Palestine Journal of Mathematics

Vol 13(4)(2024) , 170–180

Almost Monoform Modules and Related Concepts

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 16D10; Secondary 16D70, 16D80.

Keywords: Monoform modules, Almost monoform modules, Rational submodules, P-rational submodules.

The author of this article would like to thank the referees for their valuable suggestions and helpful comments.

Abstract A non-zero module M is called monoform if every non-zero submodule of M rational in M. The purpose of this article is to introduce a proper class of monoform modules , called almost monoform modules. Various properties of almost monoform modules are investigated. Analogues of many results that were satisfied in the class of monoform modules are discussed for this type of module for instance, If a module M is an almost monoform module, then all partial endomorphisms f: N⟶M with N is a non-zero submodule of M have zero kernels in their domains. Another characterization of the definition of an almost monoform module is given by using the concepts of uniform and P-polyform modules. The relationships between an almost module and some other related modules are considered such as polyform, P-polyform, fully polyform uniform, QI-monoform modules, essentially quasi-Dedekind, purely quasi-Dedikind and SQD modules. Other characterizations of almost monoform modules are shown that are analogous to those in the concept of monoform modules.

1 Introduction

Several authors studied the concept of monoform modules, such as H.H. Storrer, J. Zelmanowitiz, I.M.A. Hadi, A. Hajikarimi and A.R. Naghipour. A submodule N of M is called essential, (briefly, $N \leq_e M$) if N∩L≠0 for each non-zero submodule L of M, ([15], P.15). A submodule N of an R-module M is called rational (simply, $N \leq_r M$) if $\mathrm{Hom}_R(\frac{M}{N}, E(M))=0$, ([18], P.274), where E(M) is the injective hull of M, that is E(M) is an injective module and essential extension of M. A submodule N of M is called pure (in the sense of Anderson and Fuller) if N∩IM=IN for every ideal I of R, [5]. A non-zero R-module M is called monoform if every non-zero submodule of M is rational in M, [27]. A submodule N of M is said to be P-rational if N is a pure submodule of M and $\mathrm{Hom}_R(\frac{M}{N}, E(M))=0$, [5].

The main goal of this paper is to give a new class of modules contained properly in the class of monoform modules. It is named almost monoform modules.

This paper consists of three sections. Section 2 is devoted to investigating the important properties of such type of module that are analogous to the results which are known in the concept of the monoform module, among these results are the following:

1. If a module M is an almost monoform module, then all partial endomorphisms f: N⟶M with N is a non-zero pure submodule of M have zero kernels in their domains (i.e f is monomorphism). See Proposition 2.9.

2. If M is an almost monoform module then for each non-zero homomorphism f: M⟶E(M), the kernel of f is equal to 0. See Proposition 2.10.

In Section three, another characterization and partial characterization of almost monoform module are given such as the following:

1. An R-module M is an almost monoform module if and only if M is a uniform and P-polyform module. See Proposition 3.5.

2. Let R be a regular ring, then a module M is an almost monoform if and only if M is a P-uniform module and fully polyform. See Proposition 3.7.

3. Several relationships between this class of modules and other related modules are considered, such as Theorem 3.8 which states in the following:

   Over a regular ring R, the following statements are equivalent:

   a. M is a monoform module.
   b. M is a uniform module and P-polyform module.
   c. M is an almost monoform module.
   d. M is a P-uniform and fully polyform module.

4. Let M be a multiplication module with a prime annihilator. Consider the following:

   a. M is an almost monoform module.
   b. M is a purely quasi-Dedekind module.
   c. M is a quasi-Dedekind module.
   d. M is a uniform module.
   e. M is a P-uniform module.

   Then (a)⇒ (b)⇒ (c)⇒(d), and if M is fully P-essential then (d)⇒ (e).

   See Theorem 3.18.

5. A connection of almost monoform with each of prime, Pe-prime, essentially prime modules is discussed. See Theorem 3.17, Propositions 3.19 and 3.21.

6. Let R be a quasi-Dedekind ring. Consider the following:

   a. R is an almost monoform ring.
   b. R is a monoform ring.
   c. R is a polyform ring.
   d. R is a QI-monoform ring.

   Then (a)⇒(b)⇒(c)⇒ (d) See Proposition 3.23.

It is worth mentioning that all rings R in this paper are commutative with identity, and all modules are unitary left R-modules.

2 Almost Monoform Modules

In this section, a proper subclass of monoform module is introduced, and the main characteristics of the almost monoform module are given and discussed.

Definition 2.1. A non-zero module M is called almost monoform if every non-zero submodule of M is P-rational. That is every non-zero submodule of M is rational and pure in M. A ring R is called almost monoform if R is an almost monoform R-module.

Remark 2.2. Every almost monoform is monoform. Indeed, if M is almost monoform, then every non-zero submodule of M is P-rational. But every P-rational is rational, [8], so the result is obtained. In contrast, it is not vice versa, for example, the $\mathbb{Z}$-module $\mathbb{Q}$ is a monoform module, ([19], P.6), but not almost monoform since there exists a submodule $\mathbb{Z}$ of $\mathbb{Q}$ such that $\mathbb{Z}$ is not P-rational in $\mathbb{Q}$, ([8], Remark 2.2(2)).

Under certain conditions, the converse of Remark 2.2 is satisfied, as the the following proposition shows. Before that, a ring R is called regular (in the sense of Von Newmann), if for each x∈R there exists a∈R such that x=axa, ([15], P.10). An R-module M is said to be F-regular if every submodule of M is pure, [26].

Proposition 2.3. Let R be a regular ring, then any R-module M is almost monoform if and only if M is a monoform module.

Proof. The necessity is clear. For the reverse, assume that M is a monoform module, and let N be a non-zero submodule of M. Since R is a regular ring, then M is an F-regular module, [26]. This implies that N is pure, and by assumption, N is rational in M, hence N is a P-rational submodule in M.

Other examples and properties of almost monoform modules can be presented in the following.

Remarks and examples 2.4.

1. The simple module is almost monoform since the only non-zero submodule of the simple module say M is itself, and it is a rational and pure submodule of itself.
2. A semisimple module is not almost monoform, since in a semisimple module say M, all submodules are direct summands, so the only P-rational submodule in M is itself, ([26], Remark 2.3(8)).

   Recall that a non-zero module M is called pure simple if the only pure submodules of M are (0) and M itself, [12].

3. If M is pure simple then M is not almost monoform module, since the only P-rational submodule of M is itself, ([8], Remark 2.3(9)). That is M does not contain any proper non-zero P-rational submodules.

   An R-module M is called F-regular if every submodule of M is pure [26].

4. In the class of F-regular modules, there is no difference between almost monoform and monoform modules. This follows by ([8], Remark 2.3(10)).
5. It is known that every integral domain is a monoform ring. This fact is not achieved in the almost monoform ring. For example, the Z-module Z is an integral domain but not almost monoform since Z is pure simple and by (3), Z is not almost monoform module.

   Remember that an R-module M is called uniform if each non-zero submodule of M is essential, ([15], P.85) and [10].

6. Every almost monoform module is uniform. Indeed, every non-zero submodule of almost monoform is P-rational, and by the direct implication of a P-rational submodule to an essential submodule the assertion will be achieved.
7. The Z-module Z_4 is not almost monoform, since Z_4 contains a submodule say $<\overline{2}>$ which is not P-rational submodule of Z_4.
8. The Z-module Z_{p^∞} is not almost monoform module, since obviously, the submodule $(\frac{1}{\mathrm{p}}+\mathrm{Z})$ is not P-rational in Z_{p^∞}.
9. Each almost monoform module is indecomposable.

   Proof. Suppose that M is a decomposable almost monoform module, so there exist two proper non-zero submodules A and B such that $\mathrm{M}=\mathrm{A}\bigoplus\mathrm{B}$. Now, $\mathrm{Hom}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{A}},\mathrm{E(M)})\cong\mathrm{Hom}_{\mathrm{R}}(\mathrm{B},\mathrm{E(M)})\neq0$, which is a contradiction because M is almost monoform.

10 The converse of (9) is not true in general, for example, the Z-module Q is indecomposable, but it is not almost monoform module as shown in Remark 2.2.

An R-module M is said to be Artinian if every descending chain of submodules is terminated, ([15], P.7). The following proposition guarantee the existence of almost monoform submodule in any Artinian module.

Proposition 2.5. Every non-zero Artinian module has a submodule which is almost monoform.

Proof. Let M be a non-zero Artinian module, and let N be a submodule of M. We have to show that N is an almost monoform submodule. If N is almost monoform then there is nothing to prove, otherwise, there exists a non-zero submodule N_1 of N such that N_2 is not P-rational in N. Now, if N_1 is almost monoform then we are done, if not then there exists a non-zero submodule N_2 of N_1 such that N_2 is not P-rational in N_2. If N_2 is almost monoform then we are through, otherwise, there exists a non-zero submodule N_3 of N_2 such that N_3 is not P-rational in N_2. Continuing in this process, so after a finite number of steps of descending chain of submodules:

$$N \supset N_1 \supset N_2 \supset N_3 \supset N_4 \supset \ldots\ldots\ldots\ldots$$

we must find a submodule in M which is a almost monoform, or we would have a contradiction because M is an Artinian module. Thus, N is a almost monoform submodule.

Proposition 2.6. A non-zero submodule of an almost monoform module is almost monoform.

Proof. Let N be a non-zero submodule of M, and K be a non-zero submodule of N. Since M is an almost monoform module, then $K \leq_{pr} M$. This implies that $K \leq_r M$ [8], hence $K \leq_r N$, ([18], Proposition (8.7)(2)). In addition, since K is pure in M then K is pure in N, ([26], Remark 2.7(6), P. 274).

The following proposition deals with the connection between an almost monoform ring and the endomorphism ring of R-modules. Before that, an R-module M is called a scalar if for each $f \in End_R(M)$, there exists $r \in R$ such that $f(x)=rx\ \forall ax \in M$, [24], where $End_R(M)$ is the endomorphism ring of modules.

Proposition 2.7. Let M be a faithful scalar R-module. Then $End_R(M)$ is an almost monoform ring if and only if R is an almost monoform R-module.

Proof. Since M is a faithful scalar module, then $End_R(M) \cong R$, [23], so if $End_R(M)$ is an almost monoform ring then R is an almost monoform ring and vice versa.

An R-module M is called multiplication, if every submodule of M is of the form IM, for some ideal I of R, [9, 25]. Since every finitely generated multiplication module is scalar, ([24], Corollary 1.1.11, P.12), then as a consequence of Proposition 2.6, we obtain the following.

Corollary 2.8. Let M be a finitely generated faithful and multiplication module. Then $End_R(M)$ is an almost monoform ring if and only if R is an almost monoform ring.

Proposition 2.9. Let M be an R-module, assume that $\overline{R} \equiv R/J$ where J is an ideal of R with $J \subseteq ann_R(M)$. Then M is an almost monoform R-module if and only if M is an almost monoform $\overline{R}$-module.

Proof. Suppose that M is an almost monoform R-module, so that $Hom_R(\frac{M}{N}, E(M))=0$ for all non-zero submodule N of M. Since $Hom_R(\frac{M}{N}, E(M)) = Hom_{\overline{R}}(\frac{M}{N}, E(M))$ for all $N \leq M$, ([17], P.51), in particular for each non-zero submodule N of M. Then $Hom_{\overline{R}}(\frac{M}{N}, E(M))=0$, hence M is an almost monoform $\overline{R}$-module. Similarity for the converse.

In view of [27], a monoform module is defined as all non-zero partial endomorphisms of M are monomorphisms. For the almost monoform module, we cannot prove an analogue of this result. However, one direction can be proved in the following.

Proposition 2.10. If a module M is an almost monoform module, then all partial endomorphisms $f: N \longrightarrow M$ with N is a non-zero submodule of M have zero kernels in their domains (i.e f is monomorphism.

Proof. Assume that M is an almost monoform module, let N be a non-zero submodule of M and $0 \neq f: N \to M$. Suppose that $\ker f \neq 0$. By the first isomorphism theorem $\frac{N}{\ker f} \cong f(N)$, so there is an isomorphism $\Psi: \frac{N}{\ker f} \to f(N)$. It is clear that $\Psi \neq 0$. Consider the following sequence:

$$\frac{N}{\ker f} \xrightarrow{\Psi} f(N) \xrightarrow{i} M$$

where i is the inclusion homomorphism. Since i∘Ψ≠0, then (i∘Ψ)($\frac{N}{kerf}$)≠0. Now, M is almost monoform, so kerf is a P-rational submodule, which means $Hom_R(\frac{N}{kerf}$, M)=0, ([8], Theorem 2.7). But this is impossible since i∘Ψ≠0, therefore kerf=0.

Proposition 2.11. If M is an almost monoform module then for each non-zero homomorphism f: M⟶E(M), the kernel of this homomorphism is equal to zero.

Proof. Assume that M is almost monoform and let f: M⟶E(M) be a homomorphism with kerf ≠0. We have to show that f=0. Define g: $\frac{M}{kerf}$ ⟶ E(M) by g(m+kerf)=f(m) for all m∈E(M). To show that g is well-defined, assume that m_1+kerf=m_2+kerf, m_1, m_2∈M. This implies that (m_1-m_2)∈kerf that is f(m_1-m_2)=0. Since f is a homomorphism, then f(m_1)-f(m_2)=0, hence f(m_1)=f(m_2). Moreover, since f≠0, then g≠0. That is $Hom_R(\frac{M}{kerf}$, E(M))≠0 which is a contradiction, therefore f=0. Thus, kerf=0.

3 Almost Monoform Modules and Related Concepts

In this section, the connection between almost monoform and other related concepts is considered such as polyform, P-polyform, fully polyform, uniform, purely uniform, P-uniform, Pe-prime, essentially quasi-Dedekind, purely quasi-Dedekind and SQD.

Recall that a submodule N is called P-essential in M if for every pure submodule L of M with N∩L=(0), implying that L=(0), [4]. A module M is called P-uniform if every non-zero submodule of M is P-essential, [6]. Obviously, each uniform is P-uniform.

Remark 3.1. Every almost monoform is a P-uniform module.

Proof. In an almost monoform module say M, every non-zero submodule is P-rational. Moreover, each P-rational submodule of M is P-essential, [8], therefore M is a P-uniform module.

We need to introduce the following concept.

Definition 3.2. A non-zero module M is called purely uniform if every non-zero pure submodule of M is essential in M.

Remark 3.3. Every almost monoform is a purely uniform module.

Proof. Let N be a non-zero submodule of M. Since M is almost monoform, then $N \leq_{pr} M$, hence $N \leq_e M$, [8].

An R-module M is called P-polyform if every essential submodule of M is P-rational in M. Equivalently, ($\frac{M}{N}$, E(M))=0 for every pure and essential submodule N of M, where E(M) is the injective hull of M, [7].

Proposition 3.4. Every almost monoform is a P-polyform module

Proof. Let N be an essential submodule of M. By assumption, every non-zero submodule N of M (especially, every essential submodule) is P-rational in M. Thus, M is P-polyform.

Next, we can use Proposition 3.4 to give another characterization of the definition of an almost monoform module.

Proposition 3.5. An R-module M is an almost monoform module if and only if M is a uniform and P-polyform module.

Proof. The necessity is obtained by Remark 2.4(6) and Proposition 3.4. For the sufficient direction, assume that N is a non-zero submodule of M. Because M is a uniform module then $N \leq_e M$. Furthermore, M is P-polyform implying that $N \leq_{pr} M$, hence M is almost monoform.

Recall that a module M is said to be fully polyform if every P-essential submodule of M is rational in M. That is $Hom_R(\frac{M}{N}$, E(M))=0 for every P-essential submodule N of M, [6].

Remark 3.6. Each almost monoform is a fully polyform module.

Proof. Let N$\leq_{pe}$M. Since M is almost monoform, then every non-zero (especially each P-essential) submodule of M is P-rational, and this yields N$\leq_{r}$M [8].

According to Remark 3.1 and Remark 3.6 as well as M is defined on any regular ring R, a partial characterization of almost monoform is given in the following.

Proposition 3.7. Let R be a regular ring, then a module M is almost monoform if and only if M is a P-uniform module and a fully polyform module.

Proof. The necessity follows by Remark 3.1 and Proposition 3.5. For sufficiency, let N be a non-zero submodule of M. Since M is P-uniform, then N$\leq_{pe}$M. On the other hand, M is fully polyform, so N$\leq_{r}$M. In addition, R is regular, then M is F-regular, [26], so N is pure in M. Therefore N$\leq_{pr}$M, thus M is an almost monoform module.

Theorem 3.8. Over a regular ring R, the following statements are equivalent:

(i) M is a monoform R-module.

(ii) M is a uniform module and P-polyform R-module.

(iii) M is an almost monoform R-module.

(iv) M is a P-uniform and fully polyform R-module.

Proof.
(i) ⇒ (ii): It is known that every monoform is uniform and polyform. Moreover, since R is a regular ring, then polyform implies P-polyform, ([7], Remark 2.2(10)).
(ii) ⇔ (iii): It is just Proposition 3.5.
(iii) ⇔ (iv): Because R is a regular ring, then the result follows by Proposition 3.7.
(iv) ⇒ (v): Let N be a non-zero submodule of M. Since M is P-uniform, then N$\leq_{pe}$M. In addition, M is fully polyform implying that N$\leq_{r}$M, thus M is monoform.

Compare the following theorem with ([18], Exercise 8(4), P.284].

Theorem 3.9. Let M be a nonsingular module, then M is an almost monoform module if and only if M is purely uniform.

Proof. The necessity is obvious. For the sufficiently, let N be a non-zero submodule of M. Since M is purely uniform, then N is pure and essential in M. But M is nonsingular, then N$\leq_{pr}$M, ([8] Proposition 3.4), therefore M is an almost monoform module.

A submodule N is said to be a quasi-invertible submodule of M (simply, we use the symbol N$\leq_{qu}$M) if $\mathrm{Hom_R}(\frac{M}{N}, M)$=0, and a module M is called quasi-Dedekind if every non-zero submodule of M is quasi-invertible, [20]. A submodule N of a module M is called purely quasi-invertible if N is pure and quasi-invertible, and a module M is called purely quasi-Dedekind (for simply we use the symbol N$\leq_{pqu}$M) if every proper non-zero pure submodule of M is quasi-invertible, [13].

Proposition 3.10. Every almost monoform module is purely quasi-Dedekind.

Proof. Let M be an almost monoform module, and let N be a proper non-zero pure submodule of M. By assumption, N$\leq_{pr}$M. This implies that N$\leq_{pqu}$M, [5].

A submodule N is called SQI if for each f∈$\mathrm{Hom_R}(\frac{M}{N}, M)$, then f($\frac{M}{N}$) is a small submodule of M, and a module M is called SQD if every non-zero submodule of M is an SQI submodule of M, [22], where a submodule N is called a small submodule of M if N+L≠M for every proper submodule L of M, ([15], Exercise 20, P.20).

Corollary 3.11. Every almost monoform module is SQD.

Proof. Since every almost monoform is purely quasi-Dedekind, and every purely quasi-Dedekind is an SQD module, [13], then the desired is achieved.

The converse of Corollary 3.11 is not true in general, for example, the Z-module Z_4 is SQD Z-modul, [13], while it was verified by Example 2.4(7), the module Z_4 is not almost monoform.

Furthermore, according to the direct implication between a purely quasi-Dedekind module and quasi-Dedekind, we have the following.

Corollary 3.12. Any almost monoform is a quasi-Dedekind module.

The converse of Corollary 3.12 is not true, for example, the Z-module Q is quasi-Dedekind but not almost monoform module as shown in Remark 2.2. However, that is true whenever M is a multiplication and F-regular module as the following shows. Before that, we need the following lemma which is appeared in ([20], Theorem 3.11, P.18).

Lemma 3.13. Let M be a multiplication module such that $\mathrm{ann_R(M)}$ is prime ideal of R. Then N is a quasi-invertible submodule of M if and only if N is essential submodule of M.

Proposition 3.14. Let M be an F-regular and multiplication module, then M is almost monoform if and only if M is a quasi-Dedekind module.

Proof. The necessity is obvious. For the converse, assume that M is quasi-Dedekind, and let N be a non-zero submodule of M. By assumption, $\mathrm{N}\leq_{qu}\mathrm{M}$, and since M is a multiplication, so by Lemma 3.13, $\mathrm{N}\leq_{r}\mathrm{M}$. But M is F-regular, so N is pure in M, thus $\mathrm{N}\leq_{pr}\mathrm{M}$. That is M is almost monoform.

Following [11], an R-module M is called prime if $\mathrm{ann_R}$ (M)=$\mathrm{ann_R}$(N), for every non-zero submodule N of M. Since each quasi-Dedekind module is a prime module so the following is achieved.

Corollary 3.15. Every almost monoform module is a prime module.

The converse of Corollary 3.15 is not true in general, for example, the Z-module $\mathrm{Q}\bigoplus\mathrm{Z}$ is prime, [20], While it is not almost monoform since it is not quasi-Dedekind, [20]. However, the converse is true under certain conditions, before that, we need the following lemma.

Lemma 3.16. ([8], Proposition 2.21)
Let L be a non-zero pure submodule of an R-module M. If for any $0\neq \mathrm{m}\in\mathrm{M}$, $\mathrm{ann_R}(\frac{\mathrm{M}}{\mathrm{L}}) \nsubseteq \mathrm{ann_R(m)}$, then $\mathrm{L}\leq_{pr}\mathrm{M}$.

Theorem 3.17. Let M be an R-module satisfying $\mathrm{ann_R}\left(\frac{\mathrm{M}}{\mathrm{N}}\right)\nsubseteq\mathrm{ann_R(M)}$ for every non-zero pure submodule N of M then M is a prime module if and only if M is almost monoform.

Proof. Assume that M is a prime module and let N be a non-zero pure submodule of M, $0\neq \mathrm{x}\in\mathrm{M}$. Since M is prime then $\mathrm{ann_R(M)}=\mathrm{ann_R(x)}$. But $\mathrm{ann_R}\left(\frac{\mathrm{M}}{\mathrm{N}}\right)\nsubseteq\mathrm{ann_R(M)}$, therefore $\mathrm{ann_R}\left(\frac{\mathrm{M}}{\mathrm{N}}\right)\nsubseteq\mathrm{ann_R(x)}$. Now, since N is pure in M, then by Lemma 3.16, $\mathrm{N}\leq_{pr}\mathrm{M}$. Thus, M is an almost monoform module.

An R-module M is said to be Pe-prime if $\mathrm{ann_R(N)}= \mathrm{ann_R(M)}$ for every P-essential submodule N of M [6].

Proposition 3.18. Any almost monoform module is Pe-prime.

Proof. By Remark 2.2, each almost monoform is a monoform module, and monoform implies to fully polyform module ([6], Proposition 3.2). In addition, every fully polyform module is a Pe-prime module ([6], Corollary 3.10).

Remember that a module M is called fully P-essential if every P-essential submodule of M is essential [8].

Theorem 3.19. Let M be a multiplication module with a prime annihilator. Consider the following:

(i) M is an almost monoform module.

(ii) M is a purely quasi-Dedekind module.

(iii) M is a quasi-Dedekind module.

(iv) M is a uniform module.

(v) M is a P-uniform module.

Then (i)⇒(ii)⇒(iii)⇒(iv), and if M is fully P-essential then (iv)⇒(v).

Proof.
(i)⇒(ii): Let N be a non-zero submodule of M, by the assumption $N \leq_{pr} M$. This means N is a rational and pure submodule of M. On the other hand, every rational is quasi-invertible [20]. Therefore, $N \leq_{pqu} M$. Thus, M is purely quasi-Dedekind.
(ii)⇒ (iii): It is followed by [13].
(iii)⇒(iv): Let N be a non-zero submodule of M. Since M is a quasi-Dedekind then $N \leq_{qu} M$. In contrast, M is a multiplication with a prime annihilator, so by Lemma 3.13, $N \leq_{e} M$, that is M is a uniform module.
(iv)⇒ (v): It follows from the definition of a fully P-essential module.

A module M is called essentially quasi-Dedekind if every essential submodule of M is quasi-invertible [16].

Proposition 3.20. Every almost monoform module is an essentially quasi-Dedekind module.

Proof. Since every almost monoform is monoform and the last concept implying to essentially quasi-Dedekind, [14]. Thus, the desired has been achieved.

An R-module M is said to be essentially prime if $ann_R(N) = ann_R(M)$ for every essential submodule N of M [14]. Since every essentially quasi-Dedekind module is essentially prime, then the following can be deduced.

Corollary 3.21. Every almost monoform is an essentially prime module.

Theorem 3.22. For any R-module M, consider the following:

(i) M is almost monoform.

(ii) M is purely quasi-Dedekind.

(iii) M is quasi-Dedekind.

(iv) M is essentially quasi-Dedekind.

(v) M is essentially prime.

(vi) M is Pe-prime.

Then (i)⇒ (ii)⇒(iii)⇒(iv)⇒(v)⇒(vi).

Proof.
(i)⇒ (ii): It is just Proposition 3.10.
(ii)⇒(iii): [13].
(iii)⇒(iv): [14].
(iv)⇒(v): [14].
(v)⇒(vi): The result follows by the direct implication between essential and P-essential submodules [6].

This section ends with the following result which is achieved only in the category of rings. Before that, an R-module M is said to be quasi-invertibility (simply, QI-monoform) module if every non-zero quasi-invertible submodule is rational in M [1].

Proposition 3.23. Let R be a quasi-Dedekind ring. Consider the following:

(i) R is an almost monoform ring.

(ii) R is a monoform ring.

(iii) R is a polyform ring.

(iv) R is a QI-monoform ring.

Then (i)⇒ (ii)⇒ (iii) ⇒ (iv).

Proof.
(i) ⇒ (ii): It is obvious.
(ii) ⇒ (iii) ⇒ (iv): Since R is a quasi-Dedekind ring, then the result is obtained by ([1], Theorem 4.13).

Since every integral domain is quasi-Dedekind then the following desires are fulfilled.

Corollary 3.24. For any integral domain R. Consider the following:

(i) R is an almost monoform ring.

(ii) R is a monoform ring.

(iii) R is a polyform ring.

(iv) R is a QI-monoform ring.

Then (i)⇒ (ii)⇒ (iii) ⇒ (iv).

Following [3],, a submodule N of an R-module M is St-closed (simply, N$\leq_{stc}$M), if N has no proper semi-essential extensions in M, where a submodule N is said to be semi-essential if N∩P≠0 for every non-zero prime submodule P of M, [3] . A module M is called St-polyform, if for each submodule N of M and all homomorphism f: N→M, kerf is an St-closed submodule of M, [2].

Remark 3.25. The two concepts almost monoform and St-polyform are independent. For example, the Z-module Z_5 is almost monoform but not St-polyform, [2]. In contrast, St-polyform is not almost monoform because of the purity property.

This motivates us to define the following.

Definition 3.26. An R-module M is called purely St-polyform if each semi-essential submodule of M is P-rational in M.

Remark 3.27. Every almost monoform is a purely St-polyform module.

Proof. Assume that M is an almost monoform module, that is every non-zero (hence every non-zero semi-essential submodule) of M is P-rational, so M is purely St-polyform.

Recall that an R-module M is semi-uniform if every non-zero submodule of M is semi-essential, [21].

Theorem 3.28. Let M be a uniform and essentially quasi-Dedekind module. Consider the following statements:

1. M is an Almost monoform module.
2. M is a Purely St-polyform module.
3. M is a monoform module.
4. M is a QI-monoform module.
5. M is a polyform module.

Then (1) ⇔ (2) ⇒ (3) ⇒ (4) ⇒ (5).

Proof. (1) ⇒ (2): It is just Remark 3.27.
(2) ⇒ (1): Let N be a non-zero submodule of M. Since M is is uniform then it is semi-uniform, [21]. This implies that N is semi-essential. But M is purely St-polyform, therefore $N \leq_{pr} M$, therefore M is almost monoform.
(2) ⇒ (3): Assume that N is a non-zero submodule of M. Since M is uniform (hence semi-uniform) then N is semi-essential. But M is St-polyform, then $N \leq_{pr} M$, hence $N \leq_{r} M$, [8]. Thus, M is monoform.
(3) ⇒ (4) ⇒ (5): Since R is a uniform and essentially quasi-Dedekind, then the result follows by ([1], Theorem 4.11).

Since every nonsingular module is essentially quasi-Dedekind, ([1], Remark 4.8(3)), then we deduce the following.

Corollary 3.29. Let M be a uniform and nonsingular module. Consider the following:

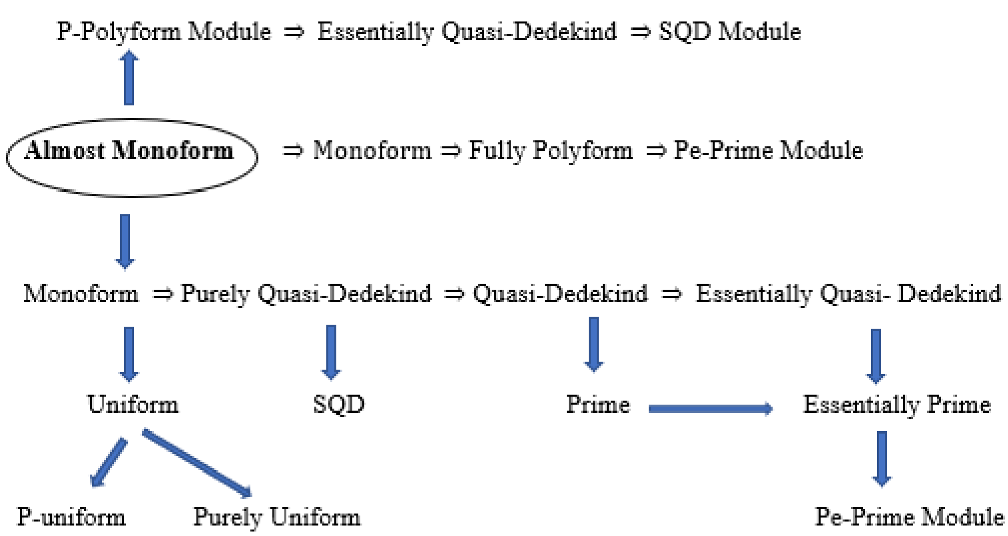
1. M is an Almost monoform module.
2. M is a Purely St-polyform module.
3. M is a monoform module.
4. M is a QI-monoform module.
5. M is a polyform module.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

4 Conclusions

In this work, the class of monoform modules has been restricted to a new class of modules. It is called almost monoform modules. The main results of this paper can be summarized as follows:

1. Several characteristics of almost monoform modules are established.
2. Sufficient conditions are given in which almost monoform and monoform modules are identical.
3. Another characterization and partial characterization of almost monoform modules are investigated.
4. Analogues of many results that were satisfied in the class of monoform modules are introduced for this type of module.
5. Many connections between almost monoform and other related concepts are discussed, such as P-uniform, P-polyform, fully polyform, quasi-Dedekind, purely quasi-Dedekind, essentially quasi-Dedekind, prime, Pe-prime and SQD modules. However, the most these relationships can be represented in the following diagram:



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Received: 2023-04-11
Accepted: 2024-09-01