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Quantum Hermite-Hadamard inequality for harmonic convex function

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Abstract This paper deals with Hermite-Hadamard inequality for harmonic convex function using q_{c_1}-integral is given. In the end, the estimation of the lower and upper bound of q_{c_1}-integral using obtained results is given.

1 Introduction

q-calculus is often known as Quantum calculus as it studies calculus without boundaries. Euler was the first to introduce the notion of q in his work Introductio in Analysin Infinitorum[1]. After Euler Jackson introduce the q-Jackson integral. It was the methodical beginning point for q-calculus. We can analyse the set of non-differentiable functions with the help of q-derivative by substituting difference operator in the classical derivative.The study of q-calculus has seen significant growth as it is helpful in physics, mechanics and mathematics. Also q-calculus has application the Mock theta functions, theory of finite differences, hypergeometric functions, Sobolev spaces, Bernoulli and Euler polynomials, umbral calculus, operator theory, gamma function theory, quantum mechanics, multiple hypergeometric functions, combinatorics, analytic number theory multiple hypergeometric functions.

In mathematics, an inequality is a difference between two values that is used to represent the relationship between two things. Simply said, a “inequality" occurs when two quantities are not equal. With the advent of calculus in the nineteenth century, the notion of inequalities and its role grew highly significant. The H-H inequality (Hermite-Hadamard inequality) is crucial to understanding convex function theory. In order for a function to be convex in an open interval of real numbers, it gives a necessary and sufficient condition [21]. Jensen's inequality, another essential inequality in the research of convex functions, is interpolated by the Hermite-Hadamard inequality as well [10].The H-H inequality has had a significant impact on integral inequalities, numerical analysis, special means theory, approximation theory, information theory, optimization theory [10]. H-H inequality for convex function is given by :

$$\Psi\left(\frac{c_1+d_1}{2}\right) \leq \frac{1}{d_1-c_1}\int_{c_1}^{d_1}\Psi(u)\,du \leq \frac{\Psi(c_1)+\Psi(d_1)}{2}. \tag{1.1}$$

Convex functions have been generalized, refined and extended to different classes, such as : m-convex function[8], s-convex function[6], P-function[5], exponential convex function[10], etc. The H-H inequality has been extended and generalised for many classes of functions by a number of mathematicians[22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

The paper is organized as follows. Section 1.1 contains the basic definitions and preliminary results which are useful for further discussion. In section 2, H-H inequality for harmonic convex function using q_{c1}-integral and related inequalities are discussed. The application of the obtained results is given in the section 3. The conclusion and the list of references are given at the end.

1.1 Preliminaries

Definition 1.1. A set $\mathbb{J}$ is harmonic convex, if

$$\frac{c_1 d_1}{\zeta c_1+(1-\zeta)d_1} \in \mathbb{J}, \quad \forall c_1, d_1 \in \mathbb{J},\ \zeta\in[0,1].$$

Işcan et al. [9] defined the harmonic convex functions.

Definition 1.2. Harmonic convex function is a function $f : \mathbb{J} \subseteq (0, +\infty) \to \mathbb{R}$, if

$$f\left(\frac{c_1 d_1}{\zeta c_1 + (1-\zeta)d_1}\right) \leq (1-\zeta)f(c_1) + \zeta f(d_1), \quad \forall c_1, d_1 \in \mathbb{J},\ \zeta \in [0,1]. \tag{1.2}$$

Let us introduced the following notation,

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad q \in (0,1).$$

The following expression,

$$D_q f(\zeta) = \frac{f(\zeta) - f(q\zeta)}{(1-q)\zeta}, \quad \zeta \neq 0. \tag{1.3}$$

is called the q-derivative of f at $\zeta \in [c_1, d_1]$.
The definite q-Jackson integral is defined as

$$\int_0^{d_1} f(\zeta)\, d_q\zeta = (1-q)d_1 \sum_{n=0}^{\infty} q^n f\big(d_1 q^n\big), \tag{1.4}$$

provided the sum converges absolutely,

and

$$\int_{c_1}^{d_1} f(\zeta)\, d_q\zeta = \int_0^{d_1} f(\zeta)\, d_q\zeta - \int_0^{c_1} f(\zeta)\, d_q\zeta. \tag{1.5}$$

Definition 1.3. For a continous function $f : [c_1, d_1] \to \mathbb{R}$, the q_{c_1}-derivative is of f at $\zeta \in [c_1, d_1]$ is defined as

$$_{c_1}D_q f(\zeta) = \frac{f(\zeta) - f(q\zeta + (1-q)c_1)}{(1-q)(\zeta - c_1)}, \quad \zeta \neq c_1. \tag{1.6}$$

The function f is said to be q_{c_1}-differntiable on $[c_1, d_1]$ if $_{c_1}D_q f(\zeta)$ exists for all $\zeta \in [c_1, d_1]$.

Definition 1.4. For a continuous function $f : [c_1, d_1] \to \mathbb{R}$, the q_{c_1}-definite integral (Reimann type q-integral) on $\zeta \in [c_1, d_1]$ is defined as

$$\begin{aligned}
\int_{c_1}^{d_1} f(\zeta)\, d_q^R\zeta &= \int_{c_1}^{d_1} f(\zeta)\, {}_{c_1}d_q\zeta \\
&= (1-q)(d_1 - c_1) \sum_{n=0}^{\infty} q^n f\big(q^n d_1 + \big(1-q^n\big)c_1\big) \\
&= (d_1 - c_1) \int_0^1 f\big((1-\zeta)c_1 + \zeta d_1\big)\, d_q\zeta. \\
&= \frac{(1-q)(d_1-c_1)}{2} \sum_{n=0}^{\infty} q^n \left[f\left(\frac{c_1+d_1}{2} + q^k\left(\frac{d_1-c_1}{2}\right)\right)\right. \\
&\quad \left. + f\left(\frac{c_1+d_1}{2}\right) - q^k\left(\frac{d_1-c_1}{2}\right)\right)\Bigg].
\end{aligned}$$

In [13], Alp et al. established the H-H inequality for convex function using q_{c_1}-definite integral

Theorem 1.5. *For convex differentiable function $f : [c_1, d_1] \to \mathbb{R}$ and $0 < q < 1$, the following inequality holds:*

$$f\left(\frac{qc_1 + d_1}{1+q}\right) \leq \frac{1}{d_1 - c_1} \int_{c_1}^{d_1} f(\zeta)\, {}_{c_1}d_q\zeta \leq \frac{qf(c_1) + f(d_1)}{1+q}. \tag{1.7}$$

Bermudo et al. [17] derived the H-H the inequality for convex function using q^{d_1}-definite integral.

Theorem 1.6. *For convex differentiable function $f : [c_1, d_1] \to \mathbb{R}$ and $0 < q < 1$, the following inequality holds:*

$$f\left(\frac{c_1 + qd_1}{1+q}\right) \leq \frac{1}{d_1 - c_1} \int_{c_1}^{d_1} f(\zeta)\, {}^{d_1}d_q\zeta \leq \frac{f(c_1) + qf(d_1)}{1+q}. \tag{1.8}$$

Using Theorem 1.5 and Theorem 1.6, next corollary can be derived easily.

Corollary 1.7. *For convex differntiable function $f : [c_1, d_1] \to \mathbb{R}$ and $0 < q < 1$, the following inequalities holds:*

$$\begin{aligned} f\left(\frac{qc_1 + d_1}{1+q}\right) + f\left(\frac{c_1 + qd_1}{1+q}\right) &\leq \frac{1}{d_1 - c_1}\left\{\int_{c_1}^{d_1} f(\zeta)\, {}_{c_1}d_q\zeta + \int_{a}^{d_1} f(\zeta)\, {}^{d_1}d_q\zeta\right\} \\ &\leq f(c_1) + f(d_1) \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} f\left(\frac{c_1 + d_1}{2}\right) &\leq \frac{1}{2(d_1 - c1)}\left\{\int_{c_1}^{d_1} f(\zeta)\, {}_{c_1}d_q\zeta + \int_{c_1}^{d_1} f(\zeta)\, {}^{d_1}d_q\zeta\right\} \\ &\leq \frac{f(c_1) + f(d_1)}{2}. \end{aligned} \tag{1.10}$$

2 Main Results

This section begins with lemma stating the relation between convex function and harmonic convex function. H-H inequality for harmonic convex function using q_{c_1}-integral is given.

Lemma 2.1. *[15] If $[c_1, d_1] \subset \mathbb{J} \subset (0, \infty)$ and if we consider the function $g : [\frac{1}{d_1}, \frac{1}{c_1}] \to \mathbb{R}$, defined by $g(\zeta) = f(\frac{1}{\zeta})$, then g is convex in the classical sense on $[\frac{1}{d_1}, \frac{1}{c_1}]$ if and only if f is harmonic convex function on $[c_1, d_1]$.*

Lemma 2.2. *[16] Let $f, g : [c_1, d_1] \subset (0, \infty) \to \mathbb{R}$ be so that $g(\zeta) = \zeta f(\zeta)$ for $\zeta \in [c_1, d_1]$. Then f is harmonic convex function on $[c_1, d_1]$ iff g is convex function on $[c_1, d_1]$.*

In the next theorem, the Hermite-Hadamard inequality for harmonically convex function is provided.

Theorem 2.3. *Let $f : [c_1, d_1] \to \mathbb{R}$ be harmonic convex differentiable function on (c_1, d_1) and $0 < q < 1$. Then*

$$\begin{aligned} \frac{qc_1 + d_1}{1+q} f\left(\frac{c_1 d_1(q+1)}{qc_1 + d_1}\right) + \left(c_1 d_1 - \left(\frac{c_1 d_1(q+1)}{qc_1 + d_1}\right)^2\right) f'\left(\frac{c_1 d_1(q+1)}{qc_1 + d_1}\right) \\ \leq \frac{1}{d_1 - c_1} \int_{c_1}^{d_1} \zeta f(\zeta)\, {}_{c_1}d_q\zeta \leq \frac{(c_1 q f(c_1) + d_1 f(d_1))}{q+1}. \end{aligned} \tag{2.1}$$

Proof. Since $f : [c_1, d_1] \to \mathbb{R}$ is harmonically convex function on $[c_1, d_1]$, the function $g(\zeta) : [c_1, d_1] \to \mathbb{R}$, $g(\zeta) = \zeta f(\zeta)$ is convex on $[c_1, d_1]$. Also g is differntiable function on (c_1, d_1), the tangent line at the point $\frac{c_1 d_1(1+q)}{qc_1 + d_1} \in [c_1, d_1]$ can be expressed as a function

$$\Psi(\zeta) = g\left(\frac{c_1 d_1(q+1)}{qc_1 + d_1}\right) + g'\left(\frac{c_1 d_1(q+1)}{qc_1 + d_1}\right)\left(\zeta - \frac{c_1 d_1(1+q)}{qc_1 + d_1}\right). \tag{2.2}$$

Also g is a convex function on $[c_1, d_1]$, then the following inequality is satisfied,

$$\Psi(\zeta) = g\left(\frac{c_1 d_1(q+1)}{qc_1 + d_1}\right) + g'\left(\frac{c_1 d_1(q+1)}{qc_1 + d_1}\right)\left(\zeta - \frac{c_1 d_1(1+q)}{qc_1 + d_1}\right) \leq g(\zeta) \tag{2.3}$$

for all $\zeta \in [c_1, d_1]$.
q_{c_1}-integrating the inequality (2.3) on $[c_1, d_1]$,

$$\begin{aligned}
&\int_{c_1}^{d_1} \Psi(\zeta)\, {}_{c_1}d_q\zeta \\
&= \int_{c_1}^{d_1} \left[g\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) + g'\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right)\left(\zeta - \frac{c_1 d_1(q+1)}{qc_1+d_1}\right)\right] {}_{c_1}d_q\zeta \\
&= (d_1-c_1) g\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) + g'\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) \int_{c_1}^{d_1} \left(\zeta - \frac{c_1 d_1(q+1)}{qc_1+d_1}\right) {}_{c_1}d_q\zeta \\
&= (d_1-c_1) g\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) + g'\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) \\
&\quad \times \left[(1-q)(d_1-c_1)\sum_{n=0}^{\infty} q^n g\Big(q^n d_1 + \big(1-q^n\big)c_1\Big) - \frac{c_1 d_1(q+1)}{qc_1+d_1}(d_1-c_1)\right] \\
&= (d_1-c_1) g\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) + g'\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right)\left[\frac{qc_1+d_1}{c_1 d_1(q+1)}(d_1-c_1)\right. \\
&\qquad\qquad \left. - \frac{c_1 d_1(q+1)}{qc_1+d_1}(d_1-c_1)\right] \\
&= (d_1-c_1)\left[\frac{c_1 d_1(q+1)}{qc_1+d_1} f\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) + \left[\frac{c_1 d_1(q+1)}{qc_1+d_1} f'\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right)\right.\right. \\
&\qquad\qquad \left.\left. + f\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right)\right] \times \left[\frac{qc_1+d_1}{1+q} - \frac{c_1 d_1(1+q)}{qc_1+d_1}\right]\right. \\
&= (d_1-c_1)\left[\frac{qc_1+d_1}{1+q} f\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) + \left(c_1 d_1 - \left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right)^2\right) f'\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right)\right] \\
&\le \int_{c_1}^{d_1} \zeta f(\zeta)_{c_1} d_q\zeta. \qquad (2.4)
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{qc_1+d_1}{1+q} f\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) + \left(c_1 d_1 - \left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right)^2\right) f'\left(\frac{c_1 d_1(q+1)}{qc_1+d_1}\right) \\
&\qquad\qquad \le \frac{1}{d_1-c_1}\int_{c_1}^{d_1} \zeta f(\zeta)_{c_1} d_q\zeta. \qquad (2.5)
\end{aligned}$$

which proves the first inequality.

For the second inequality, use the fact that $f : [c_1, d_1] \to \mathbb{R}$ is harmonically convex function on $[c_1, d_1]$, the function $g(\zeta) : [c_1, d_1] \to \mathbb{R}$, $g(\zeta) = \zeta f(\zeta)$ is convex on $[c_1, d_1]$. It implies that $g(\zeta) \le r(\zeta)$, where $r(\zeta)$ is the secant that joins the points $(c_1, g(c_1))$ and $(d_1, g(d_1))$, given as

$$\begin{aligned}
r(\zeta) &= g(d_1) + \frac{g(d_1)-g(c_1)}{d_1-c_1}(\zeta - d_1) \\
&= d_1 f(d_1) + \frac{d_1 f(d_1) - c_1 f(c_1)}{d_1-c_1}(\zeta - d_1) \\
&= \frac{c_1 d_1(f(c_1) - f(d_1))}{d_1-c_1} + \left(\frac{d_1 f(d_1) - c_1 f(c_1)}{d_1-c_1}\right)\zeta \\
\therefore g(\zeta) &\le \frac{c_1 d_1(f(c_1) - f(d_1))}{d_1-c_1} + \left(\frac{d_1 f(d_1) - c_1 f(c_1)}{d_1-c_1}\right)\zeta. \qquad (2.6)
\end{aligned}$$

q_{c_1}-integrating inequality (2.6),

$$\begin{aligned}\int_{c_1}^{d_1} \zeta f(\zeta)\, {}_{c_1}d_q\zeta &\leq \frac{c_1 d_1(f(c_1)-f(d_1))}{d_1-c_1}\int_{c_1}^{d_1} 1\, {}_{c_1}d_q\zeta \\ &+ \left(\frac{d_1 f(d_1)-c_1 f(c_1)}{d_1-c_1}\right)\int_{c_1}^{d_1} \zeta\, {}_{c_1}d_q\zeta \\ &= c_1 d_1(f(c_1)-f(d_1)) + (d_1 f(d_1) - c_1 f(c_1)\frac{qc_1+d_1}{q+1} \\ &= \frac{(c_1 q f(c_1) + d_1 f(d_1))(d_1-c_1)}{q+1}. \\ Thus, \end{aligned}$$

$$\frac{1}{d_1-c_1}\int_{c_1}^{d_1} \zeta f(\zeta)\, {}_{c_1}d_q\zeta \leq \frac{(c_1 q f(c_1) + d_1 f(d_1))}{q+1}. \tag{2.7}$$

which gives the second inequality. □

Theorem 2.4. *Let* $f : [c_1, d_1] \to \mathbb{R}$ *be harmonically convex differentiable function on* (c_1, d_1) *and* $0 < q < 1$. *Then*

$$\begin{aligned}\frac{qc_1+d_1}{1+q} f\left(\frac{c_1 d_1(q+1)}{c_1+qd_1}\right) + \left(\frac{c_1 d_1(qc_1+d_1)}{c_1+qd_1} - \left(\frac{c_1 d_1(q+1)}{c_1+qd_1}\right)^2\right) f'\left(\frac{c_1 d_1(q+1)}{c_1+qd_1}\right) \\ \leq \frac{1}{d_1-c_1}\int_{c_1}^{d_1} \zeta f(\zeta)_{c_1} d_q\zeta \leq \frac{c_1 q f(c_1) + d_1 f(d_1)}{q+1}. \end{aligned} \tag{2.8}$$

Proof. Since $f : [c_1, d_1] \to \mathbb{R}$ is harmonically convex function on $[c_1, d_1]$, the function $g(\zeta) : [c_1, d_1] \to \mathbb{R}$, $g(\zeta) = \zeta f(\zeta)$ is convex on $[c_1, d_1]$. Also g is differntiable function on (c_1, d_1), the tangent line at the point $\frac{c_1 d_1(1+q)}{c_1+qd_1} \in [c_1, d_1]$ can be expressed as a function

$$\Psi_1(\zeta) = g\left(\frac{c_1 d_1(q+1)}{c_1+qd_1}\right) + g'\left(\frac{c_1 d_1(q+1)}{c_1+qd_1}\right)\left(\zeta - \frac{c_1 d_1(q+1)}{c_1+qd_1}\right). \tag{2.9}$$

Also g is a convex function on $[c_1, d_1]$, then the following inequality is satisfied,

$$\Psi_1(\zeta) = g\left(\frac{c_1 d_1(q+1)}{c_1+qd_1}\right) + g'\left(\frac{c_1 d_1(q+1)}{c_1+qd_1}\right)\left(\zeta - \frac{c_1 d_1(q+1)}{c_1+qd_1}\right) \leq g(\zeta). \tag{2.10}$$

for all $\zeta \in [c_1, d_1]$.

q_{c_1}-integrating the inequality (2.10) on $[c_1, d_1]$, we have

$$\begin{aligned}
&\int_{c_1}^{d_1} \Psi_1(\zeta)\, {}_{c_1}d_q\zeta \\
&= \int_{c_1}^{d_1} \left[g\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) + g'\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)\left(\zeta - \frac{c_1d_1(q+1)}{c_1+qd_1}\right)\right] {}_{c_1}d_q\zeta \\
&= (d_1-c_1)g\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) + g'\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)\int_{c_1}^{d_1}\left(\zeta - \frac{c_1d_1(q+1)}{c_1+qd_1}\right) {}_{c_1}d_q\zeta \\
&= (d_1-c_1)g\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) + g'\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) \\
&\quad \times \left[(1-q)(d_1-c_1)\sum_{n=0}^{\infty} q^n\left(q^n d_1 + (1-q^n)c_1\right) - \frac{c_1d_1(q+1)}{c_1+qd_1}(d_1-c_1)\right] \\
&= (d_1-c_1)g\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) + g'\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)\left[\frac{qc_1+d_1}{q+1}(d_1-c_1)\right. \\
&\qquad\qquad \left. - \frac{c_1d_1(q+1)}{c_1+qd_1}(d_1-c_1)\right] \\
&= (d_1-c_1)\left\{\frac{c_1d_1(q+1)}{c_1+qd_1} f\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) + \left[\frac{c_1d_1(q+1)}{c_1+qd_1} f'\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)\right.\right. \\
&\qquad\qquad \left.\left. + f\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)\right] \times \left[\frac{qc_1+d_1}{1+q} - \frac{c_1d_1(1+q)}{c_1+qd_1}\right]\right\} \\
&= (d_1-c_1)\left[\frac{qc_1+d_1}{1+q} f\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)\right. \\
&\qquad\qquad \left. + \left(\frac{c_1d_1(qc_1+d_1)}{c_1+qd_1} - \left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)^2\right) f'\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)\right] \\
&\le \int_{c_1}^{d_1} \zeta f(\zeta)_{c_1}d_q\zeta.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{qc_1+d_1}{1+q} f\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) + \left(\frac{c_1d_1(qc_1+d_1)}{c_1+qd_1} - \left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)^2\right) f'\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) \\
&\le \frac{1}{d_1-c_1}\int_{c_1}^{d_1} \zeta f(\zeta)_{c_1}d_q\zeta. \qquad (2.11)
\end{aligned}$$

which proves the first inequality. The second inequality follows similarly as Theorem 2.3. □

Theorem 2.5. *Let* $f : [c_1, d_1] \to \mathbb{R}$ *be harmonically convex differentiable function on* (c_1, d_1) *and* $0 < q < 1$. *Then*

$$\begin{aligned}
&\frac{qc_1+d_1}{1+q} f\left(\frac{c_1+d_1}{2}\right) + \left[\frac{(c_1+d_1)(qc_1+d_1)}{2(1+q)} - \left(\frac{c_1+d_1}{2}\right)^2\right] f'\left(\frac{c_1+d_1}{2}\right) \\
&\qquad\qquad \le \frac{1}{d_1-c_1}\int_{c_1}^{d_1} \zeta f(\zeta)_{c_1}d_q\zeta \le \frac{(c_1qf(c_1)+d_1f(d_1))}{q+1}.
\end{aligned}$$

Proof. Since $f : [c_1, d_1] \to \mathbb{R}$ is harmonically convex function on $[c_1, d_1]$, the function $g(\zeta) : [c_1, d_1] \to \mathbb{R}$, $g(\zeta) = \zeta f(\zeta)$ is convex on $[c_1, d_1]$. Also g is differntiable function on (c_1, d_1), the tangent line at the point $\frac{c_1+d_1}{2} \in [c_1, d_1]$ can be expressed as a function

$$\Psi_2(\zeta) = g\left(\frac{c_1+d_1}{2}\right) + g'\left(\frac{c_1+d_1}{2}\right)\left(\zeta - \frac{c_1+d_1}{2}\right). \qquad (2.12)$$

Also g is a convex function on $[c_1, d_1]$, then the following inequality is satisfied,

$$\Psi_2(\zeta) = g\left(\frac{c_1+d_1}{2}\right) + g'\left(\frac{c_1+d_1}{2}\right)\left(\zeta - \frac{c_1+d_1}{2}\right) \leq g(\zeta), for\, all\, \zeta \in [c_1, d_1]. \tag{2.13}$$

q_{c_1}-integrating the inequality (2.13) on $[c_1, d_1]$,

$$\begin{aligned}
&\int_{c_1}^{d_1} \Psi_2(\zeta)\, {}_{c_1}d_q\zeta \\
&= \int_{c_1}^{d_1} \left[g\left(\frac{c_1+d_1}{2}\right) + g'\left(\frac{c_1+d_1}{2}\right)\left(\zeta - \frac{c_1+d_1}{2}\right)\right] {}_{c_1}d_q\zeta \\
&= (d_1-c_1) g\left(\frac{c_1+d_1}{2}\right) + g'\left(\frac{c_1+d_1}{2}\right) \int_{c_1}^{d_1} \left(\zeta - \frac{c_1+d_1}{2}\right) {}_{c_1}d_q\zeta \\
&= (d_1-c_1) g\left(\frac{c_1+d_1}{2}\right) + g'\left(\frac{c_1+d_1}{2}\right) \\
&\times \left[(1-q)(d_1-c_1)\sum_{n=0}^{\infty} q^n g\left(q^n d_1 + (1-q^n)c_1\right) - \frac{c_1+d_1}{2}(d_1-c_1)\right] \\
&= (d_1-c_1) g\left(\frac{c_1+d_1}{2}\right) + g'\left(\frac{c_1+d_1}{2}\right)\left(\frac{qc_1+d_1}{q+1}(d_1-c_1) - \frac{c_1+d_1}{2}(d_1-c_1)\right) \\
&= (d_1-c_1)\left[\frac{c_1+d_1}{2} f\left(\frac{c_1+d_1}{2}\right) + \left\{\frac{c_1+d_1}{2} f'\left(\frac{c_1+d_1}{2}\right) + f\left(\frac{c_1+d_1}{2}\right)\right\}\right. \\
&\qquad \left.\left(\frac{qc_1+d_1}{1+q} - \frac{c_1+d_1)}{2}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= (d_1-c_1)\left[\frac{qc_1+d_1}{1+q} f\left(\frac{c_1+d_1}{2}\right) + \left\{\frac{(c_1+d_1)(qc_1+d_1)}{2(1+q)} - \left(\frac{c_1+d_1}{2}\right)^2\right\} f'\left(\frac{c_1+d_1}{2}\right)\right] \\
&\leq \int_{c_1}^{d_1} \zeta f(\zeta)_{c_1} d_q\zeta.
\end{aligned} \tag{2.14}$$

$Thus$

$$\begin{aligned}
&\frac{qc_1+d_1}{1+q} f\left(\frac{c_1+d_1}{2}\right) + \left[\frac{(c_1+d_1)(qc_1+d_1)}{2(1+q)} - \left(\frac{c_1+d_1}{2}\right)^2\right] f'\left(\frac{c_1+d_1}{2}\right) \\
&\qquad\qquad \leq \frac{1}{d_1-c_1}\int_{c_1}^{d_1} \zeta f(\zeta)_{c_1} d_q\zeta.
\end{aligned} \tag{2.15}$$

which proves the first inequality. The second inequality follows similarly as Theorem 2.3. □

Summing up the results in Theorem 2.3, Theorem 2.4 and Theorem 2.5 yields the next corollary.

Corollary 2.6. *Let $f : [c_1, d_1] \to \mathbb{R}$ be harmonic convex differentiable function on (c_1, d_1) and $0 < q < 1$. Then*

$$max\{H_1, H_2, H_3\} \leq \frac{1}{d_1-c_1}\int_{c_1}^{d_1} f(\zeta)\, {}_{c_1}d_q\zeta \leq \frac{c_1 q f(c_1) + d_1 f(d_1)}{q+1}. \tag{2.16}$$

Where

$$H_1 = \frac{qc_1+d_1}{1+q} f\left(\frac{c_1d_1(q+1)}{qc_1+d_1}\right) + \left(c_1d_1 - \left(\frac{c_1d_1(q+1)}{qc_1+d_1}\right)^2\right) f'\left(\frac{c_1d_1(q+1)}{qc_1+d_1}\right).$$

$$H_2 = \frac{qc_1+d_1}{1+q} f\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right) + \left(\frac{c_1d_1(qc_1+d_1)}{c_1+qd_1} - \left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right)^2\right) f'\left(\frac{c_1d_1(q+1)}{c_1+qd_1}\right).$$

$$H_3 = \frac{qc_1+d_1}{1+q} f\left(\frac{c_1+d_1}{2}\right) + \left[\frac{(c_1+d_1)(qc_1+d_1)}{2(1+q)} - \left(\frac{c_1+d_1}{2}\right)^2\right] f'\left(\frac{c_1+d_1}{2}\right). \quad (2.17)$$

3 Application

Oftenly, it is not possible to find the closed form for all infinte series. So it is difficult to find q_{c_1}-integral for some function as q_{c_1}-integral contains infinite series. This can be illustrated by the following example.

3.1 Example

To find the q_{c_1}-integral of $f(\zeta) = \zeta e^{\zeta}$ in interval $[1,2]$,

$$\int_1^2 f(\zeta)\,{}_1d_q\zeta = \int_1^2 \zeta e^{\zeta}\,{}_1d_q\zeta = (1-q)(2-1)\sum_{n=0}^{\infty} q^n(q^n+1)e^{q^n+1}.$$

For particular $q = \frac{1}{2}$,

$$\int_1^2 \zeta e^{\zeta}\,{}_1d_q\zeta = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1+2^n}{2^{2n}} e^{\frac{1+2^n}{2^n}}. \quad (3.1)$$

Difficulty arises to simplify the infinte series of the right hand side of the expression (3.1). However, with the results obtainted in the Theorem 2.3, it enables us to determine the lower and upper bound of $\int_1^2 \zeta e^{\zeta}\,{}_1d_q\zeta$. By lemma 2.2, e^{ζ} is harmonic convex function on $[1,2]$ as ζe^{ζ} is convex function on $[1,2]$. By theorem 2.3,

$$\frac{5}{3}e^{\frac{6}{5}} + \frac{14}{25}e^{\frac{6}{5}} \leq \int_1^2 \zeta e^{\zeta}\,{}_1d_q\zeta \leq \frac{e+4e^2}{3}$$

$$7.3705 \leq \int_1^2 \zeta e^{\zeta}\,{}_1d_q\zeta \leq 10.7581.$$

Thus lower and upper bound of the integral $\int_1^2 \zeta e^{\zeta}\,{}_1d_q\zeta$ is 7.3705 and 10.7581 respectively.

4 Conclusion

The harmonic convexity of a function, as shown in this research study, provides the foundation for numerous mathematical inequalities. Harmonic convexity gives an analytic technique for estimating a number of q_{c_1}-integral like $\int_{c_1}^{d_1} \zeta e^{\zeta}\,{}_{c_1}d_q\zeta$, $\int_{c_1}^{d_1} \frac{1}{\zeta}\,{}_{c_1}d_q\zeta$, $\int_{c_1}^{d_1} \frac{1}{\zeta^2}\,{}_{c_1}d_q\zeta$ and $\int_{c_1}^{d_1} \frac{1}{\zeta^3}\,{}_{c_1}d_q\zeta$ using the result obtained in the section 2.

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