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Inclusion and convolution properties of a q-generalized class of convex functions

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Abstract We define and study geometric properties of functions in the class $\mathcal{KK}_q$ which is a subclass of $\mathcal{K}_q$, the class of q-convex functions. We prove that the well-known class $\mathcal{K}$ of convex univalent functions is properly contained in $\mathcal{KK}_q$. We obtain necessary and sufficient conditions for the functions to be in the class $\mathcal{KK}_q$. We also prove that the class $\mathcal{KK}_q$ is closed under convolution with convex functions.

1 Introduction

In the last few decades, the q-calculus based upon the Jackson's q-derivative, D_q, have attracted the attention of a number of researchers due to its versatile applications in the field of Mathematical and Physical Sciences. In the recent years, researchers have also shown significance of the q-calculus in the field of Machine Learning and Artificial Intelligence. In 2020, Nielsen and Sun [10] proposed new artificial neurons called q–neurons as stochastic neurons with its activation function relying on q-derivative. Recall that for a real function f Jackson ([5, 4]) defined the q-derivative of f as follows:

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & x \neq 0 \\ f'(0), & x = 0 \end{cases},$$

where $0 < q < 1$. Note that $\lim_{q \to 1^-} D_q f(x) = f'(x)$. The main advantage of q-calculus is that it eliminates the need to calculate limits of ordinary derivatives. In 1990, Ismail et al. [3] made use of the q-calculus in geometric function theory and presented a q-extension of a subclass of analytic univalent functions. To further explain the advancement of q-calculus in geometric function theory we need to introduce few standard notations.

Let $\mathcal{A}$ denote the class of analytic functions f defined in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$, thus functions in the class $\mathcal{A}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose functions are univalent in $\mathbb{E}$. A function $f \in \mathcal{A}$ is said to be starlike or convex if f maps $\mathbb{E}$ conformally onto a domain which is starlike or convex, respectively. The subclasses of $\mathcal{S}$ containing starlike (with respect to origin) and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$, respectively. For $0 \leq \alpha \leq 1$, the analytic characterizations of generalization of these classes are as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{S} \ : \ \text{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha, \ \ z \in \mathbb{E} \right\}$$

and

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{S} \ : \ \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ \ z \in \mathbb{E} \right\}.$$

For $\alpha = 0$, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$. It is well-known that $f \in \mathcal{K} \iff zf' \in \mathcal{S}^*$ and $\mathcal{K} \subsetneq \mathcal{S}^*\left(\frac{1}{2}\right)$.

Using q-derivative operator, Ismail et al. [3] introduced and studied geometric properties of the class $\mathcal{PS}_q$ which is a q-generalized class of $\mathcal{S}^*$. A function $f \in \mathcal{S}$ belongs to class $\mathcal{PS}_q$ if

$$\left| \frac{z}{f(z)}(D_q f)(z) - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \quad z \in \mathbb{E}.$$

Note that as $q \to 1^-$, the closed disk $\left|w - (1-q)^{-1}\right| \le (1-q)^{-1}$ transforms into right-half plane and the class $\mathcal{PS}_q$ becomes $\mathcal{S}^*$. Since the induction of this paper, many researchers defined and studied various geometric properties of analytic univalent functions using q-derivative. For some recent investigations we refer the reader to [6, 7, 9, 11, 12, 15, 16, 13, 20, 21] and the references cited therein. A firm footing of the usage of the q-calculus in the context of geometric function theory is provided in the book [18]. In particular, Seoudy and Aouf [17] introduced and studied the subclasses $\mathcal{S}_q^*(\alpha)$ and $\mathcal{K}_q(\alpha)$ as follows:

$$\begin{aligned} \mathcal{S}_q^*(\alpha) &:= \left\{ f \in \mathcal{S} : \ \text{Re}\left(\frac{zD_q f(z)}{f(z)}\right) > \alpha, \ \ z \in \mathbb{E} \right\} \\ \mathcal{K}_q(\alpha) &:= \left\{ f \in \mathcal{S} : \ \text{Re}\left(\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\right) > \alpha, \ \ z \in \mathbb{E} \right\}. \end{aligned}$$

For $\alpha = 0$, $\mathcal{S}_q^*(0) = \mathcal{S}_q^*$ is called the class of q-starlike functions and $\mathcal{K}_q(0) = \mathcal{K}_q$ is called the class of q-convex functions. It is well-known that $f \in \mathcal{K}_q \iff zD_q f \in \mathcal{S}_q^*$.

For two normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, their convolution, $f * F$, is given by

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n.$$

Many researchers have studied convolution properties of subclasses of analytic functions and obtained several remarkable results. In particular, Ruscheweyh and Sheil-Small [14] proved that the classes $\mathcal{S}^*$ and $\mathcal{K}$ are closed under convolution with convex functions. Although extensive work has been done in geometric function theory using q-calculus but some basic questions have not been answered yet. For example:

(i) What is the inclusion relation between $\mathcal{K}_q$ and $\mathcal{S}_q^*$?

(ii) How the functions in $\mathcal{K}_q$ behave under convolution with functions in $\mathcal{S}_q^*$ and $\mathcal{K}_q$?

In an attempt to answer such questions, we define and study inclusion and convolution properties of a subclass, $\mathcal{KK}_q$, of $\mathcal{K}_q$.

Definition 1.1. For each q $(0 < q < 1)$, a function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{KK}_q$ if

$$\text{Re}\left(1 + \frac{z\left(D_q f(z)\right)'}{D_q f(z)}\right) > 0, \ z \in \mathbb{E}. \tag{1.2}$$

Remark 1.2. One can see that $f \in \mathcal{KK}_q$ if and only if $zD_q f \in \mathcal{S}^*$.

We prove that the well-known class $\mathcal{K}$ of convex univalent functions is properly contained in $\mathcal{KK}_q$ and the class $\mathcal{KK}_q$ is properly contained in $\mathcal{K}_q$, the class of q-convex functions. We obtain necessary and sufficient conditions for the functions to be in the class $\mathcal{KK}_q$. We also prove that the class $\mathcal{KK}_q$ is closed under convolution with convex univalent functions.

2 Main Results

For any $f \in \mathcal{A}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, the q-derivative D_q of f can be written as:

$$zD_q f(z) = \frac{f(z) - f(qz)}{1-q} = \sum_{n=1}^{\infty} \frac{1-q^n}{1-q} a_n z^n = \sum_{n=1}^{\infty} [n]_q a_n z^n = f(z) * L_q(z),$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

is the q-integer number and

$$L_q(z) = \frac{z}{(1-z)(1-qz)}.$$

Note that as $q \to 1^-$, $[n]_q \to n$ and $L_q(z) \to k(z) = z/(1-z)^2$.

The proof of our first main result relies on the following lemma developed by Ruscheweyh and Sheil-Small [14].

Lemma 2.1. *Let F and G be analytic in $\mathbb{E}$ with $F(0) = G(0) = 0$. If F is convex and G is starlike, then for each analytic function f satisfying Re$f(z) > 0$ in $\mathbb{E}$, we have*

$$\text{Re}\left(\frac{(F * Gf)(z)}{(F * G)(z)}\right) > 0, \quad z \in \mathbb{E}.$$

We are now in position to state and prove our first main result.

Theorem 2.2. $\mathcal{K} \subsetneq \mathcal{K}\mathcal{K}_q$.

Proof. Using the fact that $zf'(z) = f(z) * k(z)$ and $zD_q f(z) = f(z) * L_q(z)$, we have

$$\begin{aligned}
\text{Re}\left(1 + \frac{z\,(D_q f(z))'}{D_q f(z)}\right) &= \text{Re}\left(\frac{(zD_q f(z))'}{D_q f(z)}\right) \\
&= \text{Re}\left(\frac{z\,(zD_q f(z))'}{zD_q f(z)}\right) \\
&= \text{Re}\left(\frac{f(z) * L_q(z) * k(z)}{f(z) * L_q(z)}\right) \\
&= \text{Re}\left(\frac{f(z) * L_q(z)\left(\frac{L_q(z) * k(z)}{L_q(z)}\right)}{f(z) * L_q(z)}.\right) \qquad (2.1)
\end{aligned}$$

Since L_q is a starlike function of order $(1-q)/2(1+q)$ (see [11]), thus

$$\text{Re}\left(\frac{L_q(z) * k(z)}{L_q(z)}\right) = \text{Re}\left(\frac{zL_q'(z)}{L_q(z)}\right) > \frac{1-q}{2(1+q)} > 0, \quad z \in \mathbb{E}. \qquad (2.2)$$

If $f \in \mathcal{K}$ then using Lemma 2.1 in conjunction with conditions (2.1) and (2.2) we have

$$\text{Re}\left(1 + \frac{z\,(D_q f(z))'}{D_q f(z)}\right) > 0, \; z \in \mathbb{E},$$

i.e., $f \in \mathcal{K}\mathcal{K}_q$ and thus $\mathcal{K} \subseteq \mathcal{K}\mathcal{K}_q$. In order to prove that $\mathcal{K} \subsetneq \mathcal{K}\mathcal{K}_q$ we present a function which is in the class $\mathcal{K}\mathcal{K}_q$ but not in the class $\mathcal{K}$. Let $f_0(z) = z + \dfrac{1}{2[2]_q} z^2 \in \mathcal{K}\mathcal{K}_q$. One can easily verify that

$$\text{Re}\left(1 + \frac{z\,(D_q f_0(z))'}{D_q f_0(z)}\right) = \text{Re}\left(\frac{1+z}{1+\frac{1}{2}z}\right) > 0, \quad z \in \mathbb{E}.$$

Further, using the fact that for all $0 < q < 1$, $\dfrac{1}{2[2]_q} = \dfrac{1}{2(1+q)} > \dfrac{1}{4}$, in conjunction with the condition that $z + az^2 \in \mathcal{K}$ iff $|a| \leq \frac{1}{4}$, one can conclude that $f_0 \notin \mathcal{K}$. Hence, $\mathcal{K} \subsetneq \mathcal{K}\mathcal{K}_q$. □

Remark 2.3. Keeping in mind condition (2.2), we can state Theorem 2.2 in a generalized form as follows:

$$\mathcal{K} \subsetneq \mathcal{KK}_q\left(\frac{1-q}{2(1+q)}\right). \tag{2.3}$$

Further, note that if $q \to 0^+$ then $D_q f(z) \to \dfrac{f(z)}{z}$ and consequently condition (2.3) reduces to

$$\mathcal{K} \subsetneq \mathcal{S}^*\left(\frac{1}{2}\right).$$

Our next result is an immediate consequence of the geometric interpretation of starlike functions provided in [19]. For the sake of completeness, we include the detailed proof here.

Theorem 2.4. $\mathcal{S}^* \subset \mathcal{S}_q^*$

Proof. Let $f \in \mathcal{S}^*$. Then, for any fixed $\theta \in \mathbb{R}$ and $z = re^{i\theta}$ in $\mathbb{E}$, the condition

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,$$

can be rewritten as

$$\text{Re}\left\{\frac{r}{f(re^{i\theta})}\frac{\partial f(re^{i\theta})}{\partial r}\right\} \geq 0, \quad 0 \leq r < 1.$$

Further, if $f = u + iv$, then the above condition is equivalent to

$$u\frac{\partial u}{\partial r} + v\frac{\partial v}{\partial r} \geq 0, \quad 0 \leq r < 1,$$

which implies that the function $|f(re^{i\theta})| = \sqrt{u^2+v^2}$ is strictly increasing with r, $0 \leq r < 1$. Thus,

$$|f(qz)| \leq |f(z)| \text{ for } 0 < q < 1,$$

and

$$\text{Re}\left(\frac{zD_q f(z)}{f(z)}\right) = \text{Re}\left(\frac{z}{f(z)}\frac{f(z)-f(qz)}{z-qz}\right) = \frac{1}{1-q}\text{Re}\left(1 - \frac{f(qz)}{f(z)}\right) \geq 0.$$

Hence, $f \in \mathcal{S}_q^*$. This completes the proof. □

Corollary 2.5. $\mathcal{KK}_q \subsetneq \mathcal{K}_q$.

Proof. Let $f \in \mathcal{KK}_q$. Then, $zD_q f(z) \in \mathcal{S}^*$ and consequently, $zD_q f(z) \in \mathcal{S}_q^*$, which implies that $f \in \mathcal{K}_q$. Moreover, one can easily verify that $f_1(z) = z + \dfrac{1}{[2]_q^2}z^2 \in \mathcal{K}_q$ but $f_1 \notin \mathcal{KK}_q$. Hence, $\mathcal{KK}_q \subsetneq \mathcal{K}_q$. □

Remark 2.6. $\mathcal{K} \subsetneq \mathcal{KK}_q \subsetneq \mathcal{K}_q$.

In the next result, we will obtain a necessary and sufficient condition for functions $f \in \mathcal{S}$ to be in the class $\mathcal{KK}_q$.

Theorem 2.7. $f \in \mathcal{KK}_q$ *if and only if*

$$f(z) * \frac{z}{(1-z)(1-qz)} * \left[\frac{z\left(1+\frac{\xi-1}{2}z\right)}{(1-z)^2}\right] \neq 0, \quad |\xi| = 1, \ \ 0 < |z| < 1. \tag{2.4}$$

Proof. A necessary and sufficient condition for a function $f \in \mathcal{S}$ to be in the class $\mathcal{KK}_q$ is that

$$\text{Re}\left(1 + \frac{z\left(D_q f(z)\right)'}{D_q f(z)}\right) > 0, \ z \in \mathbb{E}.$$

Since $1+\frac{z\left(D_q f(z)\right)'}{D_q f(z)} = 1$ at $z=0$, therefore, the above condition is equivalent to

$$1+\frac{z\left(D_q f(z)\right)'}{D_q f(z)} \neq \frac{\xi-1}{\xi+1}, \quad |\xi|=1,\ \xi\neq -1,\ 0<|z|<1.$$

By simple algebraic calculation, we obtain the required condition, i.e.,

$$\begin{aligned} 0 &\neq \frac{1}{2}\left\{(\xi+1)z\left(zD_q f(z)\right)' - (\xi-1)zD_q f(z)\right\} \\ &= \frac{1}{2}\left\{f(z) * \frac{z}{(1-z)(1-qz)} * \left[(\xi+1)\frac{z}{(1-z)^2} - (\xi-1)\frac{z}{1-z}\right]\right\} \\ &= f(z) * \frac{z}{(1-z)(1-qz)} * \left[\frac{z\left(1+\frac{\xi-1}{2}z\right)}{(1-z)^2}\right]. \end{aligned}$$

□

In the next result, we obtain another sufficient condition for the functions to be in the class $\mathcal{KK}_q$.

Theorem 2.8. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$. If*

$$\sum_{n=2}^{\infty} n[n]_q\, |a_n| \leq 1, \tag{2.5}$$

then, $f \in \mathcal{KK}_q$.

Proof. Substituting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the left hand side of (2.4), we get

$$\begin{aligned} &\left| f(z) * \frac{z}{(1-z)(1-qz)} * \left[\frac{z\left(1+\frac{\xi-1}{2}z\right)}{(1-z)^2}\right]\right| \\ &\quad= \left|\left(z+\sum_{n=2}^{\infty}[n]_q a_n z^n\right) * \left[\frac{z\left(1+\frac{\xi-1}{2}z\right)}{(1-z)^2}\right]\right| \\ &\quad= \left|\left(z+\sum_{n=2}^{\infty}[n]_q a_n z^n\right) * \left[z+\sum_{n=2}^{\infty}\left(\frac{\xi+1}{2}n-\frac{\xi-1}{2}\right)z^n\right]\right| \\ &\quad= \left|z+\sum_{n=2}^{\infty}[n]_q\left(\frac{\xi+1}{2}n-\frac{\xi-1}{2}\right)a_n z^n\right| \\ &\quad\geq |z|\left(1-\sum_{n=2}^{\infty}[n]_q\left|\frac{\xi+1}{2}n-\frac{\xi-1}{2}\right||a_n||z|^{n-1}\right) \\ &\quad\geq |z|\left(1-\sum_{n=2}^{\infty}[n]_q\left(\frac{|\xi|}{2}(n-1)+\frac{1}{2}(n+1)\right)|a_n|\right) \\ &\quad= |z|\left(1-\sum_{n=2}^{\infty}n[n]_q|a_n|\right). \end{aligned}$$

This last expression is nonnegative under given hypothesis (2.5) and so in view of Theorem 2.7, $f \in \mathcal{KK}_q$. □

In this last part of the paper, we explore convolution properties of functions in the class $\mathcal{KK}_q$. We prove that the class $\mathcal{KK}_q$ is closed under convolution with the functions in class $\mathcal{K}$.

Theorem 2.9. *If $f \in \mathcal{K}$ and $g \in \mathcal{KK}_q$, then $f * g \in \mathcal{KK}_q$.*

Proof. Setting $h(z) = f(z) * g(z)$, we have $zD_q h(z) = f(z) * zD_q g(z)$. Since $zD_q g(z) \in \mathcal{S}^*$ and $f \in \mathcal{K}$, thererfore $f(z) * zD_q g(z) \in \mathcal{S}^*$, that is, $zD_q h(z) \in \mathcal{S}^*$ and consequently, $h = f * g \in \mathcal{KK}_q$. □

For $0 < q < 1$, define

$$F_q(z) = \frac{1}{1-q} \log\left(\frac{1-qz}{1-z}\right) = z + \sum_{n=2}^{\infty} \frac{[n]_q}{n} z^n, \quad z \in \mathbb{E}.$$

Since $F_q \in \mathcal{K}$ (see [1]), therefore, by setting $f = F_q$ in Theorem 2.9, we obtain the following corollory.

Corollary 2.10. *If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{KK}_q$, then $(F_q * g)(z) = z + \sum_{n=2}^{\infty} \frac{[n]_q}{n} b_n z^n \in \mathcal{KK}_q$.*

For $f \in \mathcal{S}$ and Re $c > 0$, Bernardi [2] defined an integral operator $\mathfrak{L}_c[f]$, as

$$\mathfrak{L}_c[f](z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \quad z \in \mathbb{E}.$$

One can see that the Bernardi integral operator $\mathfrak{L}_c[f](z) = (\mathcal{F}_c * f)(z)$ where $\mathcal{F}_c(z) = z + \sum_{n=2}^{\infty} \frac{1+c}{n+c} z^n \in \mathcal{K}$ (see [8]). Thus, by setting $f = \mathcal{F}_c$ in Theorem 2.9, we obtain the following corollory.

Corollary 2.11. *If $g \in \mathcal{KK}_q$, then for Re $c > 0$, $\mathfrak{L}_c[g] \in \mathcal{KK}_q$.*

By using above results, one can obtain a number of functions in the class $\mathcal{KK}_q$. We close this paper by presenting two such functions.

(i) If $g_1(z) = z + \frac{z^3}{[3]_q} + \frac{z^5}{[5]_q} + \cdots$, then, $zD_q g_1(z) = z + z^3 + z^5 + \cdots = \frac{z}{1-z^2} \in \mathcal{S}^*$. Therefore, by making use of Corollary 2.10, we get that the function

$$(g_1 * F_q)(z) = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots \in \mathcal{KK}_q.$$

(ii) If $g_2(z) = z + \sum_{n=2}^{\infty} \frac{n}{[n]_q} z^n$, then, $zD_q g_2(z) = \frac{z}{(1-z)^2} \in \mathcal{S}^*$. Therefore, by making use of Corollary 2.10, we get that the function

$$(g_2 * F_q)(z) = z + z^2 + z^3 + \cdots = \frac{z}{1-z} \in \mathcal{KK}_q.$$

3 Concolusion

In this paper, we defined an analytic subclass $\mathcal{KK}_q$ using q-derivatives that necessarily contains the class of convex functions. We investigated various geometric properties of functions in the class $\mathcal{KK}_q$ using the technique of convolution.

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