j}$, $j \in \{1, ..., q\}$ and thus the vertices u_1 and u_2. Let c be a packing coloring of $\mu(T)$. As $diam(\mu(T)) = 3$, the only colors that c can use twice or more are 1 and 2.

First, if $c(w) = 1$, then $c(x_{1,i}) \neq 1$ for all $i \in \{1, ..., p\}$, $c(x_{2,j}) \neq 1$ for all $j \in \{1, ..., q\}$, $c(u_1) \neq 1$ and $c(u_2) \neq 1$, and in this case, we can use the color 1 on the vertices $v_{1,i}$, $i \in \{1, ..., p\}$ and $v_{2,j}$, $j \in \{1, ..., q\}$ at most $(p+q)$ times, and therefore color 1 appears $p+q+1$ times on $\mu(T)$. The remaining vertices get different colors from 1. So in this case, we have $\chi_p(\mu(T)) \geq (2(p+q)+5) - (p+q+1) + 1 = p+q+5$. Suppose that $c(w) \neq 1$. In this case, we can color the vertices $x_{1,i}$, $i \in \{1, ..., p\}$, $x_{2,j}$, $j \in \{1, ..., q\}$, as well as u_1 and u_2 with the color 1. This results in the color 1 being used a total of $p+q+2$ times. The remaining vertices require colors different from 1, giving us a total of $p+q+3$ different colors. Therefore, we have $\chi_p(\mu(T)) \geq (2(p+q)+5-(p+q+2))+1 = p+q+4$. Another way is to color the vertices $x_{1,i}$, $i \in \{1, ..., p\}$, $x_{2,j}$, $j \in \{1, ..., q\}$, $v_{1,i}$, $i \in \{1, ..., p\}$ and $v_{2,j}$, $j \in \{1, ..., q\}$ with the color 1, and in this case, the color 1 is used $2(p+q)$ times. The remaining vertices are $\{w, u_1, u_2, c_1, c_2\}$, which are pairwise at distance at most 2, and thus c assigns them pairwise different colors from 1, i.e., colors from $\{2, 3, 4, 5, 6\}$. As a result, we have $\chi_p(\mu(T)) \geq (2(p+q)+5-2(p+q))+1 = 6$. To prove that $\chi_p(\mu(T)) \leq 6$, consider the following 6-packing coloring: pose $c(x_{1,i}) = 1$, $i \in \{1, ..., p\}$, $c(x_{2,j}) = 1$, $j \in \{1, ..., q\}$, $c(c_1) = 2$, $c(c_2) = 3$, $c(u_1) = 4$, $c(u_2) = 5$ and $c(w) = 6$. Therefore $\chi_p(\mu(T)) \leq 6$, and thus $\chi_p(\mu(T)) = 6$. □

The following figure shows a packing coloring of $\mu(T)$ when $p = q = 3$, as described in the proof of Proposition 2.2.

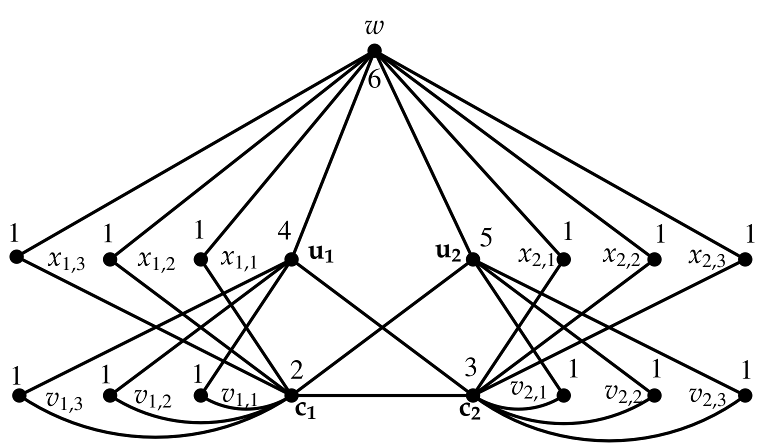

Figure 3. A packing coloring of $\mu(T)$, where T is a tree of diameter 3

In the sequel of this section, we consider the Mycielskian of the following graphs: P_n and C_n, where $n \geq 3$. We use the following notations for $\mu(P_n)$ respectively $\mu(C_n)$. The vertices of P_n and C_n are denoted by v_i, where $i \in \{1, ..., n\}$. The vertices of $\mu(P_n)$ respectively $\mu(C_n)$, which are corresponding to the vertices from $V(P_n)$ respectively $V(C_n)$, are denoted by x_i, where $i \in \{1, ..., n\}$. The common neighbor of x_i, where $i \in \{1, ..., n\}$, is denoted by w.

We begin by determining the packing chromatic number of the Mycielski graph of a path.

Theorem 2.3. *If P_n is a path of order $n \geq 2$, then*

$$\chi_p(\mu(P_n)) = \begin{cases} n - \left\lfloor \frac{n-1}{3} \right\rfloor - \left\lfloor \frac{n-2}{9} \right\rfloor - \left\lfloor \frac{n-6}{9} \right\rfloor + 1; & n \neq 3, \\ 4; & n = 3. \end{cases}$$

Proof. Let c be a packing coloring of $\mu(P_n)$. First, suppose that $n \in \{2, 3\}$. If $n = 2$, then for this graph $\mu(P_n)$ is isomorphic to C_5 and $\chi_p(C_5) = 4$. If $n = 3$, then $diam(\mu(P_n)) = 2$ and, according to Proposition 1.3, $\chi_p(\mu(P_n))$ is $\tau(P_n) + 1 = 3 + 1 = 4$ and we are done. If $n = 4$, then $diam(\mu(P_n)) = 3$, and only colors 1 and 2 can be repeated in $\mu(P_n)$. Let $c(x_i) = 1$, $i \in \{1, ..., 4\}$, $c(v_i) = 2$, $i \in \{1, 4\}$, $c(v_2) = 3$, $c(v_3) = 4$ and $c(w) = 5$, thus $\chi_p(\mu(P_n)) = 5$. If $n = 5$, then $\chi_p(\mu(P_n)) = 6$. Let $c(x_i) = 1$, $i \in \{1, ..., 5\}$, $c(v_i) = 2$, $i \in \{1, 4\}$, and the other vertices receive different colors from $\{3, 4, 5, 6\}$. Note that these cases are easy to check.

Now, suppose that $n \geq 6$. Since $diam(\mu(P_n)) = 4$, the colors which can be repeated more than once by c, are $1, 2$ and 3, and additionally, any optimal packing coloring assigns these colors to at most as possible vertices. First, if $c(w) = 2$ or 3, then for every $i \in \{1, ..., n\}$, $c(x_i) \neq 2, 3$ and $c(v_i) \neq 2, 3$ since $d(w, x_i) = 1$ and $d(w, v_i) = 2$ and in this case, color 2 or 3 is used only one time on $\mu(P_n)$. Also $c(x_i) \neq 3$ for all $i \in \{1, ..., n\}$, since $d\left(x_i, \left\{w, v_j;\ 1 \leq j \leq n\right\}\right) \leq 3$, and in this case color 3 is used only once on $\mu(P_n)$, but otherwise we can use it several times on the set $\{v_i;\ 1 \leq i \leq n\}$. Therefore $c(w) \notin \{2, 3\}$. Note that also, if $c(w) = 1$, then $c(x_i) \neq 1$, $i \in \{1, ..., n\}$ since $d(w, x_i) = 1$ and in this case, we can use also the color 1 on the vertices v_i, $i \in \{1, ..., n\}$. Furthermore, if there exists a vertex x_i colored with color 2, then this vertex is unique, since the vertices from $\{x_i; 1 \leq i \leq n\}$ are pairwise at distance 2. Let $c(x_1) = 2$ and by repeating the pattern 1312 on the vertices v_i, $i \in \{1, ..., n\}$, we obtain $n + 2$ vertices colored with colors $1, 2$ and 3 on $\mu(P_n)$. But this is not the best coloring as we will see. Therefore, it remains to distinguish two cases regarding the use of colors 1 and 2.

Case 1. Let $c(x_i) = c(v_i) = 1$, $i = 2k + 1$, $k \in \{0, ..., \lfloor \frac{n-1}{2} \rfloor\}$. i.e., $\left\lfloor \frac{n-1}{2} \right\rfloor + 1$ vertices each from the vertices x_i and v_i are colored with 1. In this case, we have $2\left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) = 2\left\lfloor \frac{n-1}{2} \right\rfloor + 2$ vertices of $\mu(P_n)$ colored with 1.

Now, we come to use the color 2. If there exists a vertex x_h, $h \in \{1, ..., n\}$ such that $c(x_h) = 2$, then this vertex colored with 2 is unique since $d(x_h, x_j) = 2$ for all j, $j \in \{1, ..., n\}$, $h \neq j$. Then, we are considering only two cases regarding the use of this color on x_i, $i \in \{1, ..., n\}$. Suppose there exists i, $i \in \{1, ..., n\}$, such that $c(x_i) = 2$, and let $c(x_1) = 2$. Then, in this case, colors $1, 2$ and 3 are used as follows. For color 1, let $c(x_i) = 1$ for $i = 2k + 3$, $k \in \{0, ..., \lfloor \frac{n-3}{2} \rfloor\}$, and so we have $\left\lfloor \frac{n-3}{2} \right\rfloor + 1$ vertices from $\{x_i;\ 1 \leq i \leq n\}$ colored with 1. To color the vertices v_i, $i \in \{1, ..., n\}$, we can use the pattern $1312, ...$ Therefore, we have $n + \left\lfloor \frac{n-3}{2} \right\rfloor + 2$ vertices of $\mu(P_n)$ colored with colors $1, 2$ and 3.

We next suppose that $c(x_2) = 2$. Then, we can color at most $\left\lfloor \frac{n-1}{2} \right\rfloor + 1$ vertices from $\{x_i;\ 1 \leq i \leq n\}$ with color 1 and to color the vertices v_i, $i \in \{1, ..., n\}$, we use the following coloring. Start the coloring of the consecutive vertices with colors 1314 and repeat the pattern 1312. Therefore, we have $n - 1$ vertices from $\{v_i;\ 1 \leq i \leq n\}$ colored by colors $1, 2$ and 3. For the vertices x_i, we have $\left\lfloor \frac{n-1}{2} \right\rfloor + 2$ vertices colored by 1 and 2. So, we have $(n - 1) + \left\lfloor \frac{n-1}{2} \right\rfloor + 2 = n + \left\lfloor \frac{n-1}{2} \right\rfloor + 1$ vertices of $\mu(P_n)$ colored with colors $1, 2$ and 3.

It is easy to check that $n+\left\lfloor\frac{n-3}{2}\right\rfloor+2=n+\left\lfloor\frac{n-1}{2}\right\rfloor+1$. Now, suppose there exists j, $3\le j\le n-2$ such that $c(x_j)=2$. Then, $c(v_i)\neq 2$ for all i, $j-2\le i\le j+2$ since $d\left(x_j,v_i\right)\le 2$. Without loss of generality assume that $j=3$ (i.e., $c(x_3)=2$), which implies that the vertices v_i, $i\in\{1,...,5\}$ cannot be colored with 2 since $d(x_3,v_i)\le 2$ for all $i\in\{1,...,5\}$, and thus the vertices v_i, $i\in\{1,...,n\}$ can be colored using the following coloring. Start the coloring of the consecutive vertices with colors 1314 and repeat the pattern 1312. Therefore, the colors 1, 2 and 3 are used in this case at most $(n-1)+\left\lfloor\frac{n-1}{2}\right\rfloor+1=n+\left\lfloor\frac{n-1}{2}\right\rfloor$ times in $\mu(P_n)$. As $n+\left\lfloor\frac{n-1}{2}\right\rfloor+1\ge n+\left\lfloor\frac{n-1}{2}\right\rfloor$, and since c is any optimal packing coloring of $\mu(P_n)$, it assigns colors $1,2$ and 3 to at most as possible vertices, and thus we can assume that $c(x_i)\neq 2$ for all $i\in\{3,...,n-2\}$. Note that, the case when $c(x_{n-1})=2$ (respectively $c(x_n)=2$) is similar to the case $c(x_2)=2$ (respectively $c(x_1)=2$).

On the other hand, we can use the color 2 only on the vertices v_i, $i\in\{1,...,n\}$. So, we repeat the pattern 1213 on the vertices v_i. Since in this case $\left\lfloor\frac{n-1}{2}\right\rfloor+1$ vertices from $\{x_i;\ 1\le i\le n\}$ can be colored with 1, we conclude that $n+\left\lfloor\frac{n-1}{2}\right\rfloor+1$ vertices of $\mu(P_n)$ can be colored with colors $1,2$ and 3.

Case 2. Let $c(x_i)=1$ for all $i\in\{1,...,n\}$. Then, n vertices are colored with 1 in $\mu(P_n)$. Hence, colors 2 and 3 are used to color the vertices v_i, $i\in\{1,...,n\}$ of the path P_n by repeating the pattern $23x2y32zt$ (starting by v_1), where x,y,z and t are distinct colors greater than 3. Now, to determine the number of vertices colored with 2 and 3 that follow this pattern, we use the following method. Let $c(v_i)=2$ for $i=3k+1$, $k\in\{0,...,\lfloor\frac{n-1}{3}\rfloor\}$. For color 3, let $c(v_i)=3$ for $i=9k+2$, $k\in\left\{0,...,\ \left\lfloor\frac{n-2}{9}\right\rfloor\right\}$ and let $c(v_i)=3$ for $i=9k+6$, $k\in\left\{0,...,\ \left\lfloor\frac{n-6}{9}\right\rfloor\right\}$. Therefore, we have $\lfloor\frac{n-1}{3}\rfloor+1$ vertices are colored with color 2 and $\left\lfloor\frac{n-2}{9}\right\rfloor+\left\lfloor\frac{n-6}{9}\right\rfloor+2$ vertices colored with color 3 in $\mu(P_n)$. Thus, in this case $n+\left\lfloor\frac{n-1}{3}\right\rfloor+\left\lfloor\frac{n-2}{9}\right\rfloor+\left\lfloor\frac{n-6}{9}\right\rfloor+3$ vertices are colored with colors $1,2$ and 3.

As a consequence of the previous cases, since $n+\left\lfloor\frac{n-1}{3}\right\rfloor+\left\lfloor\frac{n-2}{9}\right\rfloor+\left\lfloor\frac{n-6}{9}\right\rfloor+3\ge n+\left\lfloor\frac{n-1}{2}\right\rfloor+1\ge n+2$ for all $n\ge 6$, we conclude that the optimal packing coloring uses colors $1,2$ and 3, $n+\left\lfloor\frac{n-1}{3}\right\rfloor+\left\lfloor\frac{n-2}{9}\right\rfloor+\left\lfloor\frac{n-6}{9}\right\rfloor+3$ times on $\mu(P_n)$. Since $\left|V\left(\mu\left(P_n\right)\right)\right|=2n+1$, we infer that $\chi_p(\mu(P_n))\ge\left|V\left(\mu\left(P_n\right)\right)\right|-(n+\left\lfloor\frac{n-1}{3}\right\rfloor+\left\lfloor\frac{n-2}{9}\right\rfloor+\left\lfloor\frac{n-6}{9}\right\rfloor+3)+3=n-\left\lfloor\frac{n-1}{3}\right\rfloor-\left\lfloor\frac{n-2}{9}\right\rfloor-\left\lfloor\frac{n-6}{9}\right\rfloor+1$ and so, our lower bound is proved.

To prove the upper bound, we form a $\left(n-\left\lfloor\frac{n-1}{3}\right\rfloor-\left\lfloor\frac{n-2}{9}\right\rfloor-\lfloor\frac{n-6}{9}\rfloor+1\right)$–packing coloring c of a given graph. First, let $c(x_i)=1$ for $i=2k+1$, $k\in\{0,...,\lfloor\frac{n-1}{2}\rfloor\}$, $c(v_i)=2$, for $i=3k+1$, $k\in\{0,...,\lfloor\frac{n-1}{3}\rfloor\}$, $c(v_i)=3$, for $i=9k+2$, $k\in\{0,...,\lfloor\frac{n-2}{9}\rfloor\}$, and $c(v_i)=3$, for $i=9k+6$, $k\in\{0,...,\lfloor\frac{n-6}{9}\rfloor\}$, and the other vertices receive pairwise different colors from 1, 2 and 3, i.e., from the set $\{4,...,n-\left\lfloor\frac{n-1}{3}\right\rfloor-\left\lfloor\frac{n-2}{9}\right\rfloor-\left\lfloor\frac{n-6}{9}\right\rfloor+1\}$. Hence, the latter is indeed a packing coloring of $\mu(P_n)$, and $\chi_p(\mu(P_n))\le n-\left\lfloor\frac{n-1}{3}\right\rfloor-\left\lfloor\frac{n-2}{9}\right\rfloor-\left\lfloor\frac{n-6}{9}\right\rfloor+1$. Therefore $\chi_p(\mu(P_n))=n-\left\lfloor\frac{n-1}{3}\right\rfloor-\left\lfloor\frac{n-2}{9}\right\rfloor-\lfloor\frac{n-6}{9}\rfloor+1$, and the proof is completed. □

Figure 4 shows a graph $\mu(P_8)$ and its packing coloring, as is described in the proof of the previous theorem.

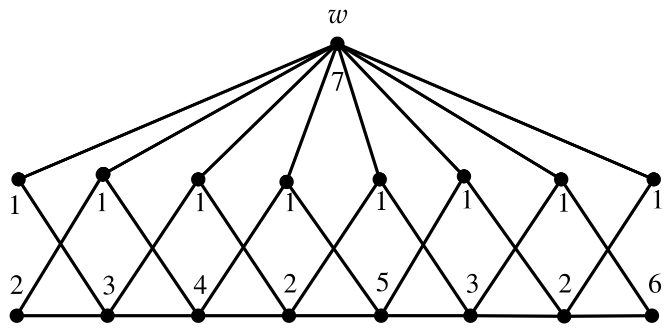

Figure 4. A packing coloring of $\mu(P_8)$

The following theorem gives the packing chromatic number of the Mycielski graph of a cycle.

Theorem 2.4. *For any cycle C_n of order $n \geq 3$,*

$$\chi_p(\mu(C_n)) = \begin{cases} n - \left\lfloor \frac{n-3}{3} \right\rfloor - \left\lfloor \frac{n-4}{9} \right\rfloor - \left\lfloor \frac{n-8}{9} \right\rfloor + 1; & n \neq 3, \\ 5; & n = 3. \end{cases}$$

Proof. Let c be a packing coloring of $\mu(C_n)$. First, suppose that $n \in \{3,4,5\}$. Then, as $diam(\mu(C_n)) = 2$, we can apply Proposition 1.3, $\chi_p(\mu(C_n)) = \tau(C_n) + 1 = n + 2$, where $\tau(\mu(C_n)) = |\{w, v_i;\ 1 \leq i \leq n\}| = n + 1$. If $n \in \{6,7\}$, then $diam(\mu(C_n)) = 3$, and in this case $\left|c^{-1}(i)\right| \leq 1$ for all $i \geq 3$. Let $c(x_i) = 1$ for $i \in \{1, ..., n\}$, $c(v_i) = 2$ for $i \in \{1,4\}$ and the other vertices of $\mu(C_n)$ receive pairwise different colors from $\{3, ..., n+1\}$. Thus $\chi_p(\mu(C_n)) = n + 1$.

Now, suppose that $n \geq 8$. Then $diam(\mu(C_n)) = 4$ and $\left|c^{-1}(i)\right| \leq 1$ for all $i \geq 4$ and thus, the only colors used twice or more by c are 1, 2 and 3. As in the proof of Theorem 2.3, it is easy to check that $c(w) \notin \{2,3\}$ and $c(x_i) \notin \{2,3\}$ (Note that $\mu(P_n)$ is a subgraph of $\mu(C_n)$). Also, in the same way, we can check that, if $c(w) = 1$, then $c(x_i) \neq 1$, since $d(w, x_i) = 1$ for all i, $i \in \{1, ..., n\}$ and in this case, the color 1 can be used on the vertices v_i, $i \in \{1, ..., n\}$, and likewise for colors 2 and 3, by repeating the pattern 1213 such that $c(v_n) \notin \{1,2\}$, since $d(v_1, v_n) = 1$ and $d(v_2, v_n) = 2$. Then, to count the number of vertices colored with 1, 2, and 3, the following method is used. Let $c(v_i) = 1$ for $i \in \{1, ..., n-1\}$, $i = 2k+1$, and $k \in \{0, ..., \lfloor \frac{n-2}{2} \rfloor\}$, and thus the color 1 can be used $\left\lfloor \frac{n-2}{2} \right\rfloor + 1$ times in the cycle C_n, keeping in mind that $c(w) = 1$. As a result, the color 1 is used $\left\lfloor \frac{n-2}{2} \right\rfloor + 2$ times in $\mu(C_n)$. For color 2, let $c(v_i) = 2$ for $i \in \{1, ..., n-1\}$, $i = 4k+2$, and $k \in \{0, ..., \lfloor \frac{n-3}{4} \rfloor\}$, and the color 2 is used $\left\lfloor \frac{n-3}{4} \right\rfloor + 1$ times. By the same method, we can determine the number of vertices colored with color 3, let $c(v_i) = 3$ for $i \in \{1, ..., n\}$, $i = 4k+4$, and $k \in \{0, ..., \lfloor \frac{n-4}{4} \rfloor\}$, and thus the color 3 can be used $\left\lfloor \frac{n-4}{4} \right\rfloor + 1$ times on the vertices v_i. In total, we obtain $\left\lfloor \frac{n-2}{2} \right\rfloor + \left\lfloor \frac{n-3}{4} \right\rfloor + \left\lfloor \frac{n-4}{4} \right\rfloor + 4$ vertices of $\mu(C_n)$ are colored with colors 1, 2 and 3.

Next, if $c(w) \neq 1$, then we can use the color 1 on the vertices x_i or on the vertices x_i and v_i at the same time for $i \in \{1, ..., n\}$ (Note that in the case when the color 1 is used on the vertices v_i without using it on x_i, it implies the use of this color on the vertex w, and so we are in the previous case). Thus, we discuss the following two cases.

Case 1. $c(x_i) = 1$, $i \in \{1, ..., n\}$, and then $\left|c^{-1}(1)\right| = n$. Colors 2 and 3 are used on the vertices v_i, $i \in \{1, ..., n\}$ by repeating the pattern $23x2y32zt$ such that $c(v_i) = 2$ for $i = 3k + 1$, $i \in \{1, ..., n-2\}$, thus $\left|c^{-1}(2)\right| = \left\lfloor \frac{n-3}{3} \right\rfloor + 1$.

For color 3, let $\begin{cases} c(v_i) = 3 \text{ pour } i = 9k+2, k \in \{0, ..., \lfloor \frac{n-4}{9} \rfloor\}; n \geq 4, \\ c(v_i) = 3 \text{ pour } i = 9k+6, k \in \{0, ..., \lfloor \frac{n-8}{9} \rfloor\}; n \geq 8. \end{cases}$, and thus $\left|c^{-1}(3)\right| = \left\lfloor \frac{n-4}{9} \right\rfloor + \left\lfloor \frac{n-8}{9} \right\rfloor + 2$. Therefore, we have $n + \left\lfloor \frac{n-3}{3} \right\rfloor + \left\lfloor \frac{n-4}{9} \right\rfloor + \left\lfloor \frac{n-8}{9} \right\rfloor + 3$ vertices of $\mu(C_n)$ are colored with colors 1, 2 and 3 in this case.

Case 2. Color 1 is used on the vertices x_i and v_i, $i \in \{1, ..., n-1\}$. Then, the optimal packing coloring c uses colors 1, 2, and 3 on $\mu(C_n)$ as follows. Let $c(x_i) = 1$ for $i = 2k+1$, $k \in \{0, ..., \lfloor \frac{n-2}{2} \rfloor\}$ and so the color 1 is used $\lfloor \frac{n-2}{2} \rfloor + 1$ times on the vertices x_i. For the vertices v_i, we can use the following coloring, by repeating the pattern 1213, such that $c(v_n) \notin \{1, 2\}$. Hence, to calculate the number of vertices of $\{v_i; 1 \leq i \leq n\}$ colored with colors 1, 2, and 3, we use this method. Let $c(v_i) = 1$, $i = 2k+1$, $k \in \{0, ..., \lfloor \frac{n-2}{2} \rfloor\}$, and therefore the color 1 is used also $\lfloor \frac{n-2}{2} \rfloor + 1$ times on the vertices v_i. For color 2, let $c(v_i) = 2$ for $i = 4k+2$, $i \in \{1, ..., n-1\}$, i.e., color 2 is used $\lfloor \frac{n-3}{4} \rfloor + 1$ times. For color 3, let $c(v_i) = 3$, $i = 4k+4$, $k \in \{0, ..., \lfloor \frac{n-4}{4} \rfloor\}$, and thus, color 3 is used $\lfloor \frac{n-4}{4} \rfloor + 1$ times on $\mu(C_n)$. Therefore, the packing coloring c uses colors 1, 2, and 3, $2\left\lfloor \frac{n-2}{2} \right\rfloor + \left\lfloor \frac{n-3}{4} \right\rfloor + \left\lfloor \frac{n-4}{4} \right\rfloor + 4$ times on $\mu(C_n)$.

By comparing the results obtained concerning the use of colors $1, 2$ and 3 on $\mu(C_n)$, it is easy to check that for every $n \geq 8$, we have $n + \left\lfloor \frac{n-3}{3} \right\rfloor + \left\lfloor \frac{n-4}{9} \right\rfloor + \left\lfloor \frac{n-8}{9} \right\rfloor + 3 \geq 2\left\lfloor \frac{n-2}{2} \right\rfloor + \left\lfloor \frac{n-3}{4} \right\rfloor + \left\lfloor \frac{n-4}{4} \right\rfloor + 4 \geq \left\lfloor \frac{n-2}{2} \right\rfloor + \left\lfloor \frac{n-3}{4} \right\rfloor + \left\lfloor \frac{n-4}{4} \right\rfloor + 4$.

As a result, and taking into consideration that $\left|V(\mu(C_n))\right| = 2n+1$, we infer that $\chi_p(\mu(C_n)) \geq (2n+1) - \left(n + \left\lfloor \frac{n-3}{3} \right\rfloor + \left\lfloor \frac{n-4}{9} \right\rfloor + \left\lfloor \frac{n-8}{9} \right\rfloor + 3\right) + 3$, and so $\chi_p(\mu(C_n)) \geq n - \left\lfloor \frac{n-3}{3} \right\rfloor - \left\lfloor \frac{n-4}{9} \right\rfloor - \left\lfloor \frac{n-8}{9} \right\rfloor + 1$. Hence, the lower bound is proven.

To prove that $\chi_p(\mu(C_n)) \leq n - \left\lfloor \frac{n-3}{3} \right\rfloor - \left\lfloor \frac{n-4}{9} \right\rfloor - \left\lfloor \frac{n-8}{9} \right\rfloor + 1$ holds for every $n \geq 8$, we form the following packing coloring. Let $c(x_i) = 1$ for $i \in \{1, ..., n\}$ and in this case we have n vertices colored with 1. To color the vertices v_i, $i \in \{1, ..., n\}$, we use the following coloring. Let $c(v_i) = 2$ for $i = 3k+1$; $k \in \left\{0, ..., \lfloor \frac{n-3}{3} \rfloor\right\}$, $i \in \{1, ..., n-2\}$. In this case $\lfloor \frac{n-3}{3} \rfloor + 1$ vertices are colored with 2. Likewise, we put $c(v_i) = 3$ for $i = 9k+2$, $k \in \left\{0, ..., \lfloor \frac{n-4}{9} \rfloor\right\}$; $n \geq 4$ and $c(v_i) = 3$ for $i = 9k+6$, $k \in \left\{0, ..., \lfloor \frac{n-8}{9} \rfloor\right\}$; $n \geq 8$, and therefore we have $\lfloor \frac{n-4}{9} \rfloor + \lfloor \frac{n-8}{9} \rfloor + 2$ vertices colored with 3. The other vertices of the subgraph C_n of $\mu(C_n)$ take pairwise different colors from $1, 2$ and 3, i.e., we have $(2n+1) - \left(n + \left(\lfloor \frac{n-3}{3} \rfloor + 1\right) + \left(\lfloor \frac{n-4}{9} \rfloor + \lfloor \frac{n-8}{9} \rfloor + 2\right)\right) = n - \lfloor \frac{n-3}{3} \rfloor - \lfloor \frac{n-4}{9} \rfloor - \lfloor \frac{n-8}{9} \rfloor - 2$ additional vertices colored with various colors from 1, 2 and 3. As a result, we have $2n+1$ vertices which correspond to the order of the graph $\mu(C_n)$. Therefore, this last coloring is a packing coloring of $\mu(C_n)$. Thus, we conclude that $\chi_p(\mu(C_n)) \leq n - \left\lfloor \frac{n-3}{3} \right\rfloor - \left\lfloor \frac{n-4}{9} \right\rfloor - \left\lfloor \frac{n-8}{9} \right\rfloor + 1$. Hence, our upper bound is proven. This completes the proof. □

Figure 5 illustrates the packing chromatic number of Mycielski of a cycle C_{10}.

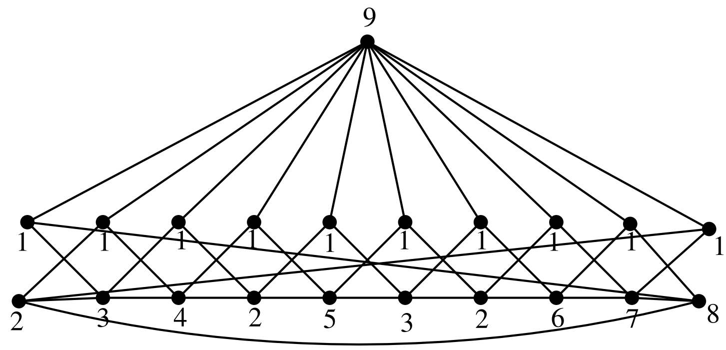

Figure 5. A packing coloring of $\mu(C_{10})$

3 Power graph

In this section, we consider the packing chromatic number of the $p-th$ power graph, with a special emphasis on the second power graph ($2-th$ power graph), for specific graphs such as the Mycielski graph of a path and a cycle. We provide the exact values of the packing chromatic numbers of the second power graph for these graphs. It is worth noting that in the proof of these theorems, we can also identify the vertex cover of each graph and subsequently use Proposition 1, given that these graphs have a diameter of 2.

First, we determine the exact value of the packing chromatic number of the second power graph of the Mycielski of a path of order $n \geq 2$ using the following theorem.

Theorem 3.1. *If $n \geq 2$, then $\chi_p((\mu(P_n))^2) = 2n - \left\lfloor \frac{n-1}{3} \right\rfloor + 1$.*

Proof. Denote by v_i, $i \in \{1, ..., n\}$ the vertices of the path P_n and by x_i, $i \in \{1, ..., n\}$ the corresponding vertices to the vertices v_i. Denote by w the vertex connected by the vertices x_i.

First, if $n = 2$, then $(\mu(P_2))^2$ is isomorphic to a complete graph K_5. Therefore $\chi_p((\mu(P_2))^2) = \chi_p(K_5) = 5$ and we are done. If $n = 3$, then $\chi_p((\mu(P_3))^2)$ is isomorphic to the complete graph K_7 and thus $\chi_p((\mu(P_3))^2) = 7$. Now, suppose that $n \geq 4$. In order to prove that $\chi_p((\mu(P_n))^2) \leq 2n - \left\lfloor \frac{n-1}{3} \right\rfloor + 1$, we form a $\left(2n - \left\lfloor \frac{n-1}{3} \right\rfloor + 1\right)-$packing coloring of $(\mu(P_n))^2$ as follows. Let $c(v_i) = 1$ for $i = 3k + 1$, $k \in \{0, ..., \left\lfloor \frac{n-1}{3} \right\rfloor\}$, and so we have $\left\lfloor \frac{n-1}{3} \right\rfloor + 1$ vertices colored with 1 and since $diam\left((\mu\ (P_n))^2\right) = 2$, we get $2n - \lfloor \frac{n-1}{3} \rfloor$ vertices with pairwise different colors from $\{2, ..., 2n - \lfloor \frac{n-1}{3} \rfloor + 1\}$. Clearly, this is a packing coloring of a considered graph, and therefore $\chi_p((\mu(P_n))^2) \leq 2n - \lfloor \frac{n-1}{3} \rfloor + 1$ for any $n \geq 4$, and our upper bound is proved.

Let us now show the lower bound $\chi_p((\mu(P_n))^2) \geq 2n - \lfloor \frac{n-1}{3} \rfloor + 1$ holds for any $n \geq 4$. Let c be any optimal packing coloring of $(\mu(P_n))^2$, where $n \geq 4$. Since $diam(\mu(P_n))^2) = 2$, the only color which can be used more than once is 1. Therefore, if $c(w) = 1$, then $c(x_i) \neq 1$ for every $i \in \{1, ..., n\}$. Similarly, since $d(w, x_i) = 1$ and $d(w, v_i) = 1$, $c(v_i) \neq 1$ for all $i \in \{1, ..., n\}$. By consequence, the color 1 is used only once on $(\mu(P_n))^2$.

However, there are two other methods for coloring the vertices of $(\mu(P_n))^2$, which are as follows. We can use the color 1 only once on the vertices x_i, $i \in \{1, ..., n\}$ (since the subgraph induced by $\{x_i,\ 1 \leq i \leq n\}$ is a complete graph). Therefore, without loss of generality assume that $c(x_1) = 1$, which implies that the color 1 is used on the vertices v_i, $i \in \{1, ..., n\}$. Let $c(v_1) = 1$ for $i = 3k + 1$, $k \in \{0, ..., \lfloor \frac{n-1}{3} \rfloor\}$ and then, the color 1 is used in this case on the vertices v_i at most $\lfloor \frac{n-1}{3} \rfloor$ times, not including the only vertex of the set $\{x_i, 1 \leq i \leq n\}$, which is also colored with 1. So we have $\left\lfloor \frac{n-1}{3} \right\rfloor + 1$ vertices colored with 1 on $(\mu(P_n))^2$. The remaining vertices of $(\mu(P_n))^2$ receive pairwise different colors from 1, i.e., color with $2n - \left\lfloor \frac{n-1}{3} \right\rfloor$ additional colors. Therefore, c uses at least $2n - \left\lfloor \frac{n-1}{3} \right\rfloor + 1$ colors.

We can also use color 1 only on the path P_n, as shown below: let $c(v_i) = 1$ for $i = 3k + 1$, $k \in \{0, ..., \lfloor \frac{n-1}{3} \rfloor\}$, and also, we have in this case $\left\lfloor \frac{n-1}{3} \right\rfloor + 1$ vertices colored with color 1. As a result, and since $\left|V\left((\mu\,(P_n))^2\right)\right| = 2n+1$, we deduce that $\chi_p((\mu\,(P_n))^2) \geq (2n+1) - \left(\left\lfloor \frac{n-1}{3} \right\rfloor + 1\right) + 1 = 2n - \left\lfloor \frac{n-1}{3} \right\rfloor + 1$, and the lower bound is proven. So, based on the above results, we conclude that $\chi_p((\mu(P_n))^2) = 2n - \lfloor \frac{n-1}{3} \rfloor + 1$. □

Figure 6 provides a packing coloring of $(\mu(P_8))^2$, described in the previous proof.

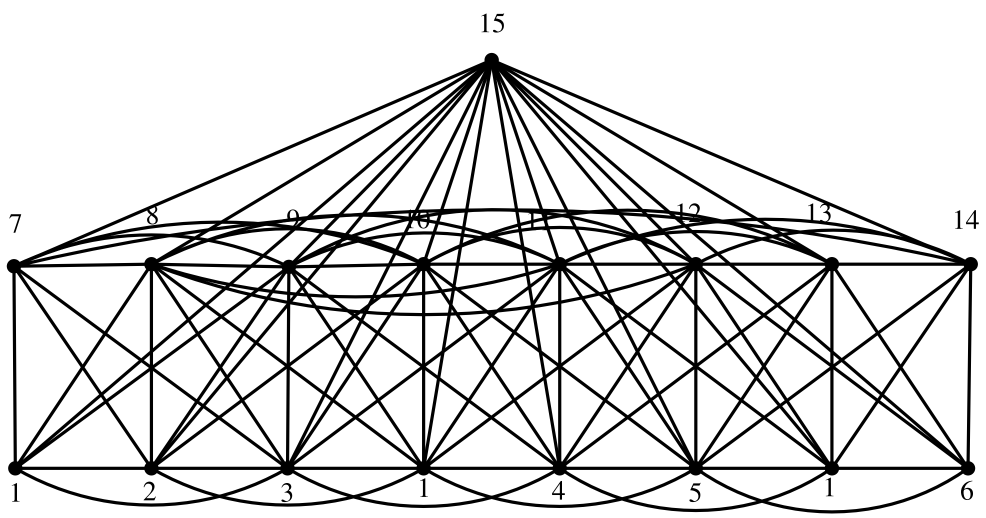

Figure 6. $(\mu(P_8))^2$ and a packing coloring of this graph

The following theorem gives the packing chromatic number for the second power graph of Mycielski of a cycle.

Theorem 3.2. *For $n \geq 3$, $\chi_p((\mu(C_n))^2) = 2n - \lfloor \frac{n-3}{3} \rfloor + 1$.*

Proof. Denote by v_i, $i \in \{1, ..., n\}$ the vertices of the cycle C_n and by x_i, $i \in \{1, ..., n\}$ the corresponding vertices to v_i. Let w be the neighbor to all vertices x_i, $i \in \{1, ..., n\}$. As $diam((\mu(C_n))^2) = 2$, the only color which can be used multiple times is color 1.

First, suppose that $n \in \{3, 4, 5\}$. The graph $(\mu(C_n))^2$ is isomorphic to K_{2n+1} and therefore $\chi_p((\mu(C_n))^2) = 2n + 1$ and we are done.

Now, suppose that $n \geq 6$. Let c be a packing coloring of $(\mu(C_n))^2$. We distinguish 3 cases.

Case 1. $c(w) = 1$. Then the other vertices of this graph take different colors from 1, i.e., $c(x_i) \neq 1$ and $c(v_i) \neq 1$ for every $i \in \{1, ..., n\}$, since $d(w, x_i) = 1$ and $d(w, v_i) = 1$, $i \in \{1, ..., n\}$, and in this case, the color 1 is used only once on $(\mu(C_n))^2$.

Case 2. $c(w) \neq 1$. If there exists $h \in \{1, ..., n\}$ such that $c(x_h) = 1$, then $c(x_i) \neq 1$ for every $i \neq h$; $i \in \{1, ..., n\}$, since $d(x_i, x_h) = 1$ (note that the subgraph formed by the vertices x_i, $i \in \{1, ..., n\}$ is a complete graph) and in this case the color 1 is used only once on the vertices x_i, $i \in \{1, ..., n\}$. Hence, without loss of generality, suppose that $c(x_1) = 1$. Then, the coloring c can assign the color 1 to at most as possible vertices from a subgraph C_n as follows. Let $c(v_i) = 1$, $i = 3k + 1$ and $k \in \{1, ..., \lfloor \frac{n-3}{3} \rfloor\}$. Therefore the color 1 is used $\lfloor \frac{n-3}{3} \rfloor$ times on the cycle C_n, and since $c(x_1) = 1$, we conclude that the color 1 is used $\lfloor \frac{n-3}{3} \rfloor + 1$ times on $(\mu(C_n))^2$.

Case 3. Use the color 1 only on the cycle C_n. Let $c(v_i) = 1$ for $i = 3k + 1$ and $k \in \{0, ..., \lfloor \frac{n-3}{3} \rfloor\}$. Therefore, we have $\lfloor \frac{n-3}{3} \rfloor + 1$ vertices colored with 1 and the other vertices of $(\mu(C_n))^2$ get pairwise different colors from 1. According to the three preceding cases, and as $\left|V\left((\mu(C_n))^2\right)\right| = 2n + 1$, we conclude that $\chi_p((\mu(C_n))^2) \geq (2n + 1) - \left(\left\lfloor \frac{n-3}{3} \right\rfloor + 1\right) + 1 = 2n - \left\lfloor \frac{n-3}{3} \right\rfloor + 1$. Now, we form the following $\left(2n - \lfloor \frac{n-3}{3} \rfloor + 1\right)$−packing coloring: let $c(v_i) = 1$ for $i = 3k+1$, $k \in \{0, ..., \lfloor \frac{n-3}{3} \rfloor\}$. The other vertices of this graph receive pairwise different colors from 1, i.e., from the set $\left\{2, ..., 2n - \lfloor \frac{n-3}{3} \rfloor + 1\right\}$. It is easy to check that c is a packing coloring and therefore $\chi_p((\mu(C_n))^2) \leq 2n - \lfloor \frac{n-3}{3} \rfloor + 1$.

Finally, we have proven that $\chi_p((\mu(C_n))^2) = 2n - \lfloor \frac{n-3}{3} \rfloor + 1$, what completes the proof. □

Figure 7 provides a packing coloring of $(\mu(C_n))^2$, as previously explained in the proof.

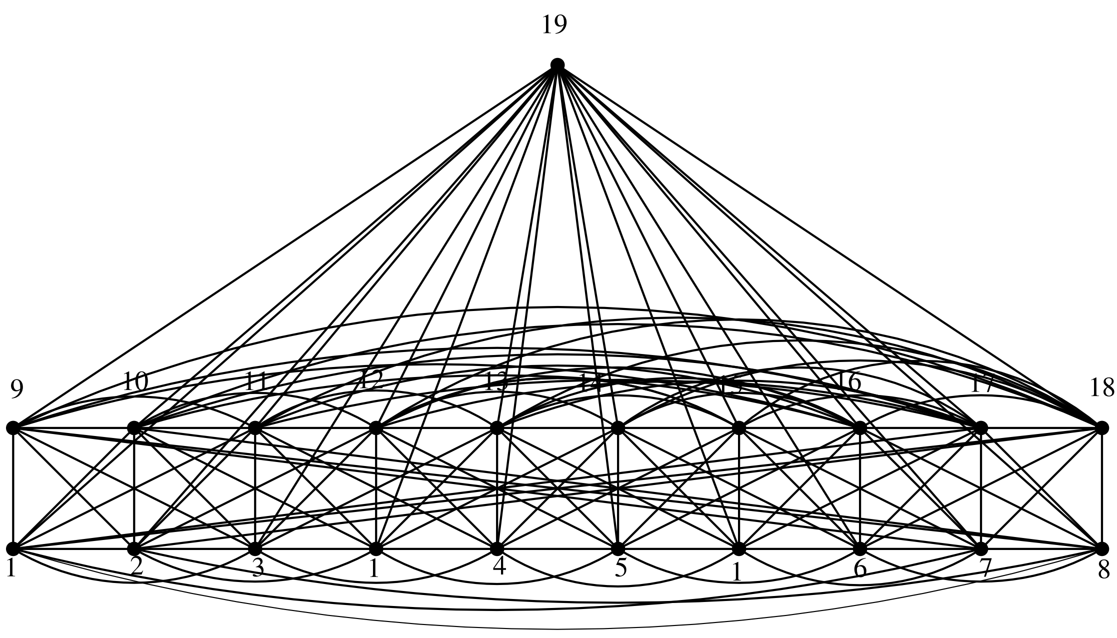

Figure 7. A packing coloring of $(\mu(C_n))^2$

4 Concluding remarks

In this paper, we investigated the packing coloring of Mycielski graphs. While the packing chromatic number of a given graph has been studied extensively in literature. As we mentioned in the first section, the packing chromatic number of the Mycielskian of paths and cycles refine recent results, where only bounds were given by the authors in [10]. Our contribution in this paper is to consider these types of graphs by enhancing these results, and determining the exact values of packing chromatic numbers for Mycielskian of paths and cycles. Since the authors of the aforementioned paper also provided bounds for the packing chromatic number of some generalized Mycielski graphs, it would be interesting to consider such a class of graphs and determine bounds or exact values for the packing chromatic number of generalized Mycielskian graphs of other classes of graphs (not necessarily paths and cycles). Another open problem that arises from the second part of our work is determining the packing chromatic number of a $p-th$ power graph for some other classes of graphs when $p \geq 3$ (not necessarily for $p = 2$).

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References

[1] A. Brandstädt, V. D. Chepoi, F. F. Dragan, Perfect elimination orderings of chordal powers of graphs, Discrete Mathematics, **158**, 273-278, (1996).

[2] J. Balogh, A. Kostochka, X. Liu, Packing chromatic number of subcubic graphs, Discrete Mathematics, **341**, 474-483, (2018).

[3] J. Balogh, A. Kostochka, X. Liu, Packing chromatic number of subdivisions of cubic graphs, Graphs and Combinatorics, **35**, 513-537, (2019).

[4] M. Barnaby, F. Raimondi, T. Chen, J. Martin, The packing chromatic number of the infinite square lattice is between 13 and 15, Discrete Applied Mathematics, **225**, 136-142, (2017).

[5] B. Brešar, J. Ferme, Packing coloring of Sierpiński-type graphs, Aequationes Mathematicae, **92**, 1091-1118, (2018).

[6] B. Brešar, J. Ferme, An infinite family of subcubic graphs with unbounded packing chromatic number, Discrete Mathematics, **341**, 2337-2342, (2018).

[7] B. Brešar, S. Klavžar, D.F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, Discrete Applied Mathematics, **155**, 2303-2311, (2007).

[8] B. Brešar, S. Klavžar, D.F. Rall, K. Wash, Packing chromatic number under local changes in a graph, Discrete Mathematics, **340**, 1110-1115, (2017).

[9] B. Brešar, S. Klavžar, D.F. Rall, K. Wash, Packing chromatic number versus chromatic and clique number, Aequationes Mathematicae, **92**, 497-513, (2018).

[10] Z. Bidine, T. Gadi and M. Kchikech, Independance number and packing coloring of generalized Mycielski graphs, Discussiones Mathematicae Graph Theory, **41**, 725-747, (2021).

[11] E. Dahlhaus and P. Duchet, On strongly chordal graphs, Ars Combinatoria, **24B**, 23-30, (1987).

[12] T. Došlić, Mycielskians and matchings, Discussiones Mathematicae Graph Theory, **25**, 261-266, (2005).

[13] J. Ekstein, P. Holub, O. Togni, The packing coloring of distance graphs $D(k,t)$, Discrete Applied Mathematics, **167**, 100-106, (2014).

[14] D.C. Fisher, P.A. McKenna and E.D. Boyer, Hamiltonicity, diameter, domina- tion, packing, and biclique partitions of Mycielski's graphs, Discrete Applied Mathematics, **84**, 93-105, (1998).

[15] J. Fiala, S. Klavžar, B. Lidický, The packing chromatic number of infinite product graphs, European Journal of Combinatorics, **30**, 1101-1113, (2009).

[16] J. Fiala and P.A. Golovach. Complexity of the packing chromatic number for trees, Discrete Applied Mathematics, **158**, 771-778, (2010).

[17] A. Finbow, D. F. Rall, On the packing chromatic number of some lattices, Discrete Applied Mathematics, **158**, 1224-1228, (2010).

[18] M. Grötschel, L. Lovász, A. Schrijver, Geometric Algorithms and Combi- natorial Optimization, 2nd edn, Springer Berlin, (1993).

[19] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris, D.F. Rall, Broadcast chromatic numbers of graphs, Ars Combinatoria, **86**, 33-49, (2008).

[20] L. Guo, R. Liu, and X. Guo, Super connectivity and super edge connectivity of the Mycielskian of a graph, Graphs and Combinatorics, **28**, 143-147, (2012).

[21] N. Gastineau, O. Togni, On the packing chromatic number of subcubic outerplanar graphs, Discrete Applied Mathematics, **255**, 209-221, (2019).

[22] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller, J.W. Thatcher (Eds.), Complexity of Computer Computations, Plenum Press, New York, 85-103, (1972).

[23] H. Kheddouci, J. F. Saclé, M. Woźniak, Packing two copies of a tree into its fourth power, Discrete Mathematics, **213**, (1-3), 169-178, (2000).

[24] M. Kim, B. Lidický, T. Masaøik, F. Pfender, Notes on complexity of packing coloring, Information Processing Letters, **137**, 6-10, (2018).

[25] S. Klavžar, D.F. Rall, Packing chromatic vertex-critical graphs, Discrete Mathematics & Theoretical Computer Science, **21**, paper #8, 18 pages, (2019).

[26] D. Korže, A. Vesel, On the packing chromatic number of square and hexagonal lattice, Ars Mathematica Contemporanea, **7**, 13-22, (2014).

[27] D. Korže, A. Vesel, (d, n)-packing colorings of infinite lattices, Discrete Applied Mathematics, **237**, 97-108, (2018).

[28] D. Korže, A. Vesel, Packing coloring of generalized Sierpiński graphs, Discrete Mathematics & Theoretical Computer Science, **21**, paper #7, 18 pages, (2019).

[29] R. Lemdani, M. Abbas, J. Ferme, Packing chromatic numbers of fnite super subdivisions of graphs, FILOMAT, **34 (10)**, 3275-3286, (2020).

[30] J. Mycielski, sur le coloriage des graphes, Colloquium Mathematicae, **3**, 161-162, (1955).

[31] A. William and S. Roy. Packing chromatic number of certain graphs, International Journal of Pure and Applied Mathematics, **87 (06)**, 731-739, (2013).

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