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Construction of continuous K-g-Frames in Hilbert C^*-Modules

A. Karara, M. Rossafi, M. Klilou and S. Kabbaj

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Corresponding Author: M. Rossafi

Abstract *In this work, we provide some constructions and the sum of new continuous K-g-frames in Hilbert C^*-Modules. We provide certaines necessary and sufficient conditions for some adjointable operators on $\mathcal{H}$, under which new continuous K-g-frames can be retrieved from those that already exist. Additionally, we discuss the sum of continuous K-g-frames, discover some of their characterizations, and offer some adjointable operators to construct new continuous K-g-frames from the previous ones.*

1 Introduction and preliminaries

*Frame theory is an active topic of mathematical research in fields such as signal processing, computer science, and more. Frames for Hilbert spaces were first introduced in **1952** by Duffin and Schaefer [7] for the study of nonharmonic Fourier series. Daubechies, Grossmann, and Meyer [6] later revised and developed them in **1986**, and they have been popularized since then.*

Recently, many mathematicians have generalized frame theory from Hilbert spaces to Hilbert C^*- modules. For detailed information on frames, we refer to [9, 10, 13, 14, 16, 17, 18, 19]. Currently, the study of continuous K-g-frames has yielded many results, which were introduced by Alizadeh, Rahimi, Osgooei, and Rahmani [1]. The study of some of their properties has been further explored in [4].

Throughout this paper (Ω,ν) be a measure space, let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules, $\{\mathcal{K}_\xi : \xi \in \Omega\}$ is a sequence of subspaces of $\mathcal{K}$, we also reserve the notation $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{K}_\xi)$ for the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ to $\mathcal{K}_\xi$ and $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{H})$ is denoted by $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. We will use $\mathcal{N}(\Theta)$ and $\mathcal{R}(\Theta)$ for the null and range space of an operator $\Theta \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$, respectively. We also denote

$$\bigoplus_{\xi\in\Omega}\mathcal{K}_\xi=\{\alpha=\{\alpha_\xi\}:\alpha_\xi\in\mathcal{K}_\xi \text{ and } \left\|\int_\Omega|\alpha_\xi|^2\mathrm{d}\nu(\xi)\right\|<\infty\}.$$

Let $f=\{f_\xi\}_{\xi\in\Omega}$ and $g=\{g_\xi\}_{\xi\in\Omega}$, the inner product is defined by $\langle f,g\rangle=\int_\Omega\langle f_\xi,g_\xi\rangle\,\mathrm{d}\nu(\xi)$, we have $\bigoplus_{\xi\in\Omega}\mathcal{K}_\xi$ is a Hilbert $\mathcal{A}$-module.

Definition 1.1. [5]. Let $\mathcal{A}$ be a Banach algebra, an involution is a map $x\to x^*$ of $\mathcal{A}$ into itself, such that, for all x and y in $\mathcal{A}$ and all scalars α the following conditions hold:

(i) $(x^*)^*=x$.

(ii) $(xy)^*=y^*x^*$.

(iii) $(\alpha x+y)^*=\bar{\alpha}x^*+y^*$.

Definition 1.2. [5]. A C^*-algebra $\mathcal{A}$ is a Banach algebra with involution, such that :

$$\|x^*x\|=\|x\|^2,$$

for every x in $\mathcal{A}$.

Definition 1.3. [11]. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatibles. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$. Where $||.||$ is a norm on $\mathcal{H}$ and if $\mathcal{H}$ is complete, this norm we will call it a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$.

Lemma 1.4. *[15]. Let $\mathcal{H}$ be Hilbert $\mathcal{A}$-module. If $\mathcal{T} \in End^*_{\mathcal{A}}(\mathcal{H})$, then*

$$\langle \mathcal{T}x, \mathcal{T}x\rangle \leq \|\mathcal{T}\|^2 \langle x, x\rangle, \forall x \in \mathcal{H}.$$

Lemma 1.5. *[3]. Let $\mathcal{H}$ and $\mathcal{K}$ two Hilbert $\mathcal{A}$-modules and $\mathcal{T} \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:*

(i) $\mathcal{T}$ is surjective.

(ii) $\mathcal{T}^*$ is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|\mathcal{T}^*x\| \geq m\|x\|$ for all $x \in \mathcal{K}$.

(iii) $\mathcal{T}^*$ is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle \mathcal{T}^*x, \mathcal{T}^*x\rangle \geq m'\langle x, x\rangle$ for all $x \in \mathcal{K}$.

Definition 1.6. [12] Let $\mathcal{T} \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. The **Moore-Penrose inverse** of $\mathcal{T}$ (if it exists) is an element $\mathcal{T}^\dagger \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ satisfying

- $\mathcal{T}\mathcal{T}^\dagger\mathcal{T} = \mathcal{T}$,
- $\mathcal{T}^\dagger\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger$,
- $(\mathcal{T}\mathcal{T}^\dagger)^* = \mathcal{T}\mathcal{T}^\dagger$,
- $(\mathcal{T}^\dagger\mathcal{T})^* = \mathcal{T}^\dagger\mathcal{T}$.

Lemma 1.7. *[8, 12] Let $\Theta \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. Then the following holds :*

1. *$\mathcal{R}(\Theta)$ is closed in $\mathcal{K}$ if and only if $\mathcal{R}(\Theta^*)$ is closed in $\mathcal{H}$.*
2. *$(\Theta^*)^\dagger = (\Theta^\dagger)^*$.*
3. *The orthogonal projection of $\mathcal{K}$ onto $\mathcal{R}(\Theta)$ is given by $\Theta\Theta^\dagger$.*
4. *The orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(\Theta^\dagger)$ is given by $\Theta^\dagger\Theta$.*

Lemma 1.8. *[8] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ two Hilbert $\mathcal{A}$-Modules and $\mathcal{T} \in \text{End}^*_{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}), \mathcal{T}' \in \text{End}^*_{\mathcal{A}}(\mathcal{H}_2, \mathcal{H})$ with $\overline{\mathcal{R}(\mathcal{T}^*)}$ is orthogonally complemented. Then the following assertions are equivalent:*

(i) $\mathcal{T}'(\mathcal{T}')^* \leq \lambda\mathcal{T}\mathcal{T}^*$ for some $\lambda > 0$.

(ii) There exist $\mu > 0$ such that, $\|(\mathcal{T}')^* x\| \leq \mu \|\mathcal{T}^*x\|$ for all $x \in F$.

(iii) There exists $Q \in \text{End}^*_{\mathcal{A}}(\mathcal{H}_2, \mathcal{H}_1)$ such that $\mathcal{T}' = \mathcal{T}Q$, that is, the equation $\mathcal{T}X = \mathcal{T}'$ has a solution.

(iv) $\mathcal{R}(\mathcal{T}') \subseteq \mathcal{R}(\mathcal{T})$.

Theorem 1.9. *[4] Let $\{\Upsilon_\xi\}_{\xi\in\Omega}$ be a c-g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$ with bound A. Then the bounded and linear operator $\mathcal{T}_\Upsilon : \bigoplus_{\xi\in\Omega} \mathcal{K}_\xi \longrightarrow \mathcal{H}$ weakly defined by*

$$\langle \mathcal{T}_\Upsilon F, g\rangle = \int_\Omega \langle \Upsilon^*_\xi F(\xi), g\rangle \, d\nu(\xi), \quad F \in \bigoplus_{\xi\in\Omega} \mathcal{K}_\xi, g \in \mathcal{H},$$

with $\|\mathcal{T}_\Upsilon\| \leq \sqrt{A}$.
Moreover, for every $g \in \mathcal{H}$ *and* $\xi \in \Omega$,

$$\mathcal{T}_\Upsilon^*(g)(\xi) = \Upsilon_\xi g.$$

The operators $\mathcal{T}_\Upsilon$ *and* $\mathcal{T}_\Upsilon^*$ *are called the synthesis and the analysis operator of* $\{\Upsilon_\xi\}_{\xi\in\Omega}$ *respectively.*

Definition 1.10. Let $K \in \text{End}_\mathcal{A}^*(\mathcal{H})$. A sequence $\{\Upsilon_\xi \in \text{End}_\mathcal{A}^*(\mathcal{H}, \mathcal{K}_\xi) : \xi \in \Omega\}$ is called a continuous $K - g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Omega}$ if :

a) For every $h \in \mathcal{H}$, the function $\chi : \Omega \to \mathcal{K}_\xi$ defined by $\chi(\xi) = \Upsilon_\xi h$ is measurable,

b) There exist constants $0 < A \leq B < \infty$ such that,

$$A\langle K^*h, K^*h\rangle \leq \int_\Omega \langle \Upsilon_\xi h, \Upsilon_\xi h\rangle \, \mathrm{d}\nu(\xi) \leq B\langle h, h\rangle.$$

The numbers A and B are called the lower and upper $c{-}K{-}g$-frame bounds of $\{\Upsilon_\xi \in \text{End}_\mathcal{A}^*(\mathcal{H}, \mathcal{K}_\xi)\}_{\xi\in\Omega}$, respectively. If $A = B = \delta$, the $c - K - g$-frame is called δ-tight and if $A = B = 1$, it is called a Parseval $c - K - g$-frame. If for each $h \in \mathcal{H}$,

$$\int_\Omega \langle \Upsilon_\xi h, \Upsilon_\xi h\rangle \, \mathrm{d}\nu(\xi) \leq B\langle h, h\rangle.$$

The sequence $\{\Upsilon_\xi\}_{\xi\in\Omega}$ is called a $c - K - g$-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{K}_\xi\}_{\xi\in\Omega}$.

We suppose that $\{\Upsilon_\xi\}_{\xi\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$ with frame bounds C, D. The c-K-g-frame operator $S_\Upsilon : \mathcal{H} \longrightarrow \mathcal{H}$ is weakly defined by

$$\langle S_\Upsilon f, g\rangle = \int_\Omega \left\langle \Upsilon_\xi^* \Upsilon_\xi f, g\right\rangle \mathrm{d}\nu(\xi), \quad f, g \in \mathcal{H}.$$

Furthermore,

$$CKK^* \leq S_\Upsilon \leq DI_\mathcal{H}.$$

Example 1.11. Let $\mathcal{A} = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C}\right\}$, and $\mathcal{H} = \left\{\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} : a, b \in \mathbb{C}\right\}$ which is a C^*-algebra. We define the inner product:

$$\begin{aligned} \langle ., .\rangle : \mathcal{H} \times \mathcal{H} &\to \mathcal{A} \\ (M, N) &\longmapsto M(\bar{N})^t. \end{aligned}$$

Let $\Omega = [0, 1]$ endowed with the Lebesgue measure. Moreover, for $\xi \in \Omega$, we define the operator $\Upsilon_\xi : \mathcal{H} \to \mathcal{H}$ by

$$\Upsilon_\xi(M) = \begin{pmatrix} \frac{\xi}{2}a & 0 & 0 \\ 0 & 0 & \xi b \end{pmatrix}.$$

Υ_ξ is linear, bounded, and selfadjoint. In addition, for $M \in \mathcal{H}$, we have

$$\begin{aligned} &\int_\Omega \langle \Upsilon_\xi M, \Upsilon_\xi M\rangle_\alpha \, d\mu(\xi) \\ &= \int_\Omega \begin{pmatrix} \frac{1}{4}\xi^2|a|^2 & 0 \\ 0 & \xi^2|b|^2 \end{pmatrix} d\mu(\xi) \\ &= \begin{pmatrix} \int_\Omega \frac{1}{4}\xi^2 d\mu(\xi)|a|^2 & 0 \\ 0 & \int_\Omega \xi^2 d\mu(\xi)|b|^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{12}|a|^2 & 0 \\ 0 & \frac{1}{3}|b|^2 \end{pmatrix}, \end{aligned}$$

which show that

$$\frac{1}{12}\langle A,A\rangle \leq \int_{\Omega}\langle \Upsilon_{\xi}M,\Upsilon_{\xi}M\rangle_{\alpha}\, d\mu(\xi) \leq \frac{1}{3}\langle A,A\rangle .$$

Now, we define an operator $K:\mathcal{H}\to\mathcal{H}$ by

$$K(M)=\begin{pmatrix} a & 0 & 0\\ 0 & 0 & b\end{pmatrix}.$$

We have

$$\begin{aligned}\frac{1}{12}\langle K^{*}M,K^{*}M\rangle &\leq \int_{\Omega}\langle \Upsilon_{\xi}M,\Upsilon_{\xi}M\rangle\, d\mu(\xi)\\ &\leq \frac{1}{3}\langle M,M\rangle.\end{aligned}$$

Hence, $\{\Upsilon_{\xi}\}_{\xi\in\Omega}$ is a continuous $K-g$ -frame for $\mathcal{H}$.

Definition 1.12. [2] A sequence $\{\Upsilon_{\xi}\in \operatorname{End}^{*}_{\mathcal{A}}(\mathcal{H},\mathcal{H}_{\xi}):\xi\in\Omega\}$ is c-K-g-frame for $\mathcal{H}$. If for every $g,h\in\mathcal{H}$,

$$\langle Kg,h\rangle=\int_{\Omega}\left\langle \Upsilon^{*}_{\xi}\Phi_{\xi}g,h\right\rangle \mathrm{d}\nu(\xi).$$

The c-g Bassel sequence $\{\Phi_{\xi}\}_{\xi\in\Omega}$ is called a dual c-K-g-Bessel sequence of $\{\Upsilon_{\xi}\}_{\xi\in\Omega}$.

2 Some new constructing of c-K-g-frames in Hilbert C^{*}-modules

Theorem 2.1. *Let $K\in\operatorname{End}^{*}_{\mathcal{A}}(\mathcal{H})$ and $\Theta\in\operatorname{End}^{*}_{\mathcal{A}}(\mathcal{H})$, Θ has a closed range such that $\Theta K=K\Theta$, suppose that $\{\Upsilon_{\xi}\in\operatorname{End}^{*}_{\mathcal{A}}(\mathcal{H},\mathcal{H}_{\xi})\}_{\xi\in\Omega}$ is a c- K-g-frame for $\mathcal{H}$, with bounds A and B. If $\mathcal{R}(K^{*})\cap\mathcal{N}(\Theta^{*})=\{0\}$, then for every $f\in\mathcal{H}$,*

$$A\left\|(\Theta^{*})^{\dagger}\right\|^{-2}\langle K^{*}f,K^{*}f\rangle \leq \int_{\Omega}\langle \Upsilon_{\xi}\Theta^{*}f,\Upsilon_{\xi}\Theta^{*}f\rangle \mathrm{d}\nu(\xi)\leq B\|\Theta^{*}\|^{2}\langle f,f\rangle .$$

Proof. Suppose that $\{\Upsilon_{\xi}\}_{\xi\in\Omega}$ is a c- K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\xi}\}_{\xi\in\Omega}$, with bounds A and B, we have for $f\in\mathcal{H}$

$$\int_{\Omega}\langle \Upsilon_{\xi}\Theta^{*}f,\Upsilon_{\xi}\Theta^{*}f\rangle \mathrm{d}\nu(\xi)\leq B\langle \Theta^{*}f,\Theta^{*}f\rangle \leq B\|\Theta^{*}\|^{2}\langle f,f\rangle .$$

On the other hand, since $\Theta K=K\Theta$, we have $K^{*}\Theta^{*}=\Theta^{*}K^{*}$. We assume that $\mathcal{R}(K^{*})\cap\mathcal{N}(\Theta^{*})=\{0\}$ and Θ has closed range, using Lemma **1.7** for every $f\in\mathcal{H}$,

$$\begin{aligned}\langle K^{*}f,K^{*}f\rangle &=\left\langle \Theta\Theta^{\dagger}K^{*}f,\Theta\Theta^{\dagger}K^{*}f\right\rangle\\ &=\left\langle \left(\Theta^{\dagger}\right)^{*}\Theta^{*}K^{*}f,\left(\Theta^{\dagger}\right)^{*}\Theta^{*}K^{*}f\right\rangle\\ &=\left\langle (\Theta^{*})^{\dagger}K^{*}\Theta^{*}f,(\Theta^{*})^{\dagger}K^{*}\Theta^{*}f\right\rangle\\ &\leq \left\|(\Theta^{*})^{\dagger}\right\|^{2}\langle K^{*}\Theta^{*}f,K^{*}\Theta^{*}f\rangle .\end{aligned}$$

Hence for each $f\in\mathcal{H}$,

$$\int_{\Omega}\langle \Upsilon_{\xi}\Theta^{*}f,\Upsilon_{\xi}\Theta^{*}f\rangle \mathrm{d}\nu(\xi)\geq A\langle K^{*}\Theta^{*}f,K^{*}\Theta^{*}f\rangle \geq A\left\|(\Theta^{*})^{\dagger}\right\|^{-2}\langle K^{*}f,K^{*}f\rangle .$$

□

Corollary 2.2. *Let $K\in\operatorname{End}^{*}_{\mathcal{A}}(\mathcal{H})$ such that $\overline{\mathcal{R}(K)}=\mathcal{H}$, $\Theta\in\operatorname{End}^{*}_{\mathcal{A}}(\mathcal{H})$ has closed range, and $\Theta K=K\Theta$. If $\{\Upsilon_{\xi}\Theta\}_{\xi\in\Omega}$ and $\{\Upsilon_{\xi}\Theta^{*}\}_{\xi\in\Omega}$ are both c - K-g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\xi}\}_{\xi\in\Omega}$, then $\{\Upsilon_{\xi}\}_{\xi\in\Omega}$ is a $c-K$-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\xi}\}_{\xi\in\Omega}$.*

Proof. As such, $\overline{\mathcal{R}(K)} = \mathcal{H}$, thus $\mathcal{N}(K^*) = \{0\}$ and $\mathcal{N}(K^*)^{\perp} = \mathcal{H}$. For every $f \in \mathcal{H}$, we have

$$A\langle K^*f, K^*f\rangle \leq \int_{\Omega} \langle \Upsilon_{\xi}\Theta^* f, \Upsilon_{\xi}\Theta^* f\rangle \, \mathrm{d}\nu(\xi).$$

Hence $\mathcal{N}(K^*) \supseteq \mathcal{N}(\Theta^*)$, so

$$\mathcal{H} = \mathcal{N}(K^*)^{\perp} \subseteq \mathcal{N}(\Theta^*)^{\perp} = \mathcal{R}(\Theta).$$

Consequently, Θ is surjective. Also, for every $f \in \mathcal{H}$,

$$A\langle K^*f, K^*f\rangle \leq \int_{\Omega} \langle \Upsilon_{\xi}\Theta f, \Upsilon_{\xi}\Theta f\rangle \, \mathrm{d}\nu(\xi),$$

thus $\mathcal{N}(\Theta) \subseteq \mathcal{N}(K^*) = \{0\}$. Therefore, Θ is invertible. Since $\Theta K = K\Theta$, we have $\Theta^{-1}K = K\Theta^{-1}, \mathcal{R}(K^*) \cap \mathcal{N}\left(\left(\Theta^{-1}\right)^*\right) = \{0\}$, and

$$\{\Upsilon_{\xi} : \xi \in \Omega\} = \left\{\Upsilon_{\xi}\left(\Theta^{-1}\Theta\right)^* : \xi \in \Omega\right\} = \left\{\left(\Upsilon_{\xi}\Theta^*\right)\left(\Theta^{-1}\right)^* : \xi \in \Omega\right\}.$$

Hence, according to Theorem 2.1, we conclude that $\{\Upsilon_{\xi}\}_{\xi\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$. □

Theorem 2.3. *Let $\Theta, K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ and $\{\Upsilon_{\xi} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_{\xi}) : \xi \in \Omega\}$ be a δ-tight c- K g-frame for $\mathcal{H}$. If $\Theta K = K\Theta$ and K^* is bounded below. Then the following assertions are equivalent:*

(i) Θ is surjective.

(ii) $\{\Upsilon_{\xi}\Theta^*\}_{\xi\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$.

Example 2.4. Let $\mathcal{A} = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C}\right\}$, and $\mathcal{H} = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{C}\right\}$ which is a C^*-algebra. We define the inner product :

$$\begin{aligned} \langle ., . \rangle : \mathcal{H} \times \mathcal{H} &\to \mathcal{A} \\ (M, N) &\longmapsto \det(MN) \end{aligned}$$

Let $\Omega = [0, 1]$ endowed with the Lebesgue measure. Moreover, for $\xi \in \Omega$, we define the operator $\Upsilon_{\xi} : \mathcal{H} \to \mathcal{H}$ by

$$\Upsilon_{\xi}(M) = \begin{pmatrix} \xi a & 0 \\ 0 & b \end{pmatrix}.$$

Υ_{ξ} is linear, bounded, and selfadjoint. We define also the identity operator $\Theta : \mathcal{H} \to \mathcal{H}$ by $\Theta(M) = M$.

It's clear that, Θ is surjective. We define an operator K by :

$$K(M) = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{2}b \end{pmatrix}.$$

In addition, we have $\Theta K = K\Theta$, $\Theta^* = \Theta$ and it's clear that, $K = K^*$. Moreover, we have:

$$\begin{aligned} \int_{\Omega} \langle \Upsilon_{\xi}\Theta^* M, \Upsilon_{\xi}\Theta^* M\rangle_{\alpha} \, d\mu(\xi) &= \int_{\Omega} \langle \Upsilon_{\xi} M, \Upsilon_{\xi} M\rangle_{\alpha} \, d\mu(\xi) \\ &= \int_{\Omega} \det\left(\begin{pmatrix} \xi^2|a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}\right) d\mu(\xi) \\ &= \int_{\Omega} \xi^2|a|^2|b|^2 d\mu(\xi) \\ &= \frac{1}{3}|a|^2|b|^2 \\ &= \frac{1}{3}\langle M, M\rangle. \end{aligned}$$

On the other hand, we have

$$\langle K^*\Theta^*M, K^*\Theta^*M\rangle = \langle K(M), K(M)\rangle = \det\left(\begin{pmatrix} |a|^2 & 0 \\ 0 & \frac{1}{4}|b|^2 \end{pmatrix}\right) = \frac{1}{4}|a|^2|b|^2 \leq \frac{1}{3}|a|^2|b|^2.$$

Hence,

$$\langle K^*\Theta^*M, K^*\Theta^*M\rangle \leq \int_\Omega \langle \Upsilon_\xi\Theta^*M, \Upsilon_\xi\Theta^*M\rangle_\alpha \, d\mu(\xi) \leq 2\,\langle \Theta^*M, \Theta^*M\rangle\,.$$

This show that $\{\Upsilon_\xi\Theta^*\}_{\xi\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$.

Proof. $(i) \Rightarrow (ii)$ The first part of proof is implied by Theorem 2.1.
$(ii) \Rightarrow (i)$ Suppose that for all $f \in \mathcal{H}$,

$$A\,\langle K^*f, K^*f\rangle \leq \int_\Omega \langle \Upsilon_\xi\Theta^*f, \Upsilon_\xi\Theta^*f\rangle \, \mathrm{d}\nu(\xi) \leq B\,\langle f, f\rangle\,.$$

Moreover, for every $h \in \mathcal{H}$,

$$\delta\,\langle K^*h, K^*h\rangle = \int_\Omega \langle \Upsilon_\xi h, \Upsilon_\xi h\rangle \, \mathrm{d}\nu(\xi).$$

Since $\Theta^*K^* = K^*\Theta^*$, we obtain

$$\delta\,\langle \Theta^*K^*f, \Theta^*K^*f\rangle = \int_\Omega \langle \Upsilon_\xi\Theta^*f, \Upsilon_\xi\Theta^*f\rangle \, \mathrm{d}\nu(\xi), \quad f \in \mathcal{H}.$$

Hence

$$\langle \Theta^*K^*f, \Theta^*K^*f\rangle = \delta^{-1}\int_\Omega \langle \Upsilon_\xi\Theta^*f, \Upsilon_\xi\Theta^*f\rangle \, \mathrm{d}\nu(\xi) \geq \delta^{-1}A\,\langle K^*f, K^*f\rangle\,.$$

Since K^* is bounded below, by Lemma **1.5** there exist $\alpha > 0$ such that

$$\langle K^*f, K^*f\rangle \geq \alpha\,\langle f, f\rangle\,.$$

So, we conclude that for every $f \in \mathcal{H}$,

$$\langle \Theta^*K^*f, \Theta^*K^*f\rangle \geq \delta^{-1}\alpha A\,\langle f, f\rangle\,.$$

Consequently $(K\Theta)^* = \Theta^*K^*$ is bounded below, so by Lemma 1.5, $K\Theta$ is surjective, and since K and Θ commute, implies that Θ is surjective. □

Assume that operators $\mathcal{T}, \Theta \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ and $\mathcal{T}^*$ conserve a c-K-g-frame for $\mathcal{R}(\mathcal{T})$. We establish some requirements on K, Θ and $\mathcal{T}$ in the preceding theorem such that Θ^* can also conserve the same c-K-g-frame for $\mathcal{R}(\Theta)$.

Theorem 2.5. *Let $\{\Upsilon_\xi \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_\xi) : \xi \in \Omega\}$ be a c-K-g-frame for $\mathcal{H}$. Assume that $\Theta, \mathcal{T} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ have close ranges, and $\mathcal{N}(\Theta) = \mathcal{N}(\mathcal{T})$ with $\mathcal{R}(K^*) \cap \mathcal{N}(\Theta^*) = \{0\}$ and $K\Theta\mathcal{T}^\dagger = \Theta\mathcal{T}^\dagger K$. If $\{\Upsilon_\xi\mathcal{T}^*\}_{\xi\in\Omega}$ is a c-K-g-frame for $\mathcal{R}(\mathcal{T})$, then $\{\Upsilon_\xi\Theta^*\}_{\xi\in\Omega}$ is a c- K-g-frame for $\mathcal{R}(\Theta)$.*

Proof. Assume that $K\Theta\mathcal{T}^\dagger = \Theta\mathcal{T}^\dagger K$. We define

$$P : \mathcal{R}(\mathcal{T}) \longrightarrow \mathcal{R}(\Theta),$$

by $Pf = \Theta\mathcal{T}^\dagger f$, for all $f \in \mathcal{R}(\mathcal{T})$. Since $\mathcal{N}(\mathcal{T}) = \mathcal{N}(\Theta)$, we have $\mathcal{R}\left(\mathcal{T}^\dagger\right) = \mathcal{R}\left(\Theta^\dagger\right)$. Consequently, by Lemma **1.7**, $\mathcal{N}(P) = \mathcal{N}\left(\Theta\mathcal{T}^\dagger\right) = \mathcal{N}\left(\mathcal{T}\mathcal{T}^\dagger\right) = (\mathcal{R}(\mathcal{T}))^\perp$, that implies

$$\mathcal{N}(P) = \mathcal{N}\left(\Theta\mathcal{T}^\dagger\right) \cap \mathcal{R}(\mathcal{T}) = (\mathcal{R}(\mathcal{T}))^\perp \cap \mathcal{R}(\mathcal{T}) = \{0\}.$$

Hence, P is invertible on $\mathcal{R}(\mathcal{T})$. By Lemma **1.7**,

$$\mathcal{T}^\dagger\mathcal{T} = P_{\mathcal{R}(\mathcal{T}^\dagger)} = P_{\mathcal{R}(\Theta^\dagger)} = \Theta^\dagger\Theta.$$

Furthermore,

$$P\mathcal{T} = \Theta\mathcal{T}^\dagger\mathcal{T} = \Theta\Theta^\dagger\Theta = \Theta. \tag{2.1}$$

Let C, be the lawer frame bound of $\{\Upsilon_\xi\mathcal{T}^*\}_{\xi\in\Omega}$, then for every $f \in \mathcal{R}(\Theta)$, by (2.1), we obtain

$$\begin{aligned}\int_\Omega \langle\Upsilon_\xi\Theta^* f, \Upsilon_\xi\Theta^* f\rangle \,\mathrm{d}\nu(\xi) &= \int_\Omega \langle\Upsilon_\xi\mathcal{T}^*P^* f, \Upsilon_\xi\mathcal{T}^*P^* f\rangle \,\mathrm{d}\nu(\xi)\\ &\geq C\,\langle K^*P^*f, K^*P^*f\rangle\\ &\geq C\left\|(P^*)^{-1}\right\|^{-2}\langle K^*f, K^*f\rangle .\end{aligned}$$

On the other hand, for every $f \in \mathcal{R}(\Theta)$,

$$\int_\Omega \langle\Upsilon_\xi\Theta^* f, \Upsilon_\xi\Theta^* f\rangle \,\mathrm{d}\nu(\xi) = \int_\Omega \langle\Upsilon_\xi\mathcal{T}^*P^* f, \Upsilon_\xi\mathcal{T}^*P^* f\rangle \,\mathrm{d}\nu(\xi) \leq B\,\langle P^*f, P^*f\rangle = B\|P^*\|^2\,\langle f, f\rangle .$$

Which implies that $\{\Upsilon_\xi\Theta^*\}_{\xi\in\Omega}$ is a c-K-g-frame for $\mathcal{R}(\Theta)$. □

Theorem 2.6. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ and $\{\Upsilon_\xi \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_\xi) : \xi \in \Omega\}$ be a c-g-Bessel sequence for $\mathcal{H}$, and $\mathcal{T}_\Upsilon$ be the synthesis operator of $\{\Upsilon_\xi\}_{\xi\in\Omega}$ with $\overline{\mathcal{R}(K^*)}$ is orthogonally complemented. Then the following assertions are equivalent:*

(1) $\mathcal{R}(K) = \mathcal{R}(\mathcal{T}_\Upsilon)$.

(2) There exist two constants $\lambda_1, \lambda_2 > 0$, such that for each $f \in \mathcal{H}$,

$$\frac{1}{\lambda_1}\langle K^*f, K^*f\rangle \leq \int_\Omega \langle\Upsilon_\xi f, \Upsilon_\xi f\rangle \,\mathrm{d}\nu(\xi) \leq \lambda_2\,\langle K^*f, K^*f\rangle . \tag{2.2}$$

(3) $\{\Upsilon_\xi\}_{\xi\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$ and there exists a c-g-Bessel sequence $\{\Phi_\xi\}_{\xi\in\Omega}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$ such that $\Upsilon_\xi = \Phi_\xi K^*$ for each $\xi \in \Omega$.

Proof. (1) Applying Lemma 1.8,

$$KK^* \leq \lambda_1\mathcal{T}_\Upsilon\mathcal{T}_\Upsilon^*, \quad and \quad \mathcal{T}_\Upsilon\mathcal{T}_\Upsilon^* \leq \lambda_2 KK^* \quad for\ some\ \lambda_1, \lambda_2 > 0.$$

Which implies that

$$\frac{1}{\lambda_1}KK^* \leq \mathcal{T}_\Upsilon\mathcal{T}_\Upsilon^* \leq \lambda_2 KK^*.$$

Hence, for every $f \in \mathcal{H}$,

$$\frac{1}{\lambda_1}\langle K^*f, K^*f\rangle \leq \langle\mathcal{T}_\Upsilon^* f, \mathcal{T}_\Upsilon^* f\rangle = \int_\Omega \langle\Upsilon_\xi f, \Upsilon_\xi f\rangle \,\mathrm{d}\nu(\xi) \leq \lambda_2\,\langle K^*f, K^*f\rangle .$$

(2) ⇒ (3) According to the assumed hypothesis, we have $\mathcal{T}_\Upsilon\mathcal{T}_\Upsilon^* \leq \lambda_2 KK^*$. Applying Lemma **1.8**, there exists an operator $\mathcal{Q} \in \mathrm{End}^*_{\mathcal{A}}\left(\bigoplus_{\xi\in\Omega}\mathcal{K}_\xi, \mathcal{H}\right)$ such that $\mathcal{T}_\Upsilon = K\mathcal{Q}$, which implies that $\mathcal{T}_\Upsilon^* = \mathcal{Q}^*K^*$. Now for any $h \in \mathcal{H}$ and for almost all $\xi \in \Omega$, we define:

$$\Phi_\xi h = (\mathcal{Q}^*h)(\xi).$$

Consequently,

$$\{\Upsilon_\xi(h)\}_{\xi\in\Omega} = \{\mathcal{T}_\Upsilon^* h\}_{\xi\in\Omega} = \left\{(\mathcal{Q}^*(K^*h)(\xi)\right\}_{\xi\in\Omega} = \{\Phi_\xi(K^*h)\}_{\xi\in\Omega} .$$

Hence $\Upsilon_\xi = \Phi_\xi K^*$ for almost all $\xi \in \Omega$. Hence for every $h \in \mathcal{H}$, we achieve the intended result by:

$$\int_\Omega \langle\Phi_\xi h, \Phi_\xi h\rangle \,\mathrm{d}\nu(\xi) = \int_\Omega \langle(\mathcal{Q}^*h)(\xi), (\mathcal{Q}^*h)(\xi)\rangle \,\mathrm{d}\nu(\xi) \leq \|\mathcal{Q}^*\|_2^2\,\langle h, h\rangle .$$

(3) $\Rightarrow$ (1) For every $f \in \mathcal{H}$,

$$\frac{1}{\lambda_1}\left\langle K^* f, K^* f\right\rangle \leq \int_{\Omega}\left\langle \Upsilon_{\xi} f, \Upsilon_{\xi} f\right\rangle d\nu\left(\xi\right) = \int_{\Omega}\left\langle \Phi_{\xi} K^* f, \Phi_{\xi} K^* f\right\rangle \mathrm{d}\nu(\xi) \leq \lambda_{\Phi}\left\langle K^* f, K^* f\right\rangle .$$

Where λ_{Φ} the upper bound of $\{\Phi_{\xi}\}_{\xi\in\Omega}$. Hence

$$\frac{1}{\lambda_1} K K^* \leq \mathcal{T}_{\Upsilon}\mathcal{T}_{\Upsilon}^* \leq \lambda_{\Phi} K K^* .$$

Hence

$$\mathcal{R}(K) = \mathcal{R}\left(\mathcal{T}_{\Upsilon}\right).$$

□

Given certain adjointable operators and some c-K-g-frames, the following theorem is used to construct c-K-g-frames in Hilbert C^*-modules.

Theorem 2.7. *Let $\{\Upsilon_{\xi} \in \mathrm{End}_{\mathcal{A}}^*\left(\mathcal{H}, \mathcal{H}_{\xi}\right) : \xi \in \Omega\}$ be a $c - K_1 - g$-frame for $\mathcal{H}$ and $K_1, K_2 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H})$ with $\overline{\mathcal{R}\left(K_1^*\right)}$ is orthogonally complemented.*

(1) If $\{\Upsilon_{\xi} \in \mathrm{End}_{\mathcal{A}}^*\left(\mathcal{H}, \mathcal{H}_{\xi}\right) : \xi \in \Omega\}$ is a $c - K_2 - g$-frame for $\mathcal{H}$, then it is a $c - (K_1 + K_2) - g$-frame for $\mathcal{H}$.

(2) If, in addition, $\{\Upsilon_{\xi} \in \mathrm{End}_{\mathcal{A}}^*\left(\mathcal{H}, \mathcal{H}_{\xi}\right) : \xi \in \Omega\}$ is δ-tight $c - K_1 - g$-frame, then it is a c-$K_2 - g$-frame for $\mathcal{H}$ if and only if $\mathcal{R}\left(K_2\right) \subseteq \mathcal{R}\left(K_1\right)$.

Proof. (1) Assume that $\{\Upsilon_{\xi}\}_{\xi\in\Omega}$ is a c - $K_1 - g$-frame and also $c - K_2 - g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_{\xi}\}_{\xi\in\Omega}$, then there exist $A_1, A_2, B_1, B_2 > 0$ constants such that for every $f \in \mathcal{H}$,

$$A_1\left\langle K_1^* f, K_1^* f\right\rangle \leq \int_{\Omega}\left\langle \Upsilon_{\xi} f, \Upsilon_{\xi} f\right\rangle \mathrm{d}\nu(\xi) \leq B_1\left\langle f, f\right\rangle .$$

And

$$A_2\left\langle K_2^* f, K_2^* f\right\rangle \leq \int_{\Omega}\left\langle \Upsilon_{\xi} f, \Upsilon_{\xi} f\right\rangle \mathrm{d}\nu(\xi) \leq B_2\left\langle f, f\right\rangle .$$

It follow that

$$\left(\frac{A_1}{2}\left\langle K_1^* f, K_1^* f\right\rangle + \frac{A_1}{2}\left\langle K_2^* f, K_2^* f\right\rangle\right) \leq \int_{\Omega}\left\langle \Upsilon_{\xi} f, \Upsilon_{\xi} f\right\rangle \mathrm{d}\nu(\xi) \leq \left(\frac{B_1}{2}\left\langle f, f\right\rangle + \frac{B_2}{2}\left\langle f, f\right\rangle\right).$$

We pose $\lambda_1 = \min\left\{\frac{A_1}{2}, \frac{A_2}{2}\right\}$ and $\lambda_2 = \max\left\{\frac{B_1}{2}, \frac{B_2}{2}\right\}$, since

$$\begin{aligned}\|K_1^* f\|^2 = \|K_1^* f\|^2 = \|(K_1^* + K_2^*) f - K_2^* f\|^2 &= \left\|(K_1 + K_2)^* f - K_2^* f\right\|^2 \\ &\geq \left\|(K_1 + K_2)^* f\right\|^2 - \|K_2^* f\|^2 .\end{aligned}$$

We have

$$\left\|(K_1 + K_2)^* f\right\|^2 \leq \|K_1^* f\|^2 + \|K_2^* f\|^2 .$$

Hence

$$\lambda_1\left\langle (K_1 + K_2)^* f, (K_1 + K_2)^* f\right\rangle \leq \int_{\Omega}\left\langle \Upsilon_{\xi} f, \Upsilon_{\xi} f\right\rangle \mathrm{d}\nu(\xi) \leq \lambda_2\left\langle f, f\right\rangle .$$

(2) For every $f \in \mathcal{H}$,

$$\delta\left\langle K_1^* f, K_1^* f\right\rangle = \int_{\Omega}\left\langle \Upsilon_{\xi} f, \Upsilon_{\xi} f\right\rangle \mathrm{d}\nu(\xi). \tag{2.3}$$

On the other hand, we have $\{\Upsilon_{\xi} \in \mathrm{End}_{\mathcal{A}}^*\left(\mathcal{H}, \mathcal{H}_{\xi}\right) : \xi \in \Omega\}$ is $c - K_2 - g$-frame, then there exists a $A > 0$, such that

$$\int_{\Omega}\left\langle \Upsilon_{\xi} f, \Upsilon_{\xi} f\right\rangle \mathrm{d}\nu(\xi) \geq A\left\langle K_2^* f, K_2^* f\right\rangle . \tag{2.4}$$

Hence, from (2.3) and (2.4),

$$K_2K_2^* \leq \frac{\delta}{A}K_1K_1^*.$$

Applying Lemma **1.8**, we get $\mathcal{R}(K_2) \subseteq \mathcal{R}(K_1)$.
Now to show the opposite inclusion, simply use the lemma **1.8**, there exists $\gamma > 0$, such that $K_2K_2^* \leq \gamma K_1K_1^*$. Hence for each $f \in \mathcal{H}$, we have

$$\langle K_2^*f, K_2^*f\rangle \leq \gamma \langle K_1^*f, K_1^*f\rangle = \frac{\gamma}{\delta}\int_\Omega \langle \Upsilon_\xi f, \Upsilon_\xi f\rangle \, \mathrm{d}\nu(\xi).$$

Therefore,

$$\frac{\delta}{\gamma}\langle K_2^*f, K_2^*f\rangle \leq \int_\Omega \langle \Upsilon_\xi f, \Upsilon_\xi f\rangle \, \mathrm{d}\nu(\xi).$$

This completes the proof. □

3 Sum of c-K-g-frames in Hilbert C^*-modules

In this part, we investigate the sum of these frames under the assumption that $\{\Upsilon_\xi\}_{\xi\in\Omega}$ and $\{\Phi_\xi\}_{\xi\in\Omega}$ are arbitrary.

Theorem 3.1. *Let $K_1, K_2 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H})$ have closed ranges, $\{\Upsilon_\xi\}_{\xi\in\Omega}$ and $\{\Phi_\xi\}_{\xi\in\Omega}$ are $c-K_1-g$ frame and $c-g$ Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$, respectively.*

(i) If $K_1 \geq 0$ and $\{\Phi_\xi\}_{\xi\in\Omega}$ is a $c-K_1-g$ dual for $\{\Upsilon_\xi\}_{\xi\in\Omega}$, then the sequence $\{\Upsilon_\xi + \Phi_\xi\}_{\xi\in\Omega}$ is a c- $K_1 - g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$.

(ii) If $\{\Phi_\xi\}_{\xi\in\Omega}$ is c-$(K_1 + K_2) - g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$ and $\mathcal{T}_\Upsilon\mathcal{T}_\Phi^* = 0$, then $\{\Upsilon_\xi + \Phi_\xi\}_{\xi\in\Omega}$ is a c- $(K_1 + K_2) - g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$.

Proof. (i) For every $f \in \mathcal{H}$, we have

$$\begin{aligned}\langle K_1^*f, h\rangle &= \langle f, K_1h\rangle \\ &= \overline{\langle K_1h, f\rangle}.\end{aligned}$$

Since $\{\Phi_\xi\}_{\xi\in\Omega}$ is a $c - K_1 - g$-dual of $\{\Upsilon_\xi\}_{\xi\in\Omega}$

$$\begin{aligned}\overline{\langle K_1h, f\rangle} &= \overline{\int_\Omega \left\langle \Upsilon_\xi^*\Phi_\xi h, f\right\rangle \mathrm{d}\nu(\xi)} \\ &= \int_\Omega \left\langle \Phi_\xi^*\Upsilon_\xi f, h\right\rangle \mathrm{d}\nu(\xi).\end{aligned}$$

We denote by $S_{\Upsilon+\Phi}$, the $c-g$ frame operator of $\{\Upsilon_\xi + \Phi_\xi\}_{\xi\in\Omega}$. Consequently for every $f, h \in \mathcal{H}$,

$$\begin{aligned}\langle S_{\Upsilon+\Phi}f, h\rangle &= \int_\Omega \left\langle f, (\Upsilon_\xi + \Phi_\xi)^* (\Upsilon_\xi + \Phi_\xi) h\right\rangle \mathrm{d}\nu(\xi) \\ &= \int_\Omega \left\langle (\Upsilon_\xi + \Phi_\xi)^* (\Upsilon_\xi + \Phi_\xi) f, h\right\rangle \mathrm{d}\nu(\xi) \\ &= \int_\Omega \left\langle \Upsilon_\xi^*\Upsilon_\xi f, h\right\rangle \mathrm{d}\nu(\xi) + \int_\Omega \left\langle \Phi_\xi^*\Phi_\xi f, h\right\rangle \mathrm{d}\nu(\xi) \\ &+ \int_\Omega \left\langle \Upsilon_\xi^*\Phi_\xi f, h\right\rangle \mathrm{d}\nu(\xi) + \int_\Omega \left\langle \Phi_\xi^*\Upsilon_\xi f, h\right\rangle \mathrm{d}\nu(\xi) \\ &= \langle S_\Upsilon f, h\rangle + \langle S_\Phi f, h\rangle + \langle K_1 f, h\rangle + \langle K_1^* f, h\rangle .\end{aligned}$$

Therefore

$$\begin{aligned}
\langle S_{\Upsilon+\Phi} f, f\rangle &= \int_{\Omega} \left\langle (\Upsilon_\xi + \Phi_\xi)^* (\Upsilon_\xi + \Phi_\xi) f, f \right\rangle \mathrm{d}\nu(\xi) \\
&\geq \langle S_\Upsilon f, f\rangle \\
&= \int_{\Omega} \langle \Upsilon_\xi f, \Upsilon_\xi f\rangle \mathrm{d}\nu(\xi) \\
&\geq C_\Upsilon \langle K_1^* f, K_1^* f\rangle .
\end{aligned}$$

This proves that $\{\Upsilon_\xi + \Phi_\xi\}_{\xi\in\Omega}$ has the lower frame bound. Now, we prove $\{\Upsilon_\xi + \Phi_\xi\}_{\xi\in\Omega}$ is a c-g-Bessel sequence. For every $f \in \mathcal{H}$,

$$\begin{aligned}
\int_{\Omega} \langle (\Upsilon_\xi + \Phi_\xi) f, (\Upsilon_\xi + \Phi_\xi) f\rangle \mathrm{d}\nu(\xi) &\leq \int_{\Omega} \langle \Upsilon_\xi f, \Upsilon_\xi f\rangle \mathrm{d}\nu(\xi) + \int_{\Omega} \langle \Phi_\xi f, \Phi_\xi f\rangle \mathrm{d}\nu(\xi) \\
&\leq B_1 \langle f, f\rangle + B_2 \langle f, f\rangle = (B_1 + B_2) \langle f, f\rangle .
\end{aligned}$$

(ii) For every $f \in \mathcal{H}$, since $\mathcal{T}_\Upsilon \mathcal{T}_\Phi^* = 0$, we have $\int_\Omega \left\langle \Lambda_\xi^* \Phi_\xi f, f \right\rangle \mathrm{d}\nu(\xi) = 0$ and

$$\begin{aligned}
\int_{\Omega} \langle (\Upsilon_\xi + \Phi_\xi) f, (\Upsilon_\xi + \Phi_\xi) f\rangle \mathrm{d}\nu(\xi) &= \int_{\Omega} \langle \Upsilon_\xi f, \Upsilon_\xi f\rangle \mathrm{d}\nu(\xi) + \int_{\Omega} \langle \Phi_\xi f, \Phi_\xi f\rangle \mathrm{d}\nu(\xi) \\
&\geq A_1 \langle K_1^* f, K_1^* f\rangle + A_2 \langle K_2^* f, K_2^* f\rangle \\
&\geq \lambda \left\langle (K_1 + K_2)^* f, (K_1 + K_2)^* f \right\rangle ,
\end{aligned}$$

where $\lambda = \min \{A_1, A_2\}$. □

Theorem 3.2. *Let* $K_1 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}), K_2 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{K})$ *and* $\Upsilon = \{\Upsilon_\xi\}_{\xi\in\Omega}$ *is a c-K* K_1*-g-frame and* $\{\Phi_\xi\}_{\xi\in\Omega}$ *is a c-g-Bessel sequence for* $\mathcal{H}$*. Assume that* $\Theta_1, \Theta_2 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ *and* $\Theta_1 \mathcal{T}_\Upsilon \mathcal{T}_\Phi^* \Theta_2^* + \Theta_2 \mathcal{T}_\Phi \mathcal{T}_\Upsilon^* \Theta_1^* + \Theta_2 S_\Phi \Theta_2^* \geq 0$*. If* Θ_1 *has closed range with* $\Theta_1 K_1 = K_2 \Theta_1$ *and* $\mathcal{R}(K_2^*) \cap \mathcal{N}(\Theta_1^*) = \{0\}$*, then* $\{\Upsilon_\xi \Theta_1^* + \Phi_\xi \Theta_2^*\}_{\xi\in\Omega}$ *is a c-*K_2*-g-frame for* $\mathcal{K}$ *with respect to* $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$*.*

Proof. Assume that $\{\Upsilon_\xi\}_{\xi\in\Omega}, \{\Phi_\xi\}_{\xi\in\Omega}$ be a $c - K_1 - g$-frame and $c - g$ Bessel sequence for $\mathcal{H}$ with bounds α_1, β_1 and β_2, respectively. For every $g \in \mathcal{K}$,

$$\begin{aligned}
\int_{\Omega} \langle (\Upsilon_\xi \Theta_1^* + \Phi_\xi \Theta_2^*) g, (\Upsilon_\xi \Theta_1^* + \Phi_\xi \Theta_2^*) g\rangle \mathrm{d}\nu(\xi) &= \int_{\Omega} \langle \Upsilon_\xi \Theta_1^* g, \Upsilon_\xi \Theta_1^* g\rangle \mathrm{d}\nu(\xi) + \langle \Theta_2 \mathcal{T}_\Phi \mathcal{T}_\Upsilon^* \Theta_1^* g, g\rangle \\
&\quad + \langle \Theta_1 \mathcal{T}_\Upsilon \mathcal{T}_\Phi^* \Theta_2^* g, g\rangle + \langle \Theta_2 \mathcal{T}_\Phi \mathcal{T}_\Phi^* \Theta_2^* g, g\rangle \\
&= \int_{\Omega} \langle \Upsilon_\xi \Theta_1^* g, \Upsilon_\xi \Theta_1^* g\rangle \mathrm{d}\nu(\xi) + \langle (\Theta_1 \mathcal{T}_\Upsilon \mathcal{T}_\Phi^* \Theta_2^* \\
&\quad + \Theta_2 \mathcal{T}_\Phi \mathcal{T}_\Upsilon^* \Theta_1^* + \Theta_2 S_\Phi \Theta_2^*) g, g\rangle .
\end{aligned}$$

According to the hypotheses, for every $g \in \mathcal{H}$ we get

$$\begin{aligned}
\int_{\Omega} \langle (\Upsilon_\xi \Theta_1^* + \Phi_\xi \Theta_2^*) g, (\Upsilon_\xi \Theta_1^* + \Phi_\xi \Theta_2^*) g\rangle \mathrm{d}\nu(\xi) &\geq \int_{\Omega} \langle \Upsilon_\xi \Theta_1^* g, \Upsilon_\xi \Theta_1^* g\rangle \mathrm{d}\nu(\xi) \\
&\geq \alpha_1 \langle K_1^* \Theta_1^* g, K_1^* \Theta_1^* g\rangle \\
&= \alpha_1 \langle \Theta_1^* K_2^* g, \Theta_1^* K_2^* g\rangle \\
&\geq \alpha_1 \left\| \Theta_1^\dagger \right\|^{-2} \langle K_2^* g, K_2^* g\rangle .
\end{aligned}$$

Hence, for every $g \in \mathcal{K}$,

$$\begin{aligned}
\alpha_1 \left\| \Theta_1^\dagger \right\|^{-2} \langle K_2^* g, K_2^* g\rangle &\leq \int_{\Omega} \langle (\Upsilon_\xi \Theta_1^* + \Phi_\xi \Theta_2^*) g, (\Upsilon_\xi \Theta_1^* + \Phi_\xi \Theta_2^*) g\rangle \mathrm{d}\nu(\xi) \\
&\leq \left(\beta_1 \|\Theta_1^*\|^2 + \beta_2 \|\Theta_2^*\|^2 \right) \langle g, g\rangle .
\end{aligned}$$

□

Theorem 3.3. *Let $K_1 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H})$ be closed range, $\{\Upsilon_\xi\}_{\xi\in\Omega}$ and $\{\Phi_\xi\}_{\xi\in\Omega}$ be c-K K_1-g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$. Assume that $K_2 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{K}), \Theta_1, \Theta_2 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H},\mathcal{K})$ and $\Theta_1 \mathcal{T}_\Upsilon \mathcal{T}_\Phi^* \Theta_2^* + \Theta_2 \mathcal{T}_\Phi \mathcal{T}_\Upsilon^* \Theta_1^* \geq 0$.*

(i) $P = \alpha_1\Theta_1 + \alpha_2\Theta_2, \qquad \mathcal{R}(K_2) \subseteq \mathcal{R}(P), \mathcal{R}(P^*) \subseteq \mathcal{R}(K_1).$

(ii) $\mathcal{Q} = \alpha_1\Theta_1 - \alpha_2\Theta_2, \qquad \mathcal{R}(\mathcal{Q}^*) \subseteq \mathcal{R}(K_1), \mathcal{R}(K_2) \subseteq \mathcal{R}(\mathcal{Q})$ *with* $\overline{\mathcal{R}(\mathcal{Q}^*)}$ *is orthogonally complemented.*

Let $\alpha_1, \alpha_2 > 0$, if one of (i), (ii) holds then, $\{\alpha_1\Upsilon_\xi\Theta_1^* + \alpha_2\Phi_\xi\Theta_2^*\}_{\xi\in\Omega}$ is a c- K_2-g-frame for $\mathcal{K}$ with respect to $\{\mathcal{H}_\xi\}_{\xi\in\Omega}$.

Proof. Let A_1, B_1 and A_2, B_2 be frame bounds of $\{\Upsilon_\xi\}_{\xi\in\Omega}$ and $\{\Phi_\xi\}_{\xi\in\Omega}$, respectively. It is easy to show that, for every $\alpha_1, \alpha_2 > 0$ and $g \in \mathcal{H}$,

$$\int_\Omega \langle (\alpha_1\Upsilon_\xi\Theta_1^* + \alpha_2\Phi_\xi\Theta_2^*)\, g, (\alpha_1\Upsilon_\xi\Theta_1^* + \alpha_2\Phi_\xi\Theta_2^*)\, g\rangle \,\mathrm{d}\nu(\xi) \leq \left(\alpha_1^2 B_1 \|\Theta_1^*\|^2 + \alpha_2^2 B_2 \|\Theta_2^*\|^2\right) \langle g, g\rangle .$$

On the other hand,

$$\begin{aligned}\int_\Omega \langle (\alpha_1\Upsilon_\xi\Theta_1^* + \alpha_2\Phi_\xi\Theta_2^*)\, g, (\alpha_1\Upsilon_\xi\Theta_1^* + \alpha_2\Phi_\xi\Theta_2^*)\, g\rangle \,\mathrm{d}\nu(\xi) &= \alpha_1^2 \int_\Omega \langle \Upsilon_\xi\Theta_1^* g, \Upsilon_\xi\Theta_1^* g\rangle \,\mathrm{d}\nu(\xi)\\ &+ 2\alpha_1\alpha_2 \left\langle \left(\Theta_2\mathcal{T}_\Phi\mathcal{T}_\Upsilon^*\Theta_1^* + \Theta_1\mathcal{T}_\Upsilon\mathcal{T}_\Phi^*\Theta_2^*\right) g\ , g\right\rangle\\ &+ \alpha_2^2 \int_\Omega \langle \Phi_\xi\Theta_2^* g, \Phi_\xi\Theta_2^* g\rangle \,\mathrm{d}\nu(\xi)\\ &\geq \alpha_1^2 A_1 \langle K_1^*\Theta_1^* g, \rangle + \alpha_2^2 A_2 \langle K_1^*\Theta_2^* g, K_1^*\Theta_2^* g\rangle .\end{aligned}$$

Assume condition (ii) is true. we pose

$$\lambda = \min\{A_1, A_2\},$$

According to the parallelogram law, for every $g \in \mathcal{H}_2$,

$$\begin{aligned}\alpha_1^2 A_1 \langle K_1^*\Theta_1^* g, K_1^*\Theta_1^* g\rangle + \alpha_2^2 A_2 \langle K_1^*\Theta_2^* g, K_1^*\Theta_2^* g\rangle &\geq \lambda \left(\langle \alpha_1 K_1^*\Theta_1^* g, \alpha_1 K_1^*\Theta_1^* g\rangle + \langle \alpha_2 K_1^*\Theta_2^* g, \alpha_2 K_1^*\Theta_2^* g\rangle\right)\\ &= \frac{\lambda}{2}\left(\left\langle K_1^*\left(\alpha_1\Theta_1 + \alpha_2\Theta_2\right)^* g, K_1^*\left(\alpha_1\Theta_1 + \alpha_2\Theta_2\right)^* g\right\rangle\right.\\ &\left.+ \left\langle K_1^*\left(\alpha_1\Theta_1 - \alpha_2\Theta_2\right)^* g, K_1^*\left(\alpha_1\Theta_1 - \alpha_2\Theta_2\right)^* g\right\rangle\right)\\ &\geq \frac{\lambda}{2}\langle K_1^*\mathcal{Q}^* g, K_1^*\mathcal{Q}^* g\rangle\\ &\geq \frac{\lambda}{2}\left\|K_1^\dagger\right\|^{-2}\langle \mathcal{Q}^* g, \mathcal{Q}^* g\rangle .\end{aligned}$$

Since $\mathcal{R}(\mathcal{Q}) \supseteq \mathcal{R}(K_2)$, consequently applying Lemma **1.8**, there exists $\alpha > 0$ such that

$$K_2K_2^* \leq \alpha\mathcal{Q}\mathcal{Q}^*.$$

Hence for $g \in \mathcal{K}$,

$$\langle \mathcal{Q}^* g, \mathcal{Q}^* g\rangle \geq \alpha^{-1}\langle K_2^* g, K_2^* g\rangle .$$

Consequently, for every $g \in \mathcal{H}_2$,

$$\begin{aligned}\frac{\lambda}{2}\alpha^{-1}\left\|K_1^\dagger\right\|^{-2}\langle K_2^* g, K_2^* g\rangle &\leq \int_\Omega \langle (\alpha_1\Upsilon_\xi\Theta_1^* + \alpha_2\Phi_\xi\Theta_2^*)\, g, (\alpha_1\Upsilon_\xi\Theta_1^* + \alpha_2\Phi_\xi\Theta_2^*)\, g\rangle \,\mathrm{d}\nu(\xi)\\ &\leq \left(\alpha_1^2 B_1 \|\Theta_1^*\|^2 + \alpha_2^2 B_2 \|\Theta_2^*\|^2\right)\langle g, g\rangle .\end{aligned}$$

□

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Author information

A. Karara, Department of Mathematics, Faculty of Sciences, University of Ibn Tofail, Kenitra, Morocco.
E-mail: abdelilah.karara@uit.ac.ma

M. Rossafi, Department of Mathematics Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, Fez, Morocco.
E-mail: rossafimohamed@gmail.com

M. Klilou, Department of Mathematics Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, Fez, Morocco.
E-mail: mohammed.klilou@usmba.ac.ma

S. Kabbaj, Department of Mathematics, Faculty of Sciences, University of Ibn Tofail, Kenitra, Morocco.
E-mail: samkabbaj@yahoo.fr

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