Palestine Journal of Mathematics

Vol 13(4)(2024) , 150–163

On an integral transformation for Generalized Differential Equations

Juan E. Nápoles Valdes

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 26A33, Secondary 34K37.

Keywords and phrases: Fractional derivatives e integral, generalized derivative, fractional calculus

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract *In this paper, we present some results related to obtaining and studying the solutions of generalized differential equations, using a certain generalized integral transformation.*

1 Introduction

We know that the Fractional Calculus is contemporary with the Ordinary Calculus, classical, integer order, this area, together with the Generalized Calculus are areas in expansion and continuous development nowadays. In the last 50 years they have focused the attention of pure and applied researchers and today they constitute one of the most dynamic areas of Mathematical Sciences (see [59] and [27]). Recent results can be consulted in [14, 29, 61].

In Chapter 1 of [5] the author presents a historical tour of the development of the theory of differential operators (from Newton to Caputo), including both local and global operators, providing a large number of applications, with a qualitative difference between both types of operators, and as a culmination, it presents a definition of a local derivative with a new parameter. In conclusion, we can ¿... we can conclude that the Riemann-Liouville and Caputo operators are not derivatives and, therefore, they are not fractional derivatives, but fractional operators. We agree with the result [62] that the local fractional operator is not a fractional derivative" (p.24). These local operators are new tools, as useful as the global ones and the integer ones, and which have demonstrated their usefulness and potential in modeling and different applications.

Although local fractional derivatives have been used since the 1960s, it was not until 2014 that they were formalized with the work [38], where a local derivative, called conformable, is defined as follows

Definition 1.1. Given a function $\phi : [0, +\infty) \to \mathbb{R}$, then the conformable fractional derivative of f of order α, with $0 < \alpha \le 1$, is defined by

$$T_\alpha \phi(\tau) = \lim_{\varepsilon \to 0} \frac{\phi(\tau + \varepsilon t^{(1-\alpha)}) - \phi(\tau)}{\varepsilon}, \quad t > 0. \tag{1.1}$$

Remark 1.2. If f is T-differentiable in some $0 < \alpha \le 1$, and $\underset{t\to 0^+}{lim} T_\alpha \phi(\tau)$ exists, then define $T_\alpha \phi(0) = \lim_{t\to 0^+} T_\alpha \phi(\tau)$. Additionally we have if f is differentiable then $T_\alpha \phi(\tau) = \phi'(\tau) t^{(1-\alpha)}$, of the latter we see that if $\alpha \to 1$ we obtain the classical derivative.

Later, in 2018, the authors in [28] presented a fractional local derivative of a new type (see [52]).

Definition 1.3. Let $\phi : [0, +\infty) \to \mathbb{R}$ a function. Then the N-derivative of ϕ of order α is defined by ${}_1N^{(\alpha)}\phi(\tau) = \lim_{h\to 0} \frac{\phi\left(\tau + he^{t^{-\alpha}}\right) - \phi(\tau)}{h}$ for all $t > 0, \alpha \in (0,1)$. If ϕ is α-differentiable in some $(0, a)$, and $\lim_{t\to 0+} {}_1N^{(\alpha)}\phi(\tau)$ exists, then define ${}_1N^{(\alpha)}\phi(0) = \lim_{t\to 0+} {}_1N^{(\alpha)}\phi(\tau)$

Lemma 1.4. *Let $\phi : [0, +\infty) \to \mathbb{R}$ be differentiable, then*

$$ {}_1N^{(\alpha)}\phi(\tau) = e^{t^{-\alpha}}\phi'(\tau). \tag{1.2}$$

Remark 1.5. The authors justify the "non-conformable" term with which they named it, since from (1.2) we obtain that when $\alpha \to 1$ the ordinary derivative is not obtained and, therefore, the slope of the tangent line to the curve at the point is not maintained.

A generalized derivative was defined in [53] (see also [23, 68]) in the following way.

Definition 1.6. Let $\phi : [0, +\infty) \to \mathbb{R}$, $\alpha \in (0, 1)$ and $\phi(\tau, \alpha)$ be some nonegative and absolutly continuous function on $I \times (0, 1]$. Then, the N-derivative of ϕ of order α is defined by

$$N_F^\alpha \phi(\tau) = \lim_{\varepsilon\to 0} \frac{\phi(\tau + \varepsilon F(\tau,\alpha)) - \phi(\tau)}{\varepsilon}, \quad \tau > 0. \tag{1.3}$$

If ϕ is α-differentiable in some $0 < \alpha \leq 1$, and $\underset{t\to 0^+}{lim} N_F^\alpha \phi(\tau)$ exists, then define $N_F^\alpha \phi(0) = \lim_{t\to 0^+} N_F^\alpha \phi(\tau)$.

Remark 1.7. This generalized derivative has proven its usefulness in various applications, to get an idea we recommend consulting [31, 44, 46, 47, 48, 49, 51, 54, 55, 56, 63, 64, 65, 66].

Remark 1.8. If the kernel of the previous definition is $F \equiv 1$ the classical derivative is obtained. By other hand, if we put $h = \varepsilon F(\tau, \alpha)$ in (1.3) then

$$N_F^\alpha \phi(\tau) = \lim_{h\to 0} \frac{\phi(\tau + h) - \phi(\tau)}{h} F(\tau, \alpha),$$

if f is differentiable then $N_F^\alpha \phi(\tau) = \phi'(\tau)F(\tau, \alpha)$.

Remark 1.9. It is clear that this definition encompasses both conformable and non-conformable derivatives, which have appeared in recent years (see [28, 52] and the references cited above in Remark 1.7). On the other hand, if $F(\tau, \beta) = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}$ then we obtain the Beta-derivative of [4, 7, 8].

The following is a result that distinguishes local derivatives from global classical ones.

Theorem 1.10. *Let $\alpha \in (0, 1]$, γ N-differentiable at $\tau > 0$ and ϕ differentiable at $\gamma(\tau)$ then $N_F^\alpha(\phi \circ \gamma)(\tau) = \phi'(\gamma(\tau))N_F^\alpha \gamma(\tau)$.*

Remark 1.11. From the above definition, it is not difficult to extend the order of the derivative for $0 \leq n - 1 < \alpha \leq n$ by putting

$$N_F^\alpha h(\tau) = \lim_{\varepsilon\to 0} \frac{h^{(n-1)}(\tau + \varepsilon\phi(\tau, \alpha)) - h^{(n-1)}(\tau)}{\varepsilon}. \tag{1.4}$$

If $h^{(n)}$ exists on some interval $I \subseteq \mathbb{R}$, then we have $N_F^\alpha h(\tau) = \phi(\tau, \alpha)h^{(n)}(\tau)$, with $0 \leq n - 1 < \alpha \leq n$.

The following result is the generalized version of the Mean Value Theorem.

Theorem 1.12. *Let $a_1 > 0$, and $\phi : [a, b] \to \mathbb{R}$ be a function that satisfies*

i) ϕ is a continuous function on $[a_1, a_2]$

ii) ϕ is N-differentiable on (a_1, a_2), for some $\alpha \in (0, 1]$

Then, exists $c \in (a_1, a_2)$ such that

$$N_F^\alpha \phi(c) = \left[\frac{\phi(a_2) - \phi(a_1)}{a_2 - a_1}\right] \phi(c, \alpha).$$

We can define the following associat integral (see [23, 32, 68]). Hereafter we will consider that the kernel F of the following definition is a nonnegative and absolutely continuous function on $I \times (0, 1]$.

Definition 1.13. We consider an interval $I \subseteq \mathbb{R}$, $a_1, \tau \in I$ and $\alpha \in \mathbb{R}$. J^α_{F,a_1} is the integral operator defined in the following way, for every locally integrable function ϕ on I as

$$J^\alpha_{F,a_1}(\phi)(\tau) = \int_{a_1}^{\tau} \frac{\phi(s)}{F(s,\alpha)} ds = \int_{a_1}^{\tau} \phi(s) d_F s, \tau > a_1. \tag{1.5}$$

The statement below is similar to the one well known from the classical calculus (see [23], [32]and [68]).

Theorem 1.14. *Let ψ be N-differentiable function in (τ_0, ∞) with $\alpha \in (0, 1]$. Then for all $\tau > \tau_0$ we have*

a) $J^\alpha_{F,\tau_0}(N^\alpha_F \psi(\tau)) = \psi(\tau) - \psi(\tau_0)$.

b) $N^\alpha_F \left(J^\alpha_{F,\tau_0} \psi(\tau)\right) = \psi(\tau)$.

Theorem 1.15. *(see [32]) Suppose that functions ϕ and γ satisfy the following assumptions on $[a_1, a_2]$:*

(1) ϕ, γ *are integrable functions on* $[a_1, a_2]$.

(2) Let γ be a nonnegative (or nonpositive) function on $[a_1, a_2]$.

(3) Let $m = \underset{[a_1,a_2]}{inf} |\phi(\tau)|$ *and* $M = \underset{[a_1,a_2]}{sup} |\phi(\tau)|$.

Then, there exists a number $x_0 \in [a_1, a_2]$ such that $\phi(x_0) \in [m, M]$ and

$$J^\alpha_{F,a_1}(\phi\gamma)(\tau) = \phi(x_0) J^\alpha_{F,a_1}(\gamma)(\tau) \tag{1.6}$$

An important propert (the Integration by parts) is established in the following result (see Theorem 4 of [45]).

Theorem 1.16. *Let u and v be N-differentiable function in (τ_0, ∞) with $\alpha \in (0, 1]$. Then for all $\tau > \tau_0$ we have*

$$J^\alpha_{F,\tau_0}((u N^\alpha_F v)(\tau)) = [uv(\tau) - uv(\tau_0)] - J^\alpha_{F,\tau_0}((v N^\alpha_F u)(\tau))$$

From [47] we can define the generalized partial derivatives in the following way.

Definition 1.17. We consider a real valued function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ a point whose ith component is positive. Then the generalized partial N-derivative of ϕ of order α in the point $\overrightarrow{a} = (a_1, \ldots, a_n)$ is defined by

$$N^\alpha_{F_i} \phi(\overrightarrow{a}) = \lim_{\varepsilon \to 0} \frac{\phi(a_1, .., a_i + \varepsilon F_i(a_i, \alpha), \ldots, a_n) - \phi(a_1, \ldots, a_i, ..., a_n))}{\varepsilon} \tag{1.7}$$

if it exists, is denoted $N^\alpha_{F_i,t_i} \phi(\overrightarrow{a})$, and it's called the ith generalized partial derivative of ϕ at $\overrightarrow{a}$ of order $\alpha \in (0, 1]$.

Remark 1.18. If $\phi : \mathbb{R}^n \to \mathbb{R}$ is a real valued function with n variables, with all generalized partial derivatives of the order $\alpha \in (0, 1]$ at $\overrightarrow{a}$ and each $a_i > 0$, then the generalized α-gradient of ϕ of the order $\alpha \in (0, 1]$ at $\overrightarrow{a}$ Is given by

$$\nabla^\alpha_N \phi(\overrightarrow{a}) = (N^\alpha_{t_1} \phi(\overrightarrow{a}), \ldots, N^\alpha_{t_n} \phi(\overrightarrow{a})) \tag{1.8}$$

In this paper we base the use of a generalized integral transformation, which allows us to study generalized, ordinary, and partial differential equations, reducing them to the classic ordinary differential equations. Inconsistencies in trying to use this method for classical fractional differential equations are noted below.

2 Main Results

In recent years, many results have been made to find exact solutions and study the qualitative behavior of such solutions, to problems that can be modeled mathematically by fractional or generalized derivatives. In particular, in the study of Fractional Equations and Systems (of one or more variables), several transformations have been used to convert these, in the corresponding equations or ordinary systems. Unfortunately, sometimes this technique has been used incorrectly, the central issue is the need to have a valid Chain Rule for the fractional derivative considered. Recall that one of the limitations of global fractional derivatives (that is, defined in terms of a certain integral), is the absence of said Rule of the Chain; therefore, in those studies where the Transformation was used and that involved the derivatives of Caputo, Riemann-Liouville and its "more modern" versions gave incorrect results (see [33], [40], [41], [42] and [67] for the general fundamentals).

Let the system

$$\begin{aligned} N_F^\alpha x(\tau) &= \phi(x,y) \\ N_F^\alpha y(\tau) &= \gamma(x,y) \end{aligned} \tag{2.1}$$

Proposition 2.1. *Suppose that there is a relation*

$$\xi = \int_{\tau_0}^{\tau} \frac{ds}{F(s,\alpha)} \tag{2.2}$$

where $t_0 > 0$ is a constant and $0 < \alpha < 1$, then there exists an equation transformation pair

$$\begin{aligned} N_F^\alpha x(\tau) &= \chi'(\xi) \\ N_F^\alpha y(\tau) &= \Psi'(\xi). \end{aligned} \tag{2.3}$$

Proof. From the basic properties of the N_F^α derivative (mainly the Chain Rule, see [28] and [52]), we arrive at the following. Let $x(\tau) = \chi(\xi)$, then

$$\begin{aligned} N_F^\alpha x(\tau) &= N_F^\alpha \chi(\xi) \\ &= \left[\chi'(\xi)\frac{1}{\phi(\tau,\alpha)}\right]\phi(\tau,\alpha) \\ &= \chi'(\xi). \end{aligned}$$

Analogously, we obtain the second relation of (2.7). □

Remark 2.2. From the equations (2.7), it is clear that we can convert classical derivatives into local fractional derivatives N_F^α and vice versa.

Take the relation (2.2) we arrive, from (2.1), at the following ordinary differential system

$$\begin{aligned} \chi' = \chi'(\xi) &= \phi(\chi(\xi),\Psi(\xi)) = \phi(x,y) \\ \Psi' = \Psi'(\xi) &= \gamma(\chi(\xi),\Psi(\xi)) = \gamma(x,y) \end{aligned} \tag{2.4}$$

An interesting particular case of the previous general system is

$$\begin{aligned} \chi' &= a(x,y)H[\alpha(y) - F(x)] \\ \Psi' &= -b(y)g(x) \end{aligned} \tag{2.5}$$

The first system is the classic two-dimensional autonomous second-order system and the second is a Liénard-type system, if we put $a = b = 1$*,* $H(u) = u$*,* $\alpha(y) = y$ *and* $F(x) = \int_0^x f(s)ds$*, we obtain the classic Liénard equation, one of the paradigms of Nonlinear Analysis.*

It is clear then, that under the transformation (2.2), I can convert any generalized differential equation (or system), with any of the known local derivatives, into an ordinary differential equation (or system).

Let us specify the above a little more. Consider, in addition to the system (2.1), the following systems:

$$\begin{aligned} N_F^{\alpha} x(\tau) &= \phi(x,y) \\ N_F^{\beta} y(\tau) &= \gamma(x,y) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} x'(\tau) &= \phi(x,y)\ , \\ y'(\tau) &= \gamma(x,y)\ , \end{aligned} \tag{2.7}$$

We will consider that the functions ϕ *and* γ *are good enough, to guarantee the existence and uniqueness of the solutions of both systems.*

From (2.6) we have

$$\frac{dy}{dx} = \frac{F(\tau,\beta)}{F(\tau,\alpha)}\frac{\gamma(x,y)}{\phi(x,y)}. \tag{2.8}$$

Let's take a solution $U(x,y)$ *of (2.7) and the curve* $V(x,y)$ *which is the image of* U *by applying a certain mapping* Π*. Let's see that this image,* V*, is a solution of (2.6) as follows.*

Let us consider the region $x > 0$*, if in this region a point* (x,y) *belongs to the curve* V*, it is clear that from the (2.6) we obtain*

$$\frac{N_F^{\beta} y(\tau)}{N_F^{\alpha} x(\tau)} = \frac{F(\tau,\beta)}{F(\tau,\alpha)}\frac{dy}{dx}.$$

In this way, we obtain that the curve V *in the region* $x > 0$ *is a solution of (2.6) obtained by multiplying the solution of (2.7) by a certain "factor" that contains a fractional integral of complementary order to the derivative considered, in other words, the Jacobian of the change of variables. If the solution* U *crosses the* y*-axis at some point* $(0, y_1)$ *we will have that the solution* V *crosses the* y*-axis at* $(0, \Pi(0, y_1))$*, with* $y_1 > 0$ *or* $y_1 < 0$*.*

Let's see the opposite case.

Let $V'(x,y)$ *be a solution of (2.6) passing through the point* $(0, \Pi(0, y_1))$ *and* $U'(x,y)$ *be the inverse image of* V' *by the application of* Π*. Taking into account the uniqueness of the solutions of (2.7),* U *and* U' *coincide, and then* V *and* V' *must also coincide due to the injectivity of* Π *(the Jacobian* $\frac{F(\tau,\beta)}{F(\tau,\alpha)} \neq 0$*). In this way, we have that* Π *maps the solutions of (2.7) into those of (2.6).*

The same reasoning can be applied in the region $x < 0$*.*

From the above, it is clear that under the conditions $x\phi(x,y) > 0$ *and* $x\gamma(x,y) > 0$*, for* $x \neq 0$*, then there is a* Π *isomorphism of the phase plane of the system (2.7) in the phase plane of the system (2.6), which is a one-to-one correspondence between all solutions of (2.7) and those of (2.6).*

Remark 2.3. Additional details on the existence of the Π isomorphism, for a Liénard type system (2.5), can be found in [22].

We are going to stop at some particular cases, to illustrate the use of a transformation of type (2.2).

2.1 Ordinary case

We study the oscillation of a class of Forced Generalized Differential Equations of $\alpha+\alpha$ order with damping:

$$\left.\begin{array}{c} N_F^\alpha\left[p(\tau)N_F^\alpha x(\tau)\right]+q(\tau)\phi(\tau,x(\tau),N_F^\alpha x(\tau))N_F^\alpha x(\tau)+ \\ +g(\tau,x(\tau))=h(\tau,x(\tau),N_F^\alpha x(\tau)), \end{array}\right\} \tag{2.9}$$

for $\tau\in I$, and $0<\alpha<1$, and we will consider that the functions p, q, phi, g and h satisfy the following conditions:

a) $p\in C^\alpha(I,\mathbb{R}_+)$, $q\in C(I,\mathbb{R})$, and $h\in C(K,\mathbb{R})$.

b) $f\in C(\mathbb{R},\mathbb{R})$.

c) The function $g\in C(I,\mathbb{R})$.

d) We assume that there exist continuous functions $a:\mathbb{R}\to\mathbb{R}$ and $r,s:I\to\mathbb{R}$, for all $x\neq 0$ and $t\in I$, such that $xa(x)>0$, $a'(x)\geq c>0$; and

$$\frac{g(\tau,x(\tau))}{a(x)}\geq\rho(\tau),\,\frac{h(\tau,x(\tau),N_F^\alpha x(\tau))}{a(x)}\leq\sigma(\tau)\text{ for }x\neq 0.$$

In this section, we will only consider those solutions which are continuous and continuable to I, and are not identically zero on any half-line $[\tau_1,\infty)$ for some $\tau_1>t_0$.

From the theory of ordinary differential equations we know that:

A) (2.9) is sublinear if $a(x)$ satisfies $0<\int_0^\varepsilon\frac{dz}{a(z)}<\infty$, $0<\int_0^{-\varepsilon}\frac{dz}{a(z)}<\infty$, $\varepsilon>0$,

B) (2.9) is superlinear if $a(x)$ satisfies $0<\int_\varepsilon^\infty\frac{dz}{a(z)}<\infty$, $0<\int_{-\varepsilon}^{-\infty}\frac{dz}{a(z)}<\infty$, $\varepsilon>0$,

C) (2.9) is a mixed type if $a(x)$ satisfies $0<\int_0^\infty\frac{dz}{a(z)}<\infty$, $0<\int_0^{-\infty}\frac{dz}{a(z)}<\infty$.

We will say that equation (2.9) is oscillatory if all its solutions are oscillatory. A solution $x=x(\tau)$ of (2.9) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called non-oscillatory, that is, a non-oscillatory solution is eventually positive or negative.

We now state and prove a first oscillation criterion for equation (2.9).

Theorem 2.4. *Under conditions a)-d) we suppose that*

i) p is a bounded function for $t\in I$, i.e. $0<p(\tau)\leq a_1$ with $a_1>0$,

ii) ϕ is bounded from below, i.e. $\phi(x)\geq -c$, $c>0$ in K,

iii) there exists a continuously differentiable function $u(\tau)$ on I such that $u(\tau)>0$, $u\prime(\tau)\geq 0$ and $u''(\tau)\leq 0$ on I, and $\gamma(\tau)=N_F^\alpha u(\tau)p(\tau)+cu(\tau)q(\tau)\geq 0$, $N_F^\alpha\gamma(\tau)\leq 0$ for $t\in I$,

iv) $\lim\limits_{t\to\infty}\inf\int_{t_0}^t u(z)(\rho(z)-\sigma(z))dz>-\infty$,

v) $\lim\limits_{t\to\infty}\sup\left(\int_{t_0}^t\frac{dz}{u(z)}\right)^{-1}\int_{t_0}^t\frac{1}{u(z)}\int_{t_0}^z u(v)(\rho(v)-\sigma(v))dvdz=\infty$.

Then equation (2.9) is oscillatory.

Proof. Let's take a non-oscillating solution $x=x(\tau)$ of the equation (2.9), this can be considered of constant sign, for τ large enough, that is, $x(\tau)\neq 0$ in $[T,\infty)$ for about $T\geq\tau_0$. Furthermore, we observe that the substitution $y=-x$ transforms (2.9) in an equation subject to the same conditions about the functions involved. So, there is no loss of generality to assume that $x(\tau)>0$ for all $\tau\geq T$.

Let

$$\left.\begin{array}{l} \xi_0=\int_{\tau_0}^T\frac{ds}{F(s,\alpha)},\quad \xi=\int_{\tau_0}^\tau\frac{ds}{F(s,\alpha)}, \\ U(\xi)=u(\tau),\Gamma(\xi)=\gamma(\tau),\sigma(\xi)=\sigma(\tau),\rho(\xi)=\rho(\tau),P(\xi)=p(\tau). \end{array}\right\} \tag{T}$$

Let's define the following Riccati transformation:

$$\omega(\tau)=\frac{u(\tau)p(\tau)N_F^\alpha x(\tau)}{a(x(\tau))}\text{ for all }\tau\geq T. \tag{2.10}$$

From (2.9) and the above expression (2.10) we obtain that

$$N_F^\alpha\omega(\tau)\leq\frac{\gamma(\tau)N_F^\alpha x(\tau)}{a(x(\tau))}+u(\tau)(\sigma(\tau)-\rho(\tau))-\frac{\omega^2(\tau)N_F^\alpha a(x(\tau))}{u(\tau)p(\tau)}. \tag{2.11}$$

Let $\omega(\tau) = \Omega(\xi)$. Then $N_F^\alpha \omega(\tau) = \Omega'(\xi)$ and $N_F^\alpha u(\tau) = U'(\xi)$. So, the above inequality is transformed into

$$\Omega'(\xi) \leq \frac{\Gamma(\xi)X'(\xi)}{a(X(\xi))} + U(\xi)(\sigma(\xi) - \rho(\xi)) - \frac{\Omega^2(\xi)K'(X(\xi))}{U(\xi)P(\xi)}.$$

Consequently, integrating form ξ_0 to ξ, we obtain

$$\left.\begin{array}{c} \int_{\xi_0}^{\xi} U(z)(\rho(z) - \sigma(z))dz \leq \\ \leq -\Omega(\xi) + \Omega(\xi_0) + \int_{\xi_0}^{\xi} \frac{\Gamma(z)X'(z)}{a(X(z))}dz - \int_{\xi_0}^{\xi} \frac{\Omega^2(z)a'(X(\xi))}{U(z)P(z)}dz. \end{array}\right\} \tag{2.12}$$

The integral $\int_{\xi_0}^{\xi} \frac{\Gamma(z)X'(z)}{a(X(z))}dz$ is bounded above. This can be seen by applying the Mean Value Theorem (Theorem 1.15 for $g \equiv 1$), for $\xi \geq \xi_0$ there exists $\xi_1 \in [\xi_0, \xi]$ such that $\int_{\xi_0}^{\xi} \frac{\Gamma(z)X'(z)}{a(X(z))}dz = \Gamma(\xi_1) \int_{\xi_0}^{\xi} \frac{X'(z)}{a(X(z))}dz = \Gamma(\xi_1) \int_{X(\xi_0)}^{X(\xi)} \frac{dr}{a(r)} \leq \Gamma(\xi_1) \int_{X(\xi_0)}^{+\infty} \frac{dr}{a(r)} = k_1$. It is clear that from (2.12) we obtain

$$\int_{\xi_0}^{\xi} U(z)(\rho(z) - \sigma(z))dz \leq -\Omega(\xi) + k_2 - \int_{\xi_0}^{\xi} \frac{\Omega^2(z)a'(X(\xi))}{U(z)P(z)}dz, \tag{2.13}$$

where $k_2 = k_1 + \Omega(\xi_0)$. Or, by virtue of condition i)

$$\int_{\xi_0}^{\xi} U(z)(\rho(z) - \sigma(z))dz \leq -\Omega(\xi) + k_2 - c\int_{\xi_0}^{\xi} \frac{\Omega^2(z)}{U(z)P(z)}dz. \tag{2.14}$$

Now, we consider the behavior of X'.

CASE 1. X' is oscillatory.
Then, there exists an infinite sequence $\{\xi_n\}$, with $\xi_n = \int_T^{\tau_n} \frac{ds}{F(s,\alpha)}$, such that $\xi_n \to \infty$ as $n \to \infty$ and $X'(\xi_n) = 0$. Thus, (2.14) gives

$$\int_{\xi_0}^{\xi_n} U(z)(\rho(z) - \sigma(z))dz \leq k_2 - c\int_{\xi_0}^{\xi_n} \frac{\Omega^2(z)}{U(z)P(z)}dz.$$

From iv) we get $\frac{\Omega^2(z)}{U(z)P(z)} \in L^1[\xi_0, \infty)$. Thus, there exists a positive constant N such that $\int_{\xi_0}^{\xi_n} \frac{\Omega^2(z)}{U(z)P(z)}dz \leq N$ for every $\xi \geq \xi_0$. From this and using the Schwartz inequality we obtain that $-\int_{\xi_0}^{\xi_n} \frac{\Omega(z)}{U(z)}dz \leq \sqrt{aN}\left(\int_{\xi_0}^{\xi} \frac{dz}{U(z)}\right)^{\frac{1}{2}}$. Furthermore, (2.14) gives $\int_{\xi_0}^{\xi} U(z)(\rho(z) - \sigma(z))dz \leq -\Omega(\xi) + k_2$, hence

$$\int_{\xi_0}^{\xi} \frac{1}{U(z)} \int_{\xi_0}^{z} U(v)(\rho(v) - \sigma(v))dvdz \leq -\int_{\xi_0}^{\xi} \frac{W(z)}{U(z)}dz + k_2\int_{\xi_0}^{\xi} \frac{dz}{U(z)}. \tag{2.15}$$

Assumptions iii) implies that $U(\xi) \leq \kappa + \mu\xi$ for all large ξ, where κ and μ are positive constants. This ensures that $\int_{\xi_0}^{\xi} \frac{dz}{U(z)} = \infty$. This fact and (2.15) implies that

$$\begin{array}{l} \int_{\xi_0}^{\xi} \frac{1}{U(z)} \int_{\xi_0}^{z} U(v)(\rho(v) - \sigma(v))dvdz \leq \sqrt{aN}\left(\int_{\xi_0}^{\xi} \frac{dz}{U(z)}\right)^{\frac{1}{2}} + k_2\int_{\xi_0}^{\xi} \frac{dz}{U(z)} \leq \\ \leq \left(\sqrt{aN} + k_2\right)\int_{\xi_0}^{\xi} \frac{dz}{U(z)}. \end{array}$$

Dividing by $\int_{\xi_0}^{\xi} \frac{dz}{U(z)}$ and taking the upper limit as $\xi \to \infty$, we obtain a contradiction to ix).

CASE 2. $X' > 0$ for $\xi_0 < \xi_1 \leq \xi$.
Then, it follows from (2.14) that $\int_{\xi_0}^{\xi_n} U(z)(\rho(z) - \sigma(z))dz \leq k_2$ and consequently the desired contradiction to ix).

CASE 3. $X' < 0$ for $\xi_0 < \xi_2 \leq \xi$.

If $\frac{\Omega^2(z)}{U(z)P(z)} \in L^1[\xi_0, \infty)$, then we can follow the procedure of Case 1 to arrive at a contradiction to v). Suppose now that $\frac{\Omega^2(z)}{U(z)P(z)} \notin L^1[\xi_0, \infty)$. From iv) and (2.13) we get for some constant $\beta > 0$ that

$$-\Omega(\xi) \geq \beta + \int_{\xi_2}^{\xi} \frac{\Omega^2(z)a'(X(\xi))}{U(z)P(z)} dz \tag{2.16}$$

for every $\xi \geq \xi_2$. Since $\frac{\Omega^2(z)}{U(z)P(z)} \notin L^1[\xi_0, \infty)$, there exists $\xi_3 \geq \xi_2$ such that $M = \beta + \int_{\xi_2}^{\xi_3} \frac{\Omega^2(z)a'(X(\xi))}{U(z)P(z)} dz > 0$. Thus (2.16) ensures $\Omega(\xi)$ is negative on $[\xi_0, \infty)$ and using this fact and (2.13) we have

$$-\Omega(\xi) \left[\beta + \int_{\xi_2}^{\xi} \frac{\omega^2(z)a'(X(\xi))}{U(z)P(z)} dz \right]^{-1} \geq 1.$$

This inequality yields

$$\ln \left[\frac{\beta + \int_{\xi_2}^{\xi} \frac{\Omega^2(z)K'(X(\xi))}{U(z)P(z)} dz}{M} \right] \geq \ln \frac{a(X(\xi_3))}{a(X(\xi))}.$$

From here we obtain

$$\beta + \int_{\xi_2}^{\xi} \frac{\Omega^2(z)a'(X(\xi))}{U(z)P(z)} dz \geq \frac{M^*}{a(X(\xi))}$$

for every $\xi \geq \xi_3$, with $M^* = Ma(X(\xi_3)) > 0$. Hence from (2.16) and the above inequality we derive that $-\Omega(\xi)a(X(\xi)) \geq M^*$, and therefore

$$X(\xi) \leq X(\xi_3) - M^* \int_{\xi_3}^{\xi} \frac{dz}{U(z)P(z)} \leq X(\xi_3) - \frac{M^*}{a} \int_{\xi_3}^{\xi} \frac{dz}{U(z)},$$

follows that $X(\xi) \to -\infty$ as $\xi \to \infty$, a contradiction.

This completes the proof. □

If in the previous theorem, we specify the function u used, more practical criteria are obtained to analyze the oscillatory character of the equation (2.9).

Corollary 2.5. *Equation (2.9) is oscillatory if A), ii) hold, $\gamma(\tau) = \theta\tau^{\theta-1}p(\tau) + c\tau^{\theta}q(\tau) \geq 0$ and $N_F^{\alpha}\gamma(\tau) \leq 0$ for some $\theta \in [0, 1]$,*

$\lim\limits_{\tau\to\infty} \inf \int_{\tau_0}^{\tau} z^{\theta}(\rho(z) - \sigma(z))dz > -\infty$,

$\lim\limits_{\tau\to\infty} \frac{1}{\ln \tau} \int_{\tau_0}^{\tau} \frac{dz}{z} \int_{\tau_0}^{z} v(\rho(v) - \sigma(v))dvdz > \infty$, *if* $\theta = 1$,

$\lim\limits_{\tau\to\infty} \frac{1}{\tau^{1-\theta}} \int_{\tau_0}^{\tau} \frac{dz}{z^{\omega}} \int_{\tau_0}^{z} v^{\omega}(\rho(v) - \sigma(v))dvdz > \infty$, *if* $0 \leq \theta < 1$.

Remark 2.6. The existence of function $a(x)$ is very close to the oscillatory nature of equation (2.9). So, if assumptions A), B) and i) are not fulfilled, we can exhibit equations that have nonoscillatory solutions. For example, the equation (see [49]) with $F(\tau, \alpha) = e^{t^{-\alpha}}$:

$$N_1^{\alpha} \left[e^{-\tau^{-\alpha}} N_1^{\alpha} x(\tau) \right] + 2N_1^{\alpha} x(\tau) + e^{\tau^{-\alpha}} N_1^{\alpha} x(\tau) = 0,$$

has the nonoscillatory solution $x(\tau) = e^{-\tau}$.

In the following result we do not assume that $a(x)$ satisfies condition B). So, it may be applicable for linear, sublinear or superlinear equations.

Theorem 2.7. *Suppose that A) and i) hold. Moreover, assume that*

vi) there exists a differentiable function $\phi : I \to (0, \infty)$ and continuous real functions h, H on $D := \{(\tau, z) : t \geq z \geq t_0\}$ and H has a continuous and nonpositive partial derivative on D

with respect to the second variable such that $H(\tau,t)=0$ for $t\geq t_0$, $H(\tau,z)>0$ for $t>z\geq t_0$, and

xi) $-\frac{\partial H(\tau,z)}{\partial z}=h(\tau,z)\sqrt{H(\tau,z)}$ *for all* $(\tau,z)\in D$.

Then equation (2.9) is oscillatory if

xii)
$$\left.\begin{array}{l} \underset{t\to\infty}{Lim}\sup\ \frac{1}{H(\tau,t_0)}\int_{t_0}^{t}\{\phi(z)H(\tau,z)(\rho(z)-\sigma(z))- \\ \frac{1}{4k}\left[p(z)\phi(z)\left(h(\tau,z)-\left(\frac{cq(z)}{p(z)}+\frac{\phi'(z)}{\phi(z)}\right)\sqrt{H(\tau,z)}\right)^2\right]\Big\}dz=\infty. \end{array}\right\}$$

Proof. Let $x(\tau)$ be a nonoscillatory solution of (2.9), say $x(\tau)>0$ for $t\geq t_0$. As in Theorem 2.4, we will consider the following notations:

$$\xi_0=\int_{t_0}^{T}e^{-\phi(s,\alpha)}ds,\xi=\int_{t_0}^{t}e^{-\phi(s,\alpha)}ds,$$

$$\Phi(\xi)=\phi(\tau),Q(\xi)=q(\tau),\sigma(\xi)=\sigma(\tau),\rho(\xi)=\rho(\tau),P(\xi)=p(\tau),$$

Taking $w(\tau)=\frac{\phi(\tau)p(\tau)x'(\tau)}{a(x(\tau))}$ we obtain from (2.9) that

$$N_F^{\alpha}w(\tau)\leq\phi(\tau)(\sigma(\tau)-\rho(\tau))+\left(\frac{cq(\tau)}{p(\tau)}+\frac{N_F^{\alpha}\phi(\tau)}{\phi(\tau)}\right)w(\tau)-\frac{kw^2(\tau)}{\phi(\tau)p(\tau)}.$$

Let $w(\tau)=W(\xi)$. Then $N_F^{\alpha}w(\tau)=W'(\xi)$ and $N_F^{\alpha}\phi(\tau)=\Phi'(\xi)$. So, the above inequality is transformed into

$$W'(\xi)\leq\Phi(\xi)(\sigma(\xi)-\rho(\xi))+\left(\frac{cQ(\xi)}{P(\xi)}+\frac{\Phi'(\xi)}{\Phi(\xi)}\right)W(\xi)-\frac{kW^2(\xi)}{\Phi(\xi)P(\xi)},$$

Consequently, integrating form ξ_0 to ξ, we obtain

$$\left.\begin{array}{l} -\int_{\xi_0}^{\xi}\Phi(z)H(\xi,z)(\rho(z)-\sigma(z))dz\leq-\int_{\xi_0}^{\xi}H(\xi,z)W'(z)dz+ \\ +\int_{\xi_0}^{\xi}H(\xi,z)\left(\frac{cQ(z)}{P(z)}+\frac{\Phi'(z)}{\Phi(z)}\right)W(z)dz-\int_{\xi_0}^{\xi}\frac{kH(\xi,z)W^2(z)}{\Phi(z)P(z)}dz. \end{array}\right\}\tag{2.17}$$

From the right member of this inequality you get

$$\begin{array}{l} H(\xi,\xi_0)W(\xi_0)-\int_{\xi_0}^{\xi}\left(-\frac{\partial H(\xi,z)}{\partial z}\right)W(z)dz+ \\ +\int_{\xi_0}^{\xi}H(\xi,z)\left(\frac{cQ(z)}{P(z)}+\frac{\Phi'(z)}{\Phi(z)}\right)W(z)dz-\int_{\xi_0}^{\xi}\frac{kH(\xi,z)W^2(z)}{\Phi(z)P(z)}dz= \\ =H(\xi,\xi_0)W(\xi_0)- \\ -\int_{\xi_0}^{\xi}\left[h(\xi,z)\sqrt{H(\xi,z)}-H(\xi,z)\left(\frac{cQ(z)}{P(z)}+\frac{\Phi'(z)}{\Phi(z)}\right)W(z)+\frac{kH(\xi,z)W^2(z)}{\Phi(z)P(z)}\right]dz= \\ =H(\xi,\xi_0)W(\xi_0)- \\ -\int_{\xi_0}^{\xi}\left[\sqrt{\frac{H(\xi,z)}{\Phi(z)P(z)}}W(z)-\frac{\sqrt{\Phi(z)P(z)}}{2\sqrt{k}}\left(h(\xi,z)-\sqrt{H(\xi,z)}\left(\frac{cQ(z)}{P(z)}+\frac{\Phi'(z)}{\Phi(z)}\right)\right)\right]^2dz+ \\ +\frac{1}{4k}\int_{\xi_0}^{\xi}P(z)\Phi(z)\left(h(\xi,z)-\sqrt{H(\xi,z)}\left(\frac{cQ(z)}{P(z)}+\frac{\Phi'(z)}{\Phi(z)}\right)\right)^2dz\leq \\ \leq H(\xi,\xi_0)W(\xi_0)+\frac{1}{4k}\int_{\xi_0}^{\xi}P(z)\Phi(z)\left(h(\xi,z)-\sqrt{H(\xi,z)}\left(\frac{cQ(z)}{P(z)}+\frac{\Phi'(z)}{\Phi(z)}\right)\right)^2dz. \end{array}$$

Hence

$$\left.\begin{array}{l} \frac{1}{H(\xi,\xi_0)}\int_{\xi_0}^{\xi}\{\Phi(z)H(\xi,z)(\rho(z)-\sigma(z))- \\ \frac{1}{4k}\left[P(z)\Phi(z)\left(h(\xi,z)-\left(\frac{cQ(z)}{P(z)}+\frac{\Phi'(z)}{\Phi(z)}\right)\sqrt{H(\xi,z)}\right)\right]\}dz\leq W(\xi_0), \end{array}\right\}$$

a contradiction with xii). This completes the proof. □

Corollary 2.8. *Suppose that condition xii) in the above Theorem can be replaced by*

$$\underset{t\to\infty}{Lim}\sup\ \frac{1}{H(\tau,t_0)}\int_{t_0}^{t}\phi(z)H(\tau,z)(\rho(z)-\sigma(z))dz=\infty,$$

and

$$\underset{t\to\infty}{Lim}\sup\ \frac{1}{H(\tau,t_0)}\int_{t_0}^{t}\left[p(z)\phi(z)\left(h(\tau,z)-\left(\frac{cq(z)}{p(z)}+\frac{\phi'(z)}{\phi(z)}\right)\right)^2\right]dz\leq\infty.$$

Then the conclusion of Theorem 2.7 holds.

Remark 2.9. The equation (2.9) was studied in [49], with the derivative ${}_1N^{(\alpha)}$, therefore, our results obtained are valid for both conformable and non-conformable derivatives.

Remark 2.10. The results obtained using the transformation (T) are possible because the N-derivative admits a Chain Rule, that is to say, a similar transformation can not be defined for global fractional derivatives.

Remark 2.11. In [50] the equation of order $\alpha + \alpha$ was studied:

$$N_F^{\alpha}\left[p(\tau)\phi(x(\tau))N_F^{\alpha}x(\tau)\right] + q(\tau)g(x(\tau)) = \rho(\tau, x(\tau), N_F^{\alpha}x(\tau)).$$

The generalized version of the latter can be used using the method set out above.

2.2 Partial case

We will present some examples of conformable partial differential equations, where a transformation of the type (2.2) is used, with $\phi(\tau,\alpha) = t^{\alpha+1}$, to obtain analytical solutions and we will sketch the method for a general kernel $\phi(\tau,\alpha)$.

In [69] they study the nonlinear fractional Klein-Gordon equation (conformable):

$$\frac{\partial^{2\alpha}u(\tau,x)}{\partial t^{2\alpha}} = \frac{\partial^2 u(\tau,x)}{\partial x^2} + au(\tau,x) + cu^3(\tau,x), t > 0, 0 < \alpha \leq 1 \tag{2.18}$$

using transformation:

$$u(\tau,x) = \phi(\xi), \xi = lx - \frac{\lambda t^{\alpha}}{\alpha}. \tag{2.19}$$

Using (2.19), the equation (2.18) reduces to the following ordinary differential equation:

$$\phi_{\xi\xi} = \frac{a}{\lambda^2 - l^2}\phi + \frac{c}{\lambda^2 - l^2}\phi^3. \tag{2.20}$$

Let us now consider the Generalized Equation of the Klein-Gordon type:

$$\frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}}\left(\frac{\partial^{(\alpha)}u(\tau,x)}{\partial x^{(\alpha)}}\right) = \frac{\partial^2 u(\tau,x)}{\partial x^2} + au(\tau,x) + cu^3(\tau,x), t > 0, 0 < \alpha \leq 1 \tag{2.21}$$

where for convenience, we have written $N_{F_2,t}^{\alpha}u(\tau,x) = \frac{\partial^{(\alpha)}u(\tau,x)}{\partial x^{(\alpha)}}$. Through transformation

$$u(\tau,x) = \phi(\xi), \xi = lx - \mathbb{F}(\tau,\alpha), \tag{2.22}$$

with $\mathbb{F}(\tau,\alpha) = \int_0^t \frac{ds}{\phi(s,\alpha)}$, we have

$$\begin{aligned} \frac{\partial^{(\alpha)}u(\tau,x)}{\partial t^{(\alpha)}} &= \phi(\tau,\alpha)\frac{\partial u}{\partial t} \\ &= \phi(\tau,\alpha)\frac{d\phi(\xi)}{dt}\frac{d\xi}{dt} \\ &= (-\lambda)\frac{d\xi}{dt}, \end{aligned}$$

from this we obtain

$$\begin{aligned} &\frac{\partial^{(\alpha)}}{\partial t^{(\alpha)}}\left(\frac{\partial^{(\alpha)}u(\tau,x)}{\partial t^{(\alpha)}}\right) \\ &= \frac{\partial^{(\alpha)}}{\partial t^{(\alpha)}}\left((-\lambda)\frac{d\xi}{dt}\right) \\ &= \phi(\tau,\alpha)(-\lambda)\frac{d^2\phi(\xi)}{dt^2}\frac{d\xi}{dt} \\ &= \lambda^2\frac{d^2\phi(\xi)}{dt^2}. \end{aligned}$$

So, we obtain the same ordinary differential equation (2.20), reason why the conclusions of this work, continue to be valid.

Remark 2.12. We should note that in equation (2.18) there is a notation error, instead of $\frac{\partial^{2\alpha}u(\tau,x)}{\partial t^{2\alpha}}$, should have been written $\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^\alpha u(\tau,x)}{\partial t^\alpha}\right)$, since the Law of Indices is not fulfilled, that is, the conformable derivative of order 2α, is not equal to the conformable derivative of order $\alpha+\alpha$ which is the one used in the work. That error is corrected in our generalized equation (2.21).

Remark 2.13. The results of [13, 18, 26, 36, 39, 60] can be generalized following the idea presented above. The advantage that we obtain with this formulation concerning considering a particular kernel (in these cases, conformable o beta derivative) is that we can model the process from two points of view, varying the kernel itself and the order α.

2.3 Inconsistencies

From the previous two sections, we realize that the Chain Rule plays a major role in the transformation used. In the case of classical fractional derivatives (global), this method cannot be used due to the non-existence of such a derivative of a compound function. However, in [35] a modified Riemann-Liouville derivative and a Chain Rule is stated that was later shown to be false (see [43]), so several published works were incorrect. To interested readers, I can supply these works, which we prefer not to present here.

3 Conclusions

In this work we have presented a transformation, that allows us to convert generalized differential equations into ordinary differential equations. We show the possibilities it offers in modeling, given the double dependence of the method used (of the kernel and the α order).

References

[1] T. Abdeljawad, On conformable fractional calculus. J. Comput. Appl. Math., 279, 2015, 57-66.

[2] I. Adalar, A. S. Ozkan, Inverse problems for a conformable fractional Sturm-Liouville operator, J. Inverse Ill-Posed Probl. 2020; 28(6): 775-782 https://doi.org/10.1515/jiip-2019-0058

[3] B. P. Allahverdiev, H. Tuna, Y. Yalçinkaya, Conformable fractional Sturm-Liouville equation, Math Meth Appl Sci. 2019;1-19 DOI: 10.1002/mma.5595

[4] A. Atangana, A novel model for the lassa hemorrhagic fever: deathly disease for pregnant women, Neural Comput. Appl. 26(8) (2015) 1895-1903.

[5] A. Atangana, Derivative with a New Parameter Theory, Methods and Applications, Academic Press, 2016.

[6] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel. Theory and application to heat transfer model. Therm Sci. 2016;20(2):763-769.

[7] A. Atangana, E. F. D. Goufo, On the mathematical analysis of ebola hemorrhagic fever: deathly infection disease in West African countries, Bio. Med. Res. Int. Article ID 261383, (2014) 1-7.

[8] A. Atangana, S. C. Oukouomi Noutchie, Model of breakbone fever via betaderivatives, Bio. Med. Res. Int. Article ID 523159, (2014) 1-10.

[9] D. Baleanu, COMMENTS ON: Ortigueira M., Martynyuk V., Fedula M., Machado J.A.T., The failure of certain fractional calculus operators in two physical models, in Fract. Calc. Appl. Anal. 22(2)(2019), Fract. Calc. Appl. Anal., Volume 23: Issue 1, DOI: https://doi.org/10.1515/fca-2020-0012.

[10] D. Baleanu, K. Diethelm et. al., Fractional Calculus: Models and Numerical Methods, volumen 5. Worls Scientific Publishing Co. Pte. Ltd., Series on Complexity, Nonlinearity and Chaos, Second Ed., 2017.

[11] D. Baleanu, A. Fernandez, On fractional operators and their classifications, Mathematics, 7(830)(2019).

[12] E. Baş, I. Najemadeen, Conformable Sturm-Liouville Problem with Modified Coulomb Potential, BSEU Journal of Science, 7(2), 698-713, 2020 DOI: 10.35193/bseufbd.668272

[13] A. Biswas, A. Sonmezoglu, M. Ekici, M. Mirzazadeh, Q. Zhou, S. P. Moshokoa, M. Belic, Optical soliton perturbation with fractional temporal evolution by generalized Kudryashov's method, Optik 164 (2018) 303-310

[14] *M. Bohner, O. Tunç, C. Tunç, Qualitative analysis of Caputo fractional integro-differential equations with constant delays. Comput. Appl. Math. 40 (2021), no. 6, Paper No. 214, 17 pp.*

[15] *M. Caputo, Linear model of dissipation whose Q is almost frequency independent II. Geophysical Journal International, 1967, 13 (5): 529-539.*

[16] *M. Caputo, Elasticità e Dissipazione. Bologna: Zanichelli, 1969.*

[17] *M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel. Natural Sciences Publishing Cor., Progress in Fractional Differentiation and Applications, 1:73-85, 2015.*

[18] *M. T. Darvishi, M. Najafi, B. C. Shin, CONFORMABLE FRACTIONAL SENSE OF FOAM DRAINAGE EQUATION AND CONSTRUCTION OF ITS SOLUTIONS, J. Korean Soc. Ind. Appl. Math. Vol.25, No.3, 132-148, 2021 https://doi.org/10.12941/jksiam.2021.25.132*

[19] *W. R. Derrick, S. J. Grossman, Elementary Differential Equations with Application, Second Edition, Addison-Wesley Publishing Company, Inc., de Reading, Massachusetts, E.U.A., 1981.*

[20] *R. Díaz, E. Pariguan, On hypergeometric functions and Pochhammer k-symbol. Divulg. Mat. 15(2), 179–192 (2007).*

[21] *G. Dwivedi, J. Tyagi, R. B. Verma, Stability of positive solution to fractional logistic equations, Funkcialaj Ekvacioj, 62 (2019) 61-73.*

[22] *A. Fleitas, J. A. Méndez-Bermúdez, J. E. Nápoles, J. M. Sigarreta, On fractional Liénard-type systems, Revista Mexicana de Física 65 (6) 618-625 NOVEMBER-DECEMBER 2019*

[23] *A. Fleitas, J. E. Nápoles, J. M. Rodríguez, J. M. Sigarreta, On the generalized fractional derivative, Revista de la UMA, Revista de la UMA, Volume 62, no. 2 (2021), 443-457 https://doi.org/10.33044/revuma.1930*

[24] *C.T. Fulton, L. Wu, S. Pruess, A Sturm Separation Theorem for linear 2nth order Self-adjoint Differential Equation, Journal applied Mathematics and Stochastic Analysis, 8, Number 1,1995, pp. 29-46.*

[25] *F. Gao, C. Chi, Improvement on Conformable Fractional Derivative and Its Applications in Fractional Differential Equations, Journal of Function Spaces Volume 2020, Article ID 5852414, 10 pages https://doi.org/10.1155/2020/5852414*

[26] *B. Ghanbar, M. S. Osman, D. Baleanu, Generalized exponential rational function method for extended Zakharov Kuzetsov equation with conformable derivative, Modern Physics Letters A Vol. 34 (2019) 1950155 (16 pages) DOI: 10.1142/S0217732319501554*

[27] *R. Gorenflo, F. Mainardi, Fractals and fractional calculus in continuum mechanics, 223-276, Springer, Vienna, 1997.*

[28] *P. M. Guzmán, G. Langton, L. M. Lugo, J. Medina, J., J. E. Nápoles Valdés, A new definition of a fractional derivative of local type.* J. Math. Anal.*,* **9:2***, 88-98 (2018).*

[29] *P. M. Guzmán, Luciano M. Lugo Motta Bittencurt, Juan E. Nápoles ValdNés, A Note on Stability of Certain Liénard Fractional Equation, International Journal of Mathematics and Computer Science, 14(2019), no. 2, 301-315.*

[30] *P. M. Guzmán, L. M. Lugo, J. E. Nápoles V., On the stability of solutions of fractional non conformable differential equations, Stud. Univ. Babes-Bolyai Math. 65(2020), No. 4, 495-502 DOI: 10.24193/subbmath.2020.4.02*

[31] *P. M. Guzmán, L. M. Lugo, J. E. Nápoles V., A note on the qualitative behavior of some nonlinear local improper conformable differential equations, J. Frac Calc & Nonlinear Sys. (2020)1(1): 13-20 doi:10.48185/jfcns.v1i1.48*

[32] *P. M. Guzmán, L. M. Lugo, J. E. Nápoles Valdés, M. Vivas, On a New Generalized Integral Operator and Certain Operating Properties, Axioms 2020, 9, 69; doi:10.3390/axioms9020069*

[33] *J. H. He, S. K. Elagan, Z. B. Li, Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus, Phys. Lett. A 376, 257-259 (2012)*

[34] *D. Hinton, Sturm's 1836 oscillation results evolution of the theory, in: Sturm–Liouville theory. Past and present, Selected survey articles based on lectures presented at a colloquium and workshop in Geneva, Italy, September 15–19, 2003 to commemorate the 200th anniversary of the birth of Charles François Sturm, Birkhäuser, Basel, 2005, pp.1-27.*

[35] *G. Jumarie, Modifed Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, Computers and Mathematics with Applications, 51(9) (2006), 1367-1376.*

[36] *M. Kaplan, Applications of two reliable methods for solving a nonlinear conformable time-fractional equation, Opt Quant Electron (2017) 49:312 DOI 10.1007/s11082-017-1151-z*

[37] *U. N. Katugampola, A new fractional derivative with classical properties, e-print. arXiv preprint arXiv:1410.6535 8 (2014), https://arxiv.org/abs/1410.6535v2.*

[38] *R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, Journal of Computationaland Applied Mathematics, vol. 264, pp. 65-70, 2014.*

[39] *V. S. Kumar, H. Rezazadeh, M. Eslami, F. Izadi, M. S. Osman, Jacobi Elliptic Function Expansion Method for Solving KdV Equation with Conformable Derivative and Dual-Power Law Nonlinearity, Int. J. Appl. Comput. Math (2019) 5:127 https://doi.org/10.1007/s40819-019-0710-3*

[40] *Z. B. Li, An extended fractional complex transform, Int. J. Nonlinear Sci. Numer. Simul. 11, 0335-0337 (2010)*

[41] *Z. B. Li, J. H. He, Fractional complex transform for fractional differential equations, Math. Comput. Appl. 15, 970-973 (2010).*

[42] *Z. B. Li, J. H. He, Application of the fractional complex transform to fractional differential equations, Nonlinear Sci. Lett. A 2, 121-126 (2011)*

[43] *C. S. Liu, Counterexamples on Jumaries two basic fractional calculus formulae, Commun Nonlinear Sci Numer Simulat 22 (2015) 9294.*

[44] *L. M. Lugo, J. E. Nápoles V., M. Vivas-Cortez, ON THE OSCILLATORY BEHAVIOUR OF SOME FORCED NONLINEAR GENERALIZED DIFFERENTIAL EQUATION, REVISTA INVESTIGACION OPERACIONAL VOL. 42, NO. 2, 267-278, 2021*

[45] *M. Vivas-Cortez, L. M. Lugo, Juan E. Nápoles Valdés, M. E. Samei, A Multi-Index Generalized Derivative Some Introductory Notes, Appl. Math. Inf. Sci. 16, No. 6, 883-890 (2022).*

[46] *F. Martínez, P. O. Mohammed, J. E. Nápoles Valdés, Non Conformable Fractional Laplace Transform, Kragujevac Journal of Mathematics Volume 46(3), 341–354.*

[47] *F. Martínez, J. E. Nápoles Valdés, Towards a Non-Conformable Fractional Calculus of N-variables, Journal of Mathematics and Applications, No 43, 87-98 (2020)*

[48] *F. Martínez, J. E. Nápoles Valdés, A NOTE ON THE ASYMPTOTIC PROPERTIES OF A GENERALIZED DIFFERENTIAL EQUATIONS, JFCA-2022/13(1), 30-41*

[49] *J. E. Nápoles Valdés, Oscillatory Criteria for Some non Conformable Differential Equation with Damping, Discontinuity, Nonlinearity, and Complexity 10(3) (2021) 461-469*

[50] *J. E. Nápoles Valdés, ON THE OSCILLATORY CHARACTER OF SOME NON CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATION, submited*

[51] *J. E. Nápoles Valdés, Y. S. Gasimov, A. R. Aliyeva, On the Oscillatory Behavior of Some Generalized Differential Equation, Punjab University Journal of Mathematics, Vol. 53(1)(2021) 71-82*

[52] *J. E. Nápoles V., P. M. Guzmán, L. M. Lugo, Some New Results on Nonconformable Fractional Calculus, Advances in Dynamical Systems and Applications, Volume 13, Number 2, pp. 167-175 (2018).*

[53] *J. E. Nápoles, P. M. Guzmán, L. M. Lugo, A. Kashuri, The local non-conformable derivative and Mittag–Leffler function, Sigma J Eng & Nat Sci 38 (2), 2020, 1007-1017.*

[54] *J. E. Nápoles V., M. N. Quevedo, On the Oscillatory Nature of Some Generalized Emden-Fowler Equation, Punjab University Journal of Mathematics, Vol. 52(6)(2020) pp. 97-106*

[55] *J. E. Nápoles V., M. N. Quevedo, A. R. Gómez P., ON THE ASYMPTOTIC BEHAVIOR OF A GENERALIZED NONLINEAR EQUATION, Sigma J. Eng & Nat Sci 38 (4), 2020, 2109-2121*

[56] *J. E. Nápoles V., J. M. Rodríguez, J. M. Sigarreta, New Hermite-Hadamard Type Inequalities Involving Non-Conformable Integral Operators, Symmetry 2019, 11, 1108; doi:10.3390/sym11091108*

[57] *J. E. Nápoles V., C. l. Tunc, ON THE BOUNDEDNESS AND OSCILLATION OF NON-CONFORMABLE LIENARD EQUATION, Journal of Fractional Calculus and Applications Vol. 11(2) July 2020, pp. 92-101.*

[58] *J. E. Nápoles V., M. Vivas-Cortez, A note on the Picone's Identity, Journal of Nonlinear Sciences, to appear.*

[59] *K. Oldham, J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order, Academic Press, USA, 1974.*

[60] *Y. Pandir, Y. Gurefe, T. Akturk, New Soliton Solutions of The Nonlinear Radhakrishnan-Kundu-Lakshmanan Equation With the Beta-Derivative, preprint https://doi.org/10.21203/rs.3.rs-866518/v1*

[61] *O. Tunç, Ö. Atan, C. Tunç, J. C. Yao, Qualitative analyses of integro-fractional differential equations with Caputo derivatives and retardations via the Lyapunov–Razumikhin method. Axioms 10 (2), 58, 2021.*

[62] *S. Umarov, S. Steinberg, Variable order differential equations with piecewise constant order-function and diffusion with changing modes, Z. Anal. Anwend. 28 (4) (2009) 431-450.*

[63] *M. Vivas-Cortez, A. Fleitas, P. M. Guzmán, J. E. Nápoles V., J. J. Rosales, Newton's Law of Cooling with Generalized Conformable Derivatives, Symmetry 2021, 13, 1093. https://doi.org/10.3390/sym13061093*

[64] *M. Vivas-Cortez, O. J. Larreal B., J. E. Nápoles V., EXTREMAL SOLUTION TO GENERALIZED DIFFERENTIAL EQUATIONS UNDER INTEGRAL BOUNDARY CONDITION, Journal of Mathematical Control Science and Applications, Vol. 7 No. 1 (January-June 2021), 47-56*

[65] *M. Vivas-Cortez, L. M. Lugo, J. E. Nápoles Valdés, On a Generalized Laplace Transform, Appl. Math. Inf. Sci. 15, No. 5, 667-675 (2021) http://dx.doi.org/10.18576/amis/150516*

[66] *M. Vivas-Cortez, J. E. Nápoles Valdés, J. E. Hernández Hernández, J. Velasco Velasco, O. Larreal, On Non Conformable Fractional Laplace Transform, Appl. Math. Inf. Sci. 15, No. 4, 403-409 (2021) http://dx.doi.org/10.18576/amis/150401*

[67] *Xiao-Jun Yang, The Zero-mass Renormalization Group Differential Equations and Limit Cycles in Non-smooth Initial Value Problems, Prespacetime Journal, July 2012, Vol. 3, Issue 9, 913-923.*

[68] *D. Zhao, M. Luo, General conformable fractional derivative and its physical interpretation, Calcolo, 54: 903-917, 2017. DOI 10.1007/s10092-017-0213-8.*

[69] *W. Zhu, Y. Xia, B. Zhang, Y. Bai, Exact Traveling Wave Solutions and Bifurcations of the Time-Fractional Differential Equations with Applications, International Journal of Bifurcation and Chaos, Vol. 29, No. 3 (2019) 1950041 (24 pages) DOI: 10.1142/S021812741950041X*

Author information

Juan E. Nápoles Valdes, Universidad Nacional del Nordeste, FaCENA, Corrientes and Universidad Tecnológica Nacional, FRRE, Resistencia, Chaco, Argentina.
E-mail: jnapoles@exa.unne.edu.ar

Received: 2023-04-07
Accepted: 2024-05-07