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EQUILIBRIA FOR SET-VALUED MAPS

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Abstract This paper illustrates a simple and direct method for establishing equilibrium and co-equilibrium theorems for set-valued maps based on fundamental fixed point and coincidence theorems for Ky Fan or Kakutani types maps. Recall that an *equilibrium* for a set-valued map $\Phi : E \supseteq X \rightrightarrows E$ (E a vector space) is an element $\bar{x} \in X$ such that $0_E \in \Phi(\bar{x})$; while, if E is a Hilbert space, a *co-equilibrium for* Φ is a point $\bar{x} \in X$ of normality, that is, $\Phi(\bar{x}) \cap N_X(\bar{x}) \neq \emptyset$ where $N_X(x)$ is the normal cone to X at $x \in X$ (understood in an appropriate sense). A co-equilibrium for a set-valued map Φ amounts to a solution of a variational inequality constrained by Φ. Our main results include a new simple proof of the Ky Fan-Halpern equilibrium theorem (see [F, H]) and an extension of that equilibrium result for an $\mathbf{H}^*$ map (upper hemicontinuous map with non-empty closed convex values) defined on a paracompact closed (instead of compact) convex set X in a locally convex topological vector space E subject to a partial boundary tangency condition on a compact subset of X. For non-convex domains, we discuss an equilibrium theorem in [BK] for an $\mathbf{H}^*$ map defined on a compact lipschitzian neighbourhood retract with non-trivial Euler characteristic. Finally, assuming that E is a Hilbert space, we show the existence of a co-equilibrium for a compact $\mathbf{H}^*$ map $\Phi : E \supseteq X \rightrightarrows E$ defined on a closed sleek set $X \subset E$ (which may not be convex) given that the pair (X, E) has the equilibrium property for tangential $\mathbf{H}^*$ maps.

1 Introduction

We briefly describe the matters under consideration as well as the concepts used in this work.

Set-valued maps (simply called *maps*) are denoted by capital Greek letters and double arrows $\rightrightarrows$. The *inverse* of a map $\Phi : X \rightrightarrows Y$ is $\Phi^{-1} : Y \rightrightarrows X$ with $x \in \Phi^{-1}(y) \Leftrightarrow y \in \Phi(x)$.

Topological spaces and topological vector spaces are assumed to be Hausdorff. Vector spaces are assumed to be real for simplicity. $\mathcal{N}(0_E)$ denotes a neighborhood basis of suitable open neighborhoods of the origin in a topological vetor space E.

Recall:

- a *fixed point* for a map $\Phi : X \rightrightarrows Y$ with $X \subseteq Y$ is an element $x_0 \in X$ with $x_0 \in \Phi(x_0)$.
- An *equilibrium (point)* for a map $\Phi : X \rightrightarrows E$, X being a non-empty set and E a real vector space, is an element $x_0 \in X$ with $0_E \in \Phi(x_0)$. When $X \subseteq E$, an equilibrium point for Φ is a fixed point $x_0 \in \{x_0\} - \Phi(x_0)$ for the field $I - \Phi$ (I, the identity mapping on E).
- A *coincidence (point)* between two maps $\Phi, \Psi : X \rightrightarrows Y$ is an element $x_0 \in X$ with $\Phi(x_0) \cap \Psi(x_0) \neq \emptyset$. (A fixed point or an equilibrium point is a particular instance of a coincidence point.)

The celebrated *Brouwer's theorem*[1] highlights the essential roles of the topology/geometry of the domain and the regularity of the mapping for the existence of a fixed point. In section 2, we describe the types of maps and domains under consideration in this work together with their fundamental fixed point properties. These include a Schauder-Tychonoff type theorem for so-called

[1] A continuous mapping $f : X \longrightarrow X$ of a non-empty convex compact subset X in a finite dimensional Euclidean space E has a fixed point $x_0 = f(x_0)$.

Ky Fan maps that plays a central role in the proof of the main equilibrium theorem of section 3. We also prove a general coincidence principle between so-called *Ky Fan maps* and *Kakutani maps* defined on convex domains with a weakened compactness condition in section 2. This principle contains the classical *Ky Fan-Browder fixed point theorem* as well as the *Kakutani's fixed point theorem*[2].

As illustrated by the *Intermediate Value Theorem* (IVT) of B. Bolzano (see [B1]), the existence of a *zero* (an *equilibrium,* in the terminology of dynamical systems) $0 = f(x_0)$ for a function $f : X := [a, b] \longrightarrow \mathbb{R}$ (a, b extended reals) follows from (a) the continuity of the mapping f, and (b) the sign condition $f(a)f(b) \leq 0$. From a geometric perspective, this sign condition can be phrased as a *tangency condition* $f(a) \in [0, +\infty) = T_X(a)$ and $f(b) \in (-\infty, 0] = T_X(b)$, where, for any $x \in \partial X, T_X(x) := cl(\bigcup_{t>0} \frac{1}{t}(X - x))$ is the tangent cone (of convex analysis) to X at the boundary point x. Such geometric hypotheses are required in order to garantee the solvability of a nonlinear inclusion $0 \in \Phi(x_0)$ in the case of a multivalued map $\Phi : X \rightrightarrows E$ taking values in a topological vector space E of arbitrary dimension (E may be equipped with a locally convex structure, a norm, or an inner product as appropriate). In the case of a convex domain X, tangency is expressed in terms of the above tangent cone $T_X(x)$ of convex analysis. For non-convex and non-smooth domains, adapted tangency conditions are required as described in section 2.

The main equilibrium and co-equilibrium results of this work are discussed in section 3. A simpler proof of the Ky Fan-Halpern equilibrium result (Theorem 2 in [H]) is provided in Theorem 3.6. The first main result (Theorem 3.11) extends the Ky-Fan-Halpern equilibrium theorem by weakening the compactness of the convex domain X to paracompactness, and the tangency condition on the whole boundary by partial tangency over a compact subset of the boundary. The second main result (Theorem 3.20) describes a generic way to derive the existence of a co-equilibrium from that of an equilibrium for the class of $\mathbf{H}^*$ maps. Special cases include co-equilibrium results on convex and non-convex domains.

2 Concepts and Preparatory Results

We briefly describe now the concepts and results from set-valued and non-smooth analyses about maps and domains required for this work.

2.1 Continuity Concepts for Set-Valued Maps

Recall that a set-valued map of topological spaces $\Psi : X \rightrightarrows Y$ is *upper semicontinuous (usc)* at a given point $x \in X$ whenever the upper inverse $\Psi_+^{-1}(V) := \{x' \in X : \Psi(x') \subset V\}$ of any open neighborhood V of $\Psi(x)$ in Y, is open in X. The map Ψ is *usc* on X if, for any open subset V of Y, the upper inverse $\Psi_+^{-1}(V) := \{x \in X : \Psi(x) \subset V\}$ is open in X.

The *lower semicontinuity (lsc)* of Ψ at a point $x \in X$ amounts to the openess of the lower inverse $\Psi_-^{-1}(V) := \{x' \in X : \Psi(x') \cap V \neq \emptyset\}$ of any open subset V of Y such that $\Psi(x) \cap V \neq \emptyset$. The map Ψ is *lsc* on X if for any open subset V of Y the lower inverse $\Psi_-^{-1}(V) := \{x \in X : \Psi(x) \cap V \neq \emptyset\}$ is open in X.

It is quite clear that if $\Psi(x) = f(x), x \in X$, is a single-valued map, upper (or lower) semicontinuity on X amounts to continuity. Both concepts are natural adaptations of the characterization of continuity for single-valued mappings in terms of open sets[3].

These semicontinuity concepts for maps are not to be confused with semicontinuity for real functions as used below[4].

Two remarks are in order.

Remark 2.1. (1) The graph of a usc map $\Psi : X \rightrightarrows Y$ with closed domain and closed values is closed in the product $X \times Y$. Conversely, a map $\Psi : X \rightrightarrows Y$ with closed graph is usc if Y is compact (see [B1]).

[2] The latter being a set-valued version of the Brouwer's fixed point theorem. Both results are fundamental in Mathematical Economics, Game Theory and Optimization.

[3] However, only lower semicontinuity is the set-valued counterpart of the sequential characterization of classical continuity.

[4] An extended real valued function $f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ of a topological space X is *upper* (respectively *lower*) *semicontinuous* if for every $\lambda \in \mathbb{R} \cup \{\pm\infty\}$, the sub-level set $\{x \in X : f(x) < \lambda\}$ (super-level set $\{x \in X : f(x) > \lambda\}$, respectively) is open in X.

(2) Let X be a topological space and E a topological vector space with continuous dual E' and let $\Psi : X \rightrightarrows E$ be a usc map with non-empty closed convex values. Given any $p \in E'$ and any $\lambda \in \mathbb{R}$, let $x_0 \in O := \{x \in X : \sigma_{\Psi(x)}(p) < \lambda\}$. Then, for some $\lambda' \in \mathbb{R}$ with $\sigma_{\Psi(x_0)}(p) < \lambda' < \lambda$, we have $\Psi(x_0) \subset V := \{y \in E : \langle p, y\rangle < \lambda'\}$ an open set in E. By upper semicontinuity, there exists an open neighborgood U of x_0 in X such that $\Psi(x) \subset V$ for all $x \in U$, that is $\sigma_{\Psi(x_0)}(p) \leq \lambda' < \lambda \Leftrightarrow U \subset O$. We have established that for any $p \in E'$ and any $\lambda \in \mathbb{R}$ the sublevel set $\{x \in X : \sigma_{\Psi(x)}(p) < \lambda\}$ is open in X. This corresponds to the upper semicontinuity of the extended real function $x \longmapsto \sigma_{\Psi(x)}(p)$ with $p \in E'$ fixed.

Remark 2.1 (2) motivates a more general continuity concept[5] for a map $\Psi : X \rightrightarrows E$ on a topological space X with non-empty closed convex values in a topological vector space E with continuous dual E'.

Definition 2.2. Ψ is *upper hemicontinuous (uhc) on* X if for each $p \in E'$, the support functional $x \mapsto \sigma_{\Psi(x)}(p) = \sup_{y\in\Psi(x)}\langle p, y\rangle$ is upper semicontinuous as an extended real-valued function on X, i.e., $\forall\lambda \in \mathbb{R} \cup \{\infty\}$, the set $\{x \in X : \sigma_{\Psi(x)}(p) < \lambda\}$ is open in X.

Clearly, in the right context, every usc map is uhc. For the converse, it should be noted that (i) a uhc map with closed convex values has closed graph, and (ii) a closed graph locally compact map is usc (see e.g., [B1] and references there).

We recall two properties of usc maps of importance in this work.

Proposition 2.3. *(1) A usc map $\Psi : X \rightrightarrows Y$ with compact values transforms compact spaces into compact spaces.*

(2) Let $\Psi : X \rightrightarrows Y$ be a usc map from a paracompact subset X of a topological vector space E into a subset Y of a topological vector space F. Then, for any pair of open neighborhoods $(U, V) \in \mathcal{N}(0_E) \times \mathcal{N}(0_F)$, there exists a map $\hat{\Psi} = \hat{\Psi}_{U,V} : X \rightrightarrows Y$ whose graph is open such that:

$$\Psi(x) \subseteq \hat{\Psi}(x) \subseteq (\Psi((x + U) \cap X) + V) \cap Y, \forall x \in X.$$

If in addition Ψ is convex valued and Y is convex in F, a locally convex space, then $\hat{\Psi}$ is convex valued as well.

Proof. To prove (1) assume the topological space X compact and let $\nu := \{V_i : i \in I\}$ be an open cover of $\Psi(X)$. For each $x \in X$, $\Psi(x)$ is covered by a finite collection $\{V_i : i \in I(x) \subset I\}$. By upper semicontinuity, there exists an open neighborhood U_x of x in X with $\Psi(U_x) \subset O_x = \bigcup_{i\in I(x)} V_i$. Being compact, $X = \bigcup_{j=1}^n U_{x_j}$ and hence, $\Psi(X) = \Psi(\bigcup_{j=1}^n U_{x_j}) \subset \bigcup_{j=1}^n O_{x_j} = \bigcup_{j=1}^n \bigcup_{i\in I(x_j)} V_i$, a finite union of members of ν.

To establish (2) let $(U, V) \in \mathcal{N}(0_E) \times \mathcal{N}(0_F)$ be given. By upper semicontinuity of Ψ, for each $x \in X$, there is an open neighborhood $U_x \in \mathcal{N}(0_E)$ contained in U such that $\Psi((x + U_x) \cap X) \subset (\Psi(x) + V) \cap Y$. Let $\{O_i\}_{i\in I}$ be a point-finite open refinement of the cover $\{x + U_x\}_{x\in X}$, i.e., for every $i \in I$, $O_i \subset x_i + U_{x_i}$ for some $x_i \in X$ and the set $I(x) := \{i \in I : x \in O_i\}$ is finite for each $x \in X$. Define the map $\hat{\Psi} = \hat{\Psi}_{U,V} : X \rightrightarrows Y$ by putting:

$$\hat{\Psi}(x) := \bigcap\nolimits_{i\in I(x)} (\Psi(x_i) + V) \cap Y, x \in X.$$

Clearly, $\Psi(x) \subseteq \hat{\Psi}(x)$ for all $x \in X$. Moreover, for every $x \in X$ and every $i \in I(x)$ we have $\hat{\Psi}(x) \subseteq \Psi(x_i) + V \subseteq \Psi(x + U) + V$. If $x' \in \bigcap_{i\in I(x)}(x_i + U_{x_i})$ then $I(x) \subseteq I(x')$; consequently, $\hat{\Psi}(x) \subseteq \hat{\Psi}(\bigcap_{i\in I(x)}(x_i + U_{x_i})) \subseteq \hat{\Psi}(x)$. Finally, for any given $x \in X$, the open set $\bigcap_{i\in I(x)} O_i \times \hat{\Psi}(x)$ is an open set around $\{x\} \times \hat{\Psi}(x)$ in $X \times Y$, i.e., $\hat{\Psi}$ has an open graph.

Finally, if for each $x \in X$, $\Psi(x)$ is a a non-empty convex subset of Y, then $\hat{\Psi}(x) := \bigcap_{i\in I(x)}(\Psi(x_i) + V) \cap Y$ is a finite intersection of convex open sets in Y. □

In this work, we are focused on two classes of maps that generalize the so-called *Ky Fan* or *Kakutani* maps important in fixed point theory and its applications.

[5] Due, as far as we can tell, to B. Cornet [C].

Ky Fan Maps and Their Fixed Points

The class of $\mathbf{\Phi}^*$ *maps* introduced in [BDG 2] contains the so-called *Ky Fan maps* important in convex analysis.

Definition 2.4. Let X be a topological space and Y a subset of a real vector space. A map $\Phi : X \rightrightarrows Y$ is said to be of class $\mathbf{\Phi}^*$ whenever:

(i) there exists a map $\tilde{\Phi} : X \rightrightarrows Y$ with $\emptyset \neq \tilde{\Phi}(x) \subseteq \Phi(x)$ for every $x \in X$;

(ii) for every $y \in Y$, the set $\tilde{\Phi}^{-1}(y)$ is open in X;

(iii) for every $x \in X$, the set $\Phi(x)$ is convex in Y.

We denote by $\mathbf{\Phi}^*(X, Y)$ the class of $\mathbf{\Phi}^*$ maps from X into Y.

The class of maps whose inverses are $\mathbf{\Phi}^*$ is denoted by $\mathbf{\Phi}(Y, X) := \{\Phi : Y \rightrightarrows X : \Phi^{-1} \in \mathbf{\Phi}^*(X, Y)\}$. $\mathbf{\Phi}^*(X, X)$ is denoted $\mathbf{\Phi}^*(X)$.

Remark 2.5. (1) The map $\tilde{\Phi}$ in Definition 2.4 above is said to be an admissible selection of Φ.

(2) In case $\tilde{\Phi} = \Phi$, the class $\mathbf{\Phi}^*$ becomes the class $\mathbf{F}^*$ of better known and so-called Ky Fan maps. It has been noted that any map Φ of class $\mathbf{\Phi}^*$ admits an admissible selection of class $\mathbf{F}^*$ (see e.g., [B2]). The advantage for considering $\mathbf{\Phi}^*$ maps lies in the decoupling of the (topological) regularity property (ii) from the (algebraic/geometric) convexity property (iii).

(3) The very stringent regularity condition (ii) in Definition 2.4 (the inverse under $\tilde{\Phi}$ of any subset of Y is open in X) is a strong form of lower semicontinuity for $\tilde{\Phi}$.

$\mathbf{\Phi}^*$ maps are remarkable in that they admit continuous single-valued selections (of finite type) on paracompact (respectively, compact) domains.

Proposition 2.6. *[BDG2] Let $\Phi \in \mathbf{\Phi}^*(X, Y)$ where X is a paracompact topological space and Y is a convex subset of a topological vector space. Then Φ has a single-valued continuous selection, i.e., there exists a continuous mapping $s : X \longrightarrow Y$ with $s(x) \in \Phi(x)$ for all $x \in X$.*

If X is compact, the continuous selection s has values in a finite dimensional convex polytope in Y, i.e., there exists $\{y_1, \cdots, y_n\} \subset Y$ such that $s(X) \subset conv\{y_1, \cdots, y_n\} \subset Y$.

Proof. For each $x \in X$, there exists $y \in \tilde{\Phi}(x)$, that is, $x \in \tilde{\Phi}^{-1}(y)$. Hence, the collection of open sets $\omega := \{\tilde{\Phi}^{-1}(y) : y \in Y\}$ covers X. Let $\mathcal{O} := \{O_i : i \in I\}$ be a locally finite open refinement of ω and let $\{\lambda_i : i \in I\}$ be a continuous partition of unity subordinated to $\mathcal{O}$. For each $x \in X$, the set of essential indices $I(x) := \{i \in I : \lambda_i(x) \neq 0\}$ is finite. Moreover, $i \in I(x) \Longrightarrow x \in O_i \subset \tilde{\Phi}^{-1}(y_i)$ for some $y_i \in Y$, hence $y_i \in \tilde{\Phi}(x) \subseteq \Phi(x)$. The finite set $\{y_i : i \in I(x)\}$ together with its convex hull is contained in $\Phi(x)$ as the latter is convex.

Define a continuous mapping $s : X \longrightarrow Y$ by:

$$s(x) := \sum_{i \in I} \lambda_i(x) y_i = \sum_{i \in I(x)} \lambda_i(x) y_i, \text{ for all } x \in X.$$

Since $s(x)$ is a convex combination of $\{y_i : i \in I(x)\}$, it follows that $s(x) \in \Phi(x)$.

If X is compact, ω admits a finite subcover $\{\tilde{\Phi}^{-1}(y_i) : i = 1, \ldots, n\}$ and $s(x) = \sum_{i=1}^n \lambda_i(x) y_i \in \Phi(x) \cap conv\{y_1, \ldots, y_n\} \subset Y$. □

Remark 2.7. It should be noted that: (i) $s(X) \subseteq conv\{\tilde{\Phi}(X)\} \subseteq Y$. (ii) For every $x \in X$, $s(x) \in C(x) := conv\{y_i : i \in I(x)\}$, a finite convex polytope. Moreover, $C(x) \cap \tilde{\Phi}(x)$ contains all the vertices $\{y_i : i \in I(x)\}$ of $C(x)$. It is easily verified that the map $C : X \rightrightarrows Y$ is lower semicontinuous on X and has finite convex polytope (thus, compact) values. Hence, a $\mathbf{\Phi}^*$ map on a paracompact domain admits a lower semicontinuous selection with compact convex values of finite type. This is a way to relate the selectionability of $\mathbf{\Phi}^*$ maps to the celebrated Michael selection theorem[6].

For the sake of completeness, the paper briefly discusses the fixed point property for $\mathbf{\Phi}^*$ maps including (i) a *Schauder-Tychonoff type fixed point theorem*[7] under the additional hypothesis of

[6]The *Michael selection theorem* states that an *lsc* map on a Hausdorff topological space X with closed convex values in a Banach space admits a continuous selection if and only if X is paracompact [Mi].

[7]The *Schauder's fixed point theorem* extends the Brouwer's fixed point theorem to a continuous mapping $f : X \longrightarrow X$ where X is a convex subset in a normed space E of any dimension (finite or infinite) and by replacing the compactness of X by that of the mapping f, that is $f(X) \subseteq K$ compact $\subseteq X$; a significant improvement! Tychonoff extended this fixed point property to any locally convex topological vector space E (see e.g., [DG]).

a local convexity structure on the underlying topological vector space; and (ii) compactness conditions involving both the domain and the map weaker than the compactness of the domain.

Theorem 2.8. *Let $\Phi \in \mathbf{\Phi}^*(X)$ where X is a non-empty convex subset in a topological vector space E. Assume that any one of the following three conditions holds:*

(1) X is compact; or

(2) Φ is a compact map, that is, $\Phi(X) \subseteq K$ a compact subset of X, and E is locally convex; or

(3) There exist a compact subset K of X and a compact convex subset C of X such that the admissible selection $\tilde{\Phi}$ of Φ satisfies the condition

$$(\kappa) \qquad \forall x \in X \setminus K, \tilde{\Phi}(x) \cap C \neq \emptyset.$$

Then Φ has a fixed point.

Proof. If X is compact and convex, Proposition 2.6 establishes the existence of a continuous single-valued selection $s(x) \in \Phi(x) \cap C, \forall x \in X$, with values in a finite convex polyhedron $C := conv\{y_i : i = 1, \ldots, n\}$ contained in a finite dimensional section of X. The Brouwer's fixed point theorem applied to the restriction $s|_C : C \longrightarrow C$ provides the existence of the desired fixed point $x_0 = s(x_0) \in \Phi(x_0)$ under hypothesis (1).

To prove (2), let Δ^n be the standard $n-$simplex in $\mathbb{R}^{n+1}$ and let $K^{n+1} := K \times \cdots \times K$ be the $(n+1)-$times cartesian product of the compact subset K of X for any given non-negative integer n. Consider the continuous mapping $\varphi_n : \Delta^n \times K^{n+1} \longrightarrow X \subseteq E$ defined by the convex combination

$$\varphi_n((\alpha_i)_{i=0}^n, (y_i)_{i=0}^n) = \sum_{i=0}^{n} \alpha_i y_i.$$

The set $C_n := \varphi_n(\Delta^n \times K^{n+1})$ is a compact convex subset of X by virtue of being the continuous image of a compact set. The countable union of compact sets $C := \bigcup_{n=0}^{\infty} C_n$ is contained in X and is precisely the convex hull of the compact set K. As the set C is a regular and $\sigma-$compact topological space, it has the Lindelöf property[8]. It follows that C is paracompact (see Theorem 5.1.2 in Engelking [E]). The set X being convex, contains C. By Proposition 2.6 the restriction of Φ to C - itself a $\mathbf{\Phi}^*$ map - admits a continuous selection $s : C \longrightarrow C$ with $s(C) \subseteq \Phi(C) \subseteq K$ compact $\subseteq C$. The Schauder-Tychonoff fixed point theorem (see [DG]) garantees the existence of a fixed point $x_0 = s(x_0) \in \Phi(x_0) \subseteq K$.

Finally, assuming that (3) holds, consider a partial continuous selection $s : K \longrightarrow X$ of the restriction $\Phi|_K : K \rightrightarrows X$ taking values in a finite dimensional convex polytope $C_1 := conv\{y_1, \ldots, y_n\}$ where $\{y_1, \ldots, y_n\} \subseteq X$ (provided by Proposition 2.6). Importantly, Remark 2.7 (ii) points out that the admissible selection $\tilde{\Phi}(x)$ of $\Phi(x)$ intersects C_1 for all $x \in K$; (indeed $\tilde{\Phi}(x) \cap C_1$ contains all elements $y_i, i \in \{1, \ldots, n\}$ such that $x \in \tilde{\Phi}^{-1}(y_i)$). The convex hull $\hat{C} := conv\{C \cup C_1\}$ is a compact convex subset of X [9]. The "restriction-compression" map $\hat{\Phi} : \hat{C} \rightrightarrows \hat{C}$ defined by $\hat{\Phi}(x) := \Phi(x) \cap \hat{C}$ for all $x \in \hat{C}$ is a $\mathbf{\Phi}^*$ map with admissible selection $\tilde{\hat{\Phi}}(x) = \tilde{\Phi}(x) \cap \hat{C}$. Indeed:

- $\tilde{\hat{\Phi}}(x)$ is a convex subset of $\hat{C}$ for all $x \in \hat{C}$;
- $\emptyset \neq \tilde{\Phi}(x) \cap C_1 \subseteq \tilde{\Phi}(x) \cap \hat{C}$ for all $x \in K \cap \hat{C}$ and $\emptyset \neq \tilde{\Phi}(x) \cap C \subseteq \tilde{\Phi}(x) \cap \hat{C}$ for all $x \in \hat{C} \setminus K$ by virtue of condition (κ);
- $\tilde{\hat{\Phi}}^{-1}(y) = \tilde{\Phi}^{-1}(y) \cap \hat{C}$ an open subset of $\hat{C}$.

The map $\hat{\Phi} : \hat{C} \rightrightarrows \hat{C}$ thus satisfies hypothesis (1) of this theorem. The fixed point of $\hat{\Phi}$ is a fixed point for Φ. □

[8] The space C is T_3 and every open cover has a countable subcover.

[9] It is well established that the convex hull of a finite union of convex compact subsets of a topological vector space is compact).

Remark 2.9. (i) Under hypothesis 1, we have the generalization to $\mathbf{\Phi}^*$ maps of the celebrated Ky Fan-Browder fixed point theorem for $\mathbf{F}^*$ maps of a compact convex domain (see [Br, BDG1]).

(ii) Pending the non-equivocal affirmation of the Schauder's conjecture's resolution, dropping the local convexity in Theorem 2.8 with hypothesis (2) remains an open problem. Interestingly, the author has established in [B2] that the local convexity of the underlying topological linear space E can be dropped for any compact composition $\Phi = \Phi_n \circ \cdots \circ \Phi_1$ of of two or more $\mathbf{\Phi}^*$ maps, that is if $n \geq 2$. The problem is open for the case of one map, $n = 1$.

(iii) Hypothesis (3) in Theorem 2.8 is most interesting. It can be seen as a "coercivity" condition imposing some control on the mapping Φ outside of a compact subset of the possibly unbounded convex domain X.

Kakutani Maps and their Fixed Points

Given a topological space X and a convex set Y in a topological vector space, the class of *Kakutani maps* is defined as:

$$\mathbf{K}^*(X,Y) := \{\Psi : X \rightrightarrows Y : \Psi \text{ is usc on } X \text{ and } \emptyset \neq \Psi(x) \text{ is convex}, \forall x \in X\}.$$

The class of maps whose inverses are of class $\mathbf{K}^*$ is $\mathbf{K}(Y,X) := \{\Psi : Y \rightrightarrows X : \Psi^{-1} \in \mathbf{K}^*(X,Y)\}$. $\mathbf{K}^*(X,X)$ is denoted $\mathbf{K}^*(X)$. Unlike $\mathbf{\Phi}^*$ or Ky Fan maps, Kakutani maps do not necessarily have single-valued continuous selections[10]. Rather, on a paracompact domain, a Kakutani map admit a single-valued continuous graph approximation. This remarkable property[11] can be proven directly (using a partition of unity argument as in Proposition 2.6 above) of be simply derived as an immediate consequence of Propositions 2.3 and 2.6 put together as we now show.

Indeed, assume that X, Y are subsets of topological vector spaces E, F respectively. For any given pair of open neighborhoods $(U,V) \in \mathcal{N}(0_E) \times \mathcal{N}(0_F)$ the majorant $\hat{\Psi} = \hat{\Psi}_{U,V}$ of Ψ provided by Proposition 2.3 has open pre-images (as it has an open graph). Any of its single-valued selections (if it exists) is a so-called (U,V)*-approximative selection* of Ψ. If $\Psi \in \mathbf{K}^*(X,Y)$ is a Kakutani map, X is paracompact, and the space F is locally convex, then the majorant $\hat{\Psi}$ also has convex values (indeed, the neighborhood of the origin V in F can be considered to be convex, so that the neighborhood $(\Psi(x_i) + V) \cap Y$ of the convex set $\Psi(x_i)$ remains a convex set in Y, for every $i \in I(x)$. Thus the map $\hat{\Psi}$ has non-empty convex values and open pre-images, i.e., is a Ky Fan map defined on a paracompact domain. By Proposition 2.6, $\hat{\Psi}$ admits a continuous selection which is a continuous (U,V)-approximative selection of Ψ, that is a continuous function $s_{U,V} : X \longrightarrow Y$ verifying

$$s_{U,V}(x) \in \Psi((x+U) \cap X) + V) \cap Y, \forall x \in X.$$

We have thus linked in a very simple manner the selectionability of Ky Fan maps with the approachability of Kakutani maps as summarized in the next result.

Proposition 2.10. *If $\Psi \in \mathbf{K}^*(X,Y)$ with X a paracompact subset in a topological vector space E and Y a convex subset in a locally convex space F, then Ψ has continuous (U,V)-approximative selections for any pair U, V of open neighborhoods of the origins in E, F respectively.*

This approximation property can be used to reduce the fixed point problem for a Kakutani compact map with closed values to the Schauder fixed point theorem for continuous approximative selections in order to ascertain the existence of a net of *almost fixed points* for the map in question. The compactness of the map together with the closedness of its graph (see Remark 2.1 (1) above) and of its values would then imply the convergence of the net to a fixed point for the map. We opt to provide here a more original proof of the *Kakutani-Ky Fan-Himmelberg* fixed point theorem as an immediate consequence of the fixed point property for $\mathbf{\Phi}^*$ maps. To do this, we need an argument to pass from *almost fixed points* to fixed points.

[10] The simplest Kakutani map $\Phi : [0,1] \rightrightarrows [0,1]$ given by $\Phi(x) = \{0\}$ if $0 \leq x < 1/2$, $\Phi(x) = [0,1]$ if $x = 1/2$, and $\Phi(x) = \{1\}$ if $1/2 < x \leq 1$ has no continuous selection.

[11] This approximation property was established explicitly by A. Cellina in 1969; though the argument was contained in an early proof by J. von Neumann of his celebrated minimax theorem. See [B1, DG].

Definition 2.11. Given a cover ω of a topological space X, an $\omega-$*fixed point* for a map $\Psi : X \rightrightarrows X$ is a point $x_\omega \in X$ such that there exists a member $W \in \omega$ with $x_\omega \in W$ and $W \cap \Psi(x_\omega) \neq \emptyset$.

Given a subspace K of a topological space X, denote by $Cov_X(K)$ the family of all covers of K by open sets of X.

Lemma 2.12. *Given a regular topological space X and a usc map $\Psi : X \rightrightarrows X$ with closed values. If Ψ has an $\omega-$fixed point for all $\omega \in Cov_X(cl(\Psi(X)))$ then Ψ has a fixed point.*

Proof. Suppose that Ψ is fixed point-free. Since X is regular, points can be separated from closed sets by open sets, i.e., for each $x \in X$ there exists open sets $U_x \ni x, V_x \supset \Psi(x)$ with $U_x \cap V_x = \emptyset$. Since Ψ is usc, U_x and V_x can be chosen so that $\Psi(U_x) \subset V_x$. Clearly, Ψ cannot have an $\omega-$fixed point for the open cover $\omega = \{U_x : x \in X\}$. A contradiction. □

Theorem 2.13. *Let X be a non-empty convex subset in a locally convex topological vector space E and let $\Psi \in \mathbf{K}^*(X)$ be a compact map with closed values. Then Ψ has a fixed point.*

Proof. As in the proof of Theorem 2.8 (2), the convex hull $C := conv\{K\}$ of the compact set $K = cl(\Psi(X)) \subseteq X$ is a convex paracompact subset of X. The "restriction-compression" map $\Psi_C : C \rightrightarrows C$ defined by $\Psi_C(x) = \Psi(x) \cap C, x \in C$, is also a Kakutani map with closed values. By Proposition 2.3, given an arbitrary convex, symmetric and open neighborhood U of the origin in E, there exists a map $\Psi_{C,U} = C \rightrightarrows C$ verifying:

- $\Psi_C(x) \subseteq \Psi_{C,U}(x) \subseteq (\Psi_C((x+U) \cap C) + U) \cap C, \forall x \in C$,
- $\Psi_{C,U} \in \mathbf{F}^*(C, C)$, and
- $\Psi_{C,U}(C) \subseteq K \cap C \subseteq K$;

that is, $\Psi_{C,U}$ is an $\mathbf{F}^*$ compact U-approximation of Ψ_C. By Theorem 2.8 (2) - as the underlying space E is locally convex - $\Psi_{C,U}$ has a fixed point

$$x_U \in \Psi_{C,U}(x_U) \subseteq (\Psi_C((x+U) \cap C) + U) \cap C \subseteq \Psi(x'_U) + U$$

for some $x'_U \in (x_U + U) \cap C$, i.e., x_U is an $\omega-$fixed point of Ψ for the cover $\omega = \{(x+U) \cap X : x \in X\}$. Lemma 2.12 ends the proof. □

The classical case where X is compact and convex and $\Psi \in \mathbf{K}^*(X)$ is a particular case of Theorem 2.13. Indeed, Proposition 2.3 (1) implies that $K := \Psi(X)$ is compact, i.e., Ψ is a compact map with compact values.

2.2 A Coincidence Principle

We end this section with a generalization of a coincidence property between Ky Fan and Kakutani maps (see, e.g., [BDG1, BDG2] for earlier results and particular cases). Recall that the class $\mathbf{K}(X,Y) := \{\Psi : X \rightrightarrows Y : \Psi^{-1} \in \mathbf{K}^*(Y,X)\}$ consists of inverses of Kakutani maps.

Theorem 2.14. *($(\mathbf{\Phi}^*, \mathbf{K})$ Coincidence) Let X be a non-empty convex subset in a locally convex topological vector spaces E, Y be a non-empty convex subset of a topological vector space F. We are given two maps $\Phi \in \mathbf{\Phi}^*(X,Y)$ and $\Psi \in \mathbf{K}^*(Y,X)$ with Ψ having closed values in X. Assume that one of the following conditions holds:*

(1) X is compact - or more generally, Ψ is a compact map; or

(2) Y is compact and Ψ has compact values; or

(3) The map Ψ has compact values and the map $\Phi : X \rightrightarrows Y$ verifies the condition (κ), that is, there exist a compact subset K of X and a compact convex subset C of Y such that the admissible selection $\tilde{\Phi}$ of Φ satisfies:

$$(\kappa) \qquad \forall x \in X \setminus K, \tilde{\Phi}(x) \cap C \neq \emptyset.$$

Then the pair (Φ, Ψ^{-1}) has a coincidence point, that is, $\exists (x_0, y_0) \in X \times Y$ with $y_0 \in \Phi(x_0)$ and $x_0 \in \Psi(y_0)$.

Proof. Suppose that $\Psi : Y \rightrightarrows X$ is a compact map, that is $K := cl(\Psi(Y))$ is a compact subset of X (this is so if X is compact). The set $C := conv\{K\}$ is a paracompact subset of X. As $\Phi \in \mathbf{\Phi}^*(X,Y)$, Proposition 2.6 implies the existence of a continuous selection $s : C \longrightarrow Y$ of Φ. The composition map $\Psi \circ s : C \stackrel{s}{\longrightarrow} Y \stackrel{\Psi}{\rightrightarrows} K \hookrightarrow C$ is a compact $\mathbf{K}^*$ map with closed values defined on a convex subset of a locally convex space. It has a fixed point $x_0 \in \Psi(s(x_0))$ by Theorem 2.13. Obviously, $y_0 = s(x_0) \in \Phi(x_0)$ that is $y_0 \in \Phi(x_0) \cap \Psi^{-1}(x_0)$ and we are done.

Assume now that (2) holds. By Proposition 2.1 (1) $K := \Psi(Y)$ is a compact subset of X. As in the proof of Theorem 2.8 (2) above, the restriction of the map Φ to the paracompact subset $C := conv\{K\}$ of X has a continuous selection $s : C \longrightarrow Y, s(x) \in \Phi(x)$ for all $x \in C$. The composition map $\Psi \circ s : C \stackrel{s}{\longrightarrow} Y \stackrel{\Psi}{\rightrightarrows} C$ is a compact $\mathbf{K}^*$ map of the convex subset C of the locally convex space E. The proof concludes as above.

Finally, assume that (3) holds with $K \subseteq X$ compact and $C \subseteq Y$ convex compact such that $\tilde{\Phi}(x) \cap C \neq \emptyset$ for all $x \in X \setminus K$. The restriction $\Phi|_K : K \rightrightarrows Y$ is a $\mathbf{\Phi}^*$ map with compact domain. Thus, it admits a finite dimensional continuous selection $s : K \longrightarrow P$ with P a finite convex polytope contained in Y (Proposition 2.6). Consider the compact convex subset $\hat{C} = conv\{C \cup P\}$ of Y and the compression map $\Gamma : X \rightrightarrows \hat{C}$ given by $\Gamma(x) = \Phi(x) \cap \hat{C}$ with admissible selection $\tilde{\Gamma}(x) = \tilde{\Phi}(x) \cap \hat{C}$ where $\tilde{\Phi}$ is an admissible selection of Φ. It is readily seen that $\Gamma \in \mathbf{\Phi}^*(X, \hat{C})$. Indeed:

- $\Gamma(x)$ is a convex subset of $\hat{C}$ for all $x \in X$.
- If $x \in K$, as per Remark 2.7 (ii), then x belongs to some member $\tilde{\Phi}^{-1}(y_i), y_i \in P \subseteq \hat{C}$, of a finite open cover $\{\tilde{\Phi}^{-1}(y_i) : y_i \in Y, i = 1, \ldots, n\}$ of K. Thus, some $y_i \in \tilde{\Phi}(x) \cap \hat{C} = \tilde{\Gamma}(x)$. On the other hand, if $x \in X \setminus K$, then $\emptyset \neq \tilde{\Phi}(x) \cap C \subseteq \tilde{\Phi}(x) \cap \hat{C} = \tilde{\Gamma}(x)$ by virtue of condition (κ). All in all, $\tilde{\Gamma}(x) \neq \emptyset$ for all $x \in X$.
- for any $y \in \hat{C}$, $\tilde{\Gamma}^{-1}(y) = \tilde{\Phi}^{-1}(y)$ is an open set in X.

As $\hat{C}$ is compact and convex, we are thus in the context of part (1) of this theorem for the pair of maps $\Psi|_{\hat{C}} \in \mathbf{K}^*(\hat{C}, X)$ and $\Gamma \in \mathbf{\Phi}^*(X, \hat{C})$ which must coincide at some pair $(x_0, y_0) \in \hat{C} \times Y$. This coincidence for $\Psi|_{\hat{C}}$ and Γ is also a coincidence for the pair Ψ and Φ. □

Conjecture 2.15. The assumption that Ψ must have compact values in Theorem 2.14 can be weakened to "Ψ has closed values" under assumptions (2) and (3). This seems to be an open problem.

2.3 Tangency and Sleek Sets

As mentionned in the introduction, the existence of an equilibrium for a mapping requires the mapping to satisfy a (boundary) tangency condition. It is thus helpful to briefly describe here some tangency concepts for general domains[12].

Definition 2.16. Given a non-empty set X in a real topological vector space E and an element $x \in \overline{X}$, define:

(i) the *pseudo-tangent cone* to X at x is:

$$T_X^P(x) := cl(S_X(x)) \text{ where } S_X(x) := \bigcup_{t>0} \frac{1}{t}(X - x).$$

(ii) The *adjacent cone* to X at x is:

$$T_X^A(x) := \lim \inf_{t \downarrow 0^+} \{\frac{1}{t}(X - x)\}.$$

(iii) The *Bouligand-Severi contingent cone* to X at x is:

$$T_X^B(x) := \lim \sup_{t \downarrow 0^+} \{\frac{1}{t}(X - x)\}.$$

[12] The limits of nets of sets in Definition 2.16 are meant in the sense of *Painlevé-Kuratowski*. Given a net $\mathfrak{a} := \{A_t\}_{t \in T}$ of subsets in a Hausdorff topological space Z, the inner limit of $\mathfrak{a}$ is $\lim\inf_t A_t := \{x \in Z : \forall U \in \mathcal{N}_Z(x), \exists t_0 \in T$ such that $\forall t_0 \preceq t, U \cap A_t \neq \emptyset\}$; while the outer limit of $\mathfrak{a}$ is $\lim\sup_t A_t := \{x \in Z : \forall U \in \mathcal{N}_Z(x), \forall t \in T$ such that $\exists t \preceq t'$, $U \cap A_{t'} \neq \emptyset\}$.

(iv) The *Clarke's tangent cone* to X at x is:

$$T_X^C(x) := \lim \inf_{t\downarrow 0^+, x'\to_X x} \{\frac{1}{t}(X - x')\}.$$

Remark 2.17. (See [AF, B1]) One readily sees that for any subset X in a topological vector space and any $x \in cl(X)$, we have:

(1) Always, $T_X^C(x) \subseteq T_X^A(x) \subseteq T_X^B(x) \subseteq T_X^P(x)$.

(2) If $x \in int(X)$, then $T_X^C(x) = T_X^A(x) = T_X^B(x) = T_X^P(x) = E$.

(3) $T_X^A(x)$ is the set of all limit points of generalized sequences $\{x_t\}_{t>0}$ with $x_t \in \frac{1}{t}(X - x)$. $T_X^B(x)$ is the set of all cluster points of generalized sequences $\{x_t\}_{t>0}$ with $x_t \in \frac{1}{t}(X - x)$.

(4) While always closed, the cones $T_X^A(x), T_X^B(x)$ and $T_X^P(x)$ may not be convex. The cone $T_X^C(x)$ is always closed and convex[13].

(5) In the particular case where E is metrizable, we have the simpler sequential characterizations:

$$\begin{aligned}
T_X^P(x) &= \{v \in E : \exists v_n \to v \text{ such that } \forall n, \exists t_n > 0 \text{ with } x + t_n v_n \in X\}.\\
T_X^A(x) &= \{v \in E : \forall t_n \to 0^+, \exists v_n \to v \text{ such that } \forall n, x + t_n v_n \in X\}.\\
T_X^B(x) &= \{v \in E : \exists t_n \to 0^+, \exists v_n \to v \text{ such that } \forall n, x + t_n v_n \in X\}\\
&= \{v \in E : \liminf_{t\downarrow 0^+} d_X(x + tv) = 0\}.\\
T_X^C(x) &= \{v \in E : \forall t_n \to 0^+, \forall x_n \to_X x, \exists v_n \to v \text{ such that } \forall n, x_n + t_n v_n \in X\}\\
&= \{v \in E : \limsup_{t\downarrow 0^+, x'\to_X x} d_X(x' + tv) = 0\}.
\end{aligned}$$

(6) X is said to be differentiable at x if $T_X^A(x) = T_X^B(x) = \lim_{t\downarrow 0^+}\{\frac{1}{t}(X - x)\}$. In this case, the contingent cone $T_X^B(x)$ consists of all one-sided derivatives $x'_+(0) = \lim_{t\downarrow 0^+} \frac{x_t - x}{t}$ where $x_t \in X$ for all $0 < t$ small enough.

(7) If X is locally convex at x, that is, there exists an open neighborhood of x in E such that $X \cap U$ is convex, then

$$T_X^C(x) = T_X^A(x) = T_X^B(x) = T_X^P(x).$$

Obviously, a convex subset of a locally convex space is locally convex at each of its points.

(8) Always, $\liminf_{x'\to_X x} T_X^B(x') \subseteq T_X^C(x)$ with equality holding whenever, e.g., E is a finite dimensional space (see theorem 4.1.10 in [AF]).

Generalized normal cones are defined as negative polar cones[14] to the tangent cones above.

$$\begin{aligned}
T_X^P(x)^- &=: N_X(x) = \{p \in E' : \langle p, v\rangle \leq 0, \forall v \in T_X^P(x)\}.\\
T_X^B(x)^- &=: \widehat{N}_X(x) = \{p \in E' : \langle p, v\rangle \leq 0, \forall v \in T_X^B(x)\}.\\
T_X^C(x)^- &=: N_X^C(x) = \{p \in E' : \langle p, v\rangle \leq 0, \forall v \in T_X^C(x)\}.
\end{aligned}$$

Remark 2.18. (1) The closed convex cone of outward normals $N_X(x) = T_X^P(x)^-$ is also described as $N_X(x) = S_X(x)^- = (X - x)^- = \{p \in E' : \langle p, v\rangle \leq 0, \forall v \in (X - x)\} = \{p \in E' : \langle p, x\rangle = \sigma_X(p)\}$ where $\sigma_X(p) := \max_{v\in X}\langle p, v\rangle\}$.

(2) The closed convex cone $\widehat{N}_X(x)$ is known as the Hadamard *normal cone* (or the cone of regular normals) to X at x. The *limiting normal cone* to X at x is the cone of limiting proximal normals $\tilde{N}_X(x) = \limsup_{x'\to_X x} \widehat{N}_X(x')$[15]. The basic normal cone need not be convex. In

[13] A drawback of the convexity of the Clarke's tangent cone $T_X^C(x)$ is that it may reduce to $\{0_{E'}\}$; making the tangency condition (τ) useless for equilibrium theorems.

[14] The *negative polar cone* to a subset A of a topological real vector space E is defined as $A^- := \{p \in E' : \langle p, x\rangle \leq 0, \forall x \in A\}$ (always a closed convex cone).

[15] In case E is metrizable, it holds $\widehat{N}_X(x) = \{p \in E' : \limsup_{u\to_X x} \frac{\langle p, u - x\rangle}{d(u, x)} \leq 0\}$ and $\tilde{N}_X(x) = \{p \in E' : \exists\{x_n\}_n \subseteq X, \exists\{p_n \in \widehat{N}_X(x_n)\}_n$ with $x_n \longrightarrow x$ and $p_n \longrightarrow p\}$.

fact, the *Clarke's normal cone* to X at x is precisely $N_X^C(x) = cl(conv(\tilde{N}_X(x)))$. Clearly, $\widehat{N}_X(x) \subseteq \tilde{N}_X(x) \subseteq N_X^C(x)$.

(3) The set X is said to be (Clarke) *regular* at x if the Bouligand-Severi cone $T_X^B(x)$ and the basic normal cone $\tilde{N}_X(x)$ are mutually (negative) polars (this makes them both closed convex cones). In this case, $T_X^B(x) = T_X^A(x) = T_X^C(x)$ and $\widehat{N}_X(x) = \tilde{N}_X(x) = N_X^C(x)$. (In particular, X is differentiable at x.)

The next concept is essential for our last main result (Theorem 3.20 below).

Definition 2.19. The set X is said to be *sleek at a point* x if the tangent map $T_X^B(.) : X \rightrightarrows E$ is lower semicontinuous at x.

If X is sleek at x then $\liminf_{x' \to_X x} T_X^B(x') \subseteq T_X^C(x) \subseteq T_X^B(x) \subseteq \liminf_{x' \to_X x} T_X^B(x')$. Hence $T_X^C(x) = T_X^B(x)$ (that is, X is regular at x), $T_X^C(.)$ is lower semicontinuous at x, and $T_X^B(x)$ is also a closed convex cone (see Theorem 4.1.8 in [AF]).

Proposition 2.20. *If X is a sleek subset of a locally convex topological vector space E, then the convex and closed-valued map $N_X^C : X \rightrightarrows E'$ has a closed graph.*

Proof. For the sake of proof's simplicity, assume E is a metrizable. Note first that given a sequence of sets $\{T_n\}$ in E, we always have $\liminf_{n\to\infty} T_n \subseteq (\sigma - \limsup_{n\to\infty} T_n^-)^-$[16]. Indeed, if $x \in \liminf_{n\to\infty} T_n$, i.e., $x = \lim_n x_n, x_n \in T_n$, and $p \in \sigma - \limsup_{n\to\infty} T_n^-$, i.e. p is the weak*–limit $\sigma - \lim_{n_k} p_{n_k}$ of a subsequence $p_{n_k} \in T_{n_k}^-$, $\langle p_{n_k}, x_{n_k}\rangle \leq 0$, then $\langle p, x\rangle \leq 0$.

Let $x_n \to x$ in $X, T_n = T_X^C(x_n), T_n^- = N_X^C(x_n)$. Since X is sleek, T_X^C is lower semicontinuous, hence

$$T_X^C(x) \subseteq \liminf_{n\to\infty} T_n \subseteq (\sigma - \limsup_{n\to\infty} N_X^C(x_n))^-.$$

Thus,

$$\sigma - \limsup_{n\to\infty} N_X^C(x_n) \subseteq (\sigma - \limsup_{n\to\infty} N_X^C(x_n))^{--} \subseteq T_X^C(x)^- = N_X^C(x),$$

that is, N_X^C is weak*-upper semicontinuous. By Proposition 3 (5) in [B1], it has a weakly-closed graph Γ. Being a convex set, by Mazur's theorem Γ is also strongly closed. □

3 Main Results: Equilibria and Co-Equilibria

3.1 Equilibria in Convex Subsets of Locally Convex Spaces

In this section we provide a novel simple proof of the existence of an equilibrium for a tangential upper hemicontinuous map with closed convex values defined on a compact convex subset of a locally convex topological vector space based on the $(\mathbf{\Phi}^*, \mathbf{K})$ coincidence principle (Theorem 2.14 above). We then extend this theorem to non-compact convex domain subject to partial tangency conditions. To do this, some preparatory results are required starting with a useful analytical expression of Theorem 2.14 in the form of an alternative for systems of nonlinear inequalities.

Proposition 3.1. *Let $\Psi \in \mathbf{K}^*(Y, X)$ be a closed valued map on a non-empty convex subset Y in a topological vector space F with values in a non-empty convex subset X in a locally convex topological vector space E into a non-empty convex subset Y of a locally convex topological vector space F and let $f, \tilde{f} : X \times Y \longrightarrow \mathbb{R}$ be two functions satisfying:*

(i) for every $(x, y) \in X \times Y$, $\tilde{f}(x, y) \leq f(x, y)$.

(ii) for every fixed $y \in Y$, the partial function $x \mapsto \tilde{f}(x, y)$ is lower semicontinuous on X.

(iii) for every fixed $x \in X$, the partial function $y \mapsto f(x, y)$ is quasiconcave on Y.

Assume that one of the following conditions holds:

(1) X is compact - or more generally, Ψ is a compact map; or

(2) Y is compact and Ψ has compact values; or

[16] This is known as the *Duality Theorem* (theorem 1.1.8 in [AF]). Equality occurs, e.g., when T_n is a closed convex cone.

(3) The map Ψ has compact values and there exist a compact subset K of X and a compact convex subset C of Y such that:

$$(\kappa) \qquad \forall x \in X \setminus K, \exists y \in C \text{ such that } \tilde{f}(x,y) > 0.$$

Then, the following nonlinear alternative holds:
(A) $\exists (x_0, y_0) \in X \times Y$ with $x_0 \in \Psi(y_0)$ and $f(x_0, y_0) > 0$; or
(B) $\exists \bar{x} \in X$ such that $\tilde{f}(\bar{x}, y) \leq 0, \forall y \in Y$.

Proof. Consider the maps $\mathbf{\Phi}, \tilde{\mathbf{\Phi}} : X \rightrightarrows Y$ defined by

$$\mathbf{\Phi}(x) := \{y \in Y : f(x,y) > 0\}, \tilde{\mathbf{\Phi}}(x) := \{y \in Y : \tilde{f}(x,y) > 0\}, \forall x \in X.$$

The maps $\mathbf{\Phi}, \tilde{\mathbf{\Phi}}$ satisfy the following:

- $\tilde{\mathbf{\Phi}}(x) \subseteq \mathbf{\Phi}(x)$ for all $x \in X$ by (i).
- For all $y \in Y$, $\tilde{\mathbf{\Phi}}^{-1}(y)$ is open in X by (ii).
- For all $x \in X$, $\mathbf{\Phi}(x)$ is convex in Y by (iii).

  If $\tilde{\mathbf{\Phi}}(x) \neq \emptyset$ for all $x \in X$, then $\mathbf{\Phi}$ is a $\mathbf{\Phi}^*$ map and the coincidence Theorem 2.14 applies yielding conclusion (A). Otherwise, $\tilde{\mathbf{\Phi}}(\bar{x}) = \emptyset$ for some $\bar{x} \in X$, amounting to alternative (B).

□

Remark 3.2. There are several situations where hypothesis (iii) in Proposition 3.1 can occur. The classical case whereby $K = X$ is a compact set is obvious as it renders (iii) vacuously true. Otherwise, if neither of X nor Y is compact, one may consider the case where $X \subseteq E = \mathbb{R}^n, Y \subseteq F = \mathbb{R}^m$ and $\tilde{f} : X \times Y \longrightarrow \mathbb{R}$ verifying:

- for each fixed $x \in X$, the function $y \mapsto \tilde{f}(x,y)$ is upper semicontinuous on Y. So that on any convex compact subset C of Y, the lower semicontinuous function $\varphi_C(x) := \max_{y\in C} \tilde{f}(x,y)$ is well-defined, say $\varphi_C(x) := \tilde{f}(x, y_C)$ for some $y_C \in C$;
- for each fixed $y \in C$, some compact convex subset of Y, the function $x \mapsto \tilde{f}(x,y)$ is proper, that is, $\lim_{\|x\| \to_X +\infty} \tilde{f}(x,y) = -\infty$. So that for each compact convex subset C of Y, the closed set $X_C := \{x \in X : \varphi_C(x) \leq 0\}$ is unbounded.

Obviously, no compact subset K of X can be included in X_C.
(The interested reader can envisage a similar situation whereby E is a reflexive Banach space equipped with the weak topology and invoke the Eberlein-Smulian theorem.)

As an immediate consequence we have a novel simple proof of the first equilibrium theorem for tangential $\mathbf{K}^*$ maps on a compact convex domain in a locally convex topological vector space which is essentially the Ky Fan-Halpern equilibrium theorem. Define first,

Definition 3.3. A map $\mathbf{\Phi} : X \rightrightarrows E$ on a non-empty closed subset X in a topological vector space E is said to be tangential if it satisfies the so-called weak Halpern boundary condition[17]

$$(\tau) \qquad \forall x \in \partial X,\ \mathbf{\Phi}(x) \cap T_X(x) \neq \emptyset,$$

where $T_X(x)$ is a suitably chosen tangent cone concept (see above).

Remark 3.4. It must be noted that
(i) condition (τ) is trivially satisfied for $x \in int(X)$ whenever $T_X(x) = E$ for interior points; this is the case for all cones T^A, T^B, T^C, T^P discussed in section 2.3.
(ii) under tangency condition (τ), *Ky Fan's normality condition*

$$(KF) \qquad p \in N_X(x) \Longrightarrow f(x,p) = \inf_{y \in \mathbf{\Phi}(x)} \langle p, y \rangle \leq 0$$

always holds with $N_X(x)$ being the negative polar cone to $T_X(x)$ in (τ).

[17]In reference to B. Halpern [H]. A comparison of noteworthy tangency conditions is undertaken in [B1].

Definition 3.5. Let X be a non-empty closed subset in a topological vector space E with continuous dual E'. Define the classes

$$\mathbf{H}^*(X,E) := \{\Phi : X \rightrightarrows E : \Phi \text{ is uhc on } X \text{ and } \emptyset \neq \Phi(x) \text{ is closed and convex}, \forall x \in X\}.$$
$$\mathbf{H}^*_\tau(X,E) := \{\Phi \in \mathbf{H}^*(X,E) : \Phi \text{ is tangential on } X\}.$$

Clearly, closed valued $\mathbf{K}^*$ maps are in $\mathbf{H}^*$. For convex compact valued maps, $\mathbf{K}^* = \mathbf{H}^*$.

Keep in mind that for closed convex set, tangency condition (τ) is expressed in terms of $T_X(x) = T_X^P(x) = cl(\bigcup_{t>0} \frac{1}{t}(X - x))$ the closed convex tangent cone of convex analysis. The cone of outward normals is $N_X(x) = T_X^P(x)^-$ the negative polar cone to the tangent cone.

Theorem 3.6. *Let X be a non-empty convex compact subset in a locally convex topological vector space E. Then every tangential map $\Phi \in \mathbf{H}^*_\tau(X,E)$ has an equilibrium.*

Proof. We shall apply the proposition 3.1 with $Y = E'$, $f = \tilde{f} : X \times Y \longrightarrow \mathbb{R}$ defined by $f(x,p) = \inf_{y\in\Phi(x)}\langle p,y\rangle$, and $\Psi : E' \rightrightarrows X$ defined by

$$\Psi(p) := N_X^{-1}(p) = \{x \in X : \langle p,x\rangle = \max_{u\in X}\langle p,u\rangle\} = \arg\max_X \sigma_X(p).$$

We take notice of Remark 3.4 (ii): "tangency $(\tau) \Longrightarrow$ normality (KF)".

Since every $\mathbf{K}^*$ map is upper hemicontinuous, that is, the numerical function $x \mapsto \sigma_\Phi(x,p)$ is upper semicontinuous, it follows that for each fixed $p \in E'$, the function $x \longmapsto f(x,p)$ is lower semicontinuous on X. Obviously, for each fixed $x \in X$, the function $p \longmapsto f(x,p)$ is concave.

Moreover, as the graphs of inverse maps are identical, $graph(\Psi) = graph(N_X^{-1})$. As X being a convex set is sleek, Proposition 2.20 implies that Ψ has closed graph. But Ψ also has closed, hence compact values in X. Hence, by Remark 2.1 (1), Ψ is usc and compact valued.

Ky Fan's normality condition (KF) opposes alternative (A) of Proposition 3.1. Hence alternative (B) of that proposition holds: $\exists\hat{x} \in X$ with $\inf_{y\in\Phi(\hat{x})}\langle p,y\rangle \leq 0$ for all $p \in E'$.

If $0 \notin \Phi(\bar{x})$, since $\Phi(\bar{x})$ is non-empty, closed and convex, by the Hahn-Banach separation theorem, $\exists p \in E', \exists\alpha \in \mathbb{R}$ with $p(0) = 0 < \alpha < p(y), \forall y \in \Phi(\bar{x})$. This implies $0 < \alpha \leq \inf_{y\in\Phi(\bar{x})}\langle p,y\rangle \leq 0$, a contradiction. Thus $0 \in \Phi(\bar{x})$ and the proof is complete. □

Remark 3.7. Note that $\Psi(0) = X$ (as $0 \in N_X(x)$ for all $x \in X$). Moreover, for $p \in E' \setminus \{0\}, \Psi(p) \cap N_X(x) = \emptyset$, that is, $\Psi(p) \subseteq \partial X$. Hence, in the case where X is not compact, in order to reproduce the same proof for the existence of an equilibrium based on the preceding proposition (under hypothesis (2) or (3)), it is essential to judiciously select a subset Y of non-trivial linear form on E for which $\Psi(p)$ is a compact subset of ∂X. A simple illustration is given by the simplest case of the intermediate value theorem for a continuous real function on an unbounded closed interval $f : X = [a,+\infty) \longrightarrow E = \mathbb{R}$ satisfying:

- the boundary condition $f(a) \geq 0$ amounts to condition $(\tau) : f(a) \in T_X(a) = [0,+\infty)$;
- the "coercivity condition" (κ) reads: there exists a compact subset K of X and a compact convex subset C of $Y := (-\infty,0) \subset \mathbb{R} = E'$ such that

  $$\forall x \in X \setminus K, \exists p \in C \text{ such that } f(x,p) := pf(x) > 0.$$

  K could be any interval $[a,b] \subset X = [a,+\infty)$ and C any interval $[c,d] \subset Y = (-\infty,0)$. Condition (κ) implies that $f(x) < 0$ outside of K, in particular at $x = b+1$. The classical IVT on $[a,b+1]$ garantees an equilibrium $f(\bar{x}) = 0$, with $\bar{x} \in K \subset X$. (Such a situation occurs, e.g., when $\lim_{x\to+\infty} f(x) < 0$.)

To establish an equilibrium theorem for a tangential $\mathbf{H}^*$ map on a convex non-compact domain X, we shall adopt now a different approach based on the analytical expression of the fixed point Theorem 2.8 for $\mathbf{\Phi}^*$ maps.

Proposition 3.8. *Let X be a non-empty convex subset in a topological vector space E and $f, \tilde{f}$: $X \times X \longrightarrow \mathbb{R}$ be two functions satisfying the following conditions.*

(i) For every $(x, y) \in X \times X$, $\tilde{f}(x, y) \leq f(x, y)$.
(ii) For every fixed $y \in X$, *the partial function* $x \mapsto \tilde{f}(x, y)$ *is lower semicontinuous on* X.
(iii) For every fixed $x \in X$, *the partial function* $y \mapsto f(x, y)$ *is quasiconcave on* X.
(iv) There exist compact subsets K *and* C *of* X *with* C *convex, such that:*

$$\forall x \in X \setminus K, \exists y \in C \text{ such that } \tilde{f}(x, y) > 0.$$

Then the following alternative holds:
(A) $\exists x_0 \in X$ *with* $f(x_0, x_0) > 0$; *or*
(B) $\exists \bar{x} \in X$ *such that* $\tilde{f}(\bar{x}, y) \leq 0$ *for all* $y \in X$.

Proof. Consider the maps $\Phi : X \rightrightarrows X$, given by $\Phi(x) := \{y \in X : f(x, y) > 0\}$ and $\tilde{\Phi}(x) := \{y \in X : \tilde{f}(x, y) > 0\}$, for all $x \in X$. If conclusion (B) fails, that is, $\tilde{\Phi}(x) \neq \emptyset$ for all $x \in X$ then Φ is a $\mathbf{\Phi}^*$ map. The weaker compactness hypothesis (iv) corresponds to the condition (κ) of Theorem 2.8. The latter implies the existence of a fixed point $x_0 \in \Phi(x_0)$ and (A) holds. □

Corollary 3.9. *Let* X *be a convex subset of a topological vector space E, let* Y *be a set of functions in* $\{p : X \longrightarrow \mathbb{R} : p$ *is upper semicontinuous and quasiconcave*$\}$[18] *and let.* $s : X \longrightarrow Y$ *be a continuous function.*

Assume that the following compactness condition holds: there exist two compact subsets K *and* C *of* X *with* C *convex, such that, for* $p = s(x)$, *we have:*

$$\forall x \in X \setminus K,\ p(x) < \max_{y \in C} p(y).$$

Then,

$$\exists \bar{x} \in X \text{ such that for } \bar{p} = s(\bar{x}) \in Y \text{ we have } \bar{p}(\bar{x}) = \max_{y \in X} \bar{p}(y).$$

Proof. Let $s : X \longrightarrow Y$ be a continuous selection of Γ and consider the function $f = \tilde{f} : X \times X \longrightarrow \mathbb{R}$ defined as

$$f(x, y) = s(x)(y) - s(x)(x) \text{ for every pair } (x, y) \in X \times X.$$

Clearly $f(., y)$ is lower semicontinuous on X and $f(x, .)$ is quasiconcave on X. We check that hypothesis (iv) of Proposition 3.8 holds: $\forall x \in X \setminus K$, for $p = s(x) \in Y$, let $y \in \arg\max_C p$ (C is compact) and write $p(x) < p(y)$. Thus, $f(x, y) = s(x)(y) - s(x)(x) > 0$, satisfying (iv) of Proposition 3.8.

Obviously, alternative (A) in the conclusion of Proposition 3.8: $f(x_0, x_0) = s(x_0)(x_0) - s(x_0)(x_0) = 0 > 0$, for some $x_0 \in X$, is impossible. Thus thesis (B) of Proposition 3.8 holds, i.e., $\exists \bar{x} \in X$ with $f(\bar{x}, y) = s(\bar{x})(y) - s(\bar{x})(\bar{x}) \leq 0, \forall y \in X$. Thus, $\bar{p} = s(\bar{x})$ satisfies $\bar{p}(y) \leq \bar{p}(\bar{x}), \forall y \in X$. □

A particular instance of Corollary 3.9 corresponds to the case where s is a continuous selection of an $\mathbf{F}^*$ map $\Gamma : X \rightrightarrows Y$.

Corollary 3.10. *Let* X *be a paracompact convex subset of a topological vector space* E, *let* Y *be a convex subset of functions in* $\{p : X \longrightarrow \mathbb{R} : p$ *is upper semicontinuous and quasiconcave*$\}$, *and let* $f : X \times Y \longrightarrow \mathbb{R}$ *be a function verifying the following hypotheses.*
(i) for every fixed $p \in Y$, $x \mapsto f(x, p)$ *is lower semicontinuous on* X;
(ii) for every fixed $x \in X$, $p \longmapsto f(x, p)$ *is quasiconcave on* Y.
(iii) There exist two compact subsets C, K *of* X *with* C *convex such that:*

$$\forall x \in X \setminus K, \forall p \in Y, \text{ we have } [p(x) \geq \max_{y \in C} p(y) \Rightarrow f(x, p) \leq 0].$$

Then one of the following holds:
(A) $\exists x^* \in X$ *with* $f(x^*, p) \leq 0$ *for all* $p \in Y$; *or*
(B) $\exists \bar{x} \in K, \exists \bar{p} \in Y$ *with* $\begin{cases} f(\bar{x}, \bar{p}) > 0 \\ \bar{p}(\bar{x}) = \max_{y \in X} \bar{p}(y) \end{cases}$

[18] Equipped with a suitable topology.

Proof. The map $\Gamma : X \rightrightarrows Y$ defined by $\Gamma(x) := \{p \in Y : f(x,p) > 0\}$ has convex values by (i) and open pre-images by (ii). If (A) fails, then $\Gamma(x) \neq \emptyset$ for all $x \in X$ and Γ is therefore an $\mathbf{F}^*-$map. Having a paracompact domain X, Γ admits a continuous selection $s : X \longrightarrow Y$ by Proposition 2.6. Hypothesis (iii) implies the compactness hypothesis of Corollary 3.9, yielding (B). □

We are now ready to establish a generalization of Theorem 3.6 to possibly unbounded domains, subject to a weaker compactness condition and partial tangency (again, as X is convex, $T_X = T_X^P$ is the tangent cone of convex analysis).

Theorem 3.11. *Let X be a non-empty closed convex and paracompact subset in a locally convex t.v.s. E with continuous dual E', and $\Phi \in \mathbf{H}^*(X,E)$. Assume that Φ satisfies the following condition:*

$$(\tau_\kappa)\quad \begin{cases} \exists C, K \text{ compact subsets of } X \text{ with } C \text{ convex such that:} \\ (i)\ \Phi \text{ is tangential on } K, \text{ that is: } \forall x \in K \cap \partial X, \Phi(x) \cap T_X(x) \neq \emptyset. \\ (ii) \forall x \in X \setminus K, \forall p \in E', \text{ we have} \\ \qquad\qquad [\langle p, x\rangle \geq \max_{y\in C}\langle p,y\rangle \Rightarrow \inf_{y\in\Phi(x)}\langle p,y\rangle \leq 0]. \end{cases}$$

Then Φ has an equilibrium in X.

Proof. Let $Y = \{p|_X : p \in E'\}$ (a convex set) and define the function $f : X \times Y \longrightarrow \mathbb{R}$ by $f(x,p) := \inf_{y\in\Phi(x)}\langle p,y\rangle$ for all $(x,p) \in X \times Y$.

As in Theorem 3.6, since the support functional $x \mapsto \sigma_\Phi(x,p) := \sup_{y\in\Phi(x)}\langle p,y\rangle$ associated to Φ is upper semicontinuous, then $x \mapsto f(x,p)$ is lower semicontinuous on X. In addition, as the infimum of linear forms, $p \mapsto f(x,p)$ is concave on Y.

Tangency on K (hypothesis $(\tau_\kappa - (i))$) implies the Ky Fan normality condition:

$$\forall x \in K \cap \partial X,\ p \in N_X(x) \Longrightarrow f(x,p) \leq 0.$$

This opposes alternative (B) of Corollary 3.10. Hence alternative (A) of that corollary holds: $\exists x^* \in X$ with $\inf_{y\in\Phi(x^*)}\langle p,y\rangle \leq 0$ for all $p \in E'$.

If $0 \notin \Phi(x^*)$, by the Hahn-Banach separation theorem, $\exists p \in E', \exists\lambda \in \mathbb{R}$ with $p(0) = 0 < \alpha < p(y), \forall y \in \Phi(x^*)$. This implies $0 < \alpha \leq \inf_{y\in\Phi(x^*)}\langle p,y\rangle \leq 0$, a contradiction. Thus $0 \in \Phi(x^*)$, completing the proof. □

3.2 Equilibria in Compact Retracts

In this section, we discuss a far reaching equilibrium theorem without convexity due to [BK]. The main result requires a special continuous selection/approximation property that ties together Proposition 2.10 on the existence of continuous approximative selections to Kakutani maps and the celebrated selection theorem of Michael[19]. We provide here a version of this hybrid continuous "almost" selection property in the more general context of a paracompact topological domain equipped with a uniformity and a locally convex topological vector space as co-domain.

Let us define the class of *Michael* maps from a topological space X into a topological vector space E as:

$$\mathbf{M}^*(X,E) := \{\Psi : X \rightrightarrows E : \Psi \text{ is lsc and } \emptyset \neq \Psi(x) \text{ convex } \forall x \in X\}.$$

Proposition 3.12. *Assume that X is paracompact topological space with uniform structure $\mathcal{U}$ and let $\Psi \in \mathbf{M}^*(X,E), \Phi \in \mathbf{K}^*(X,E)$ be two maps such that $\Phi(x) \cap \Psi(x) \neq \emptyset$ for each $x \in X$. Then, $\forall U \in \mathcal{U}, \forall V \in \mathcal{N}_E(0)$, there exists a continuous mapping $s : X \longrightarrow E$ such that for every $x \in X$:*

(i) $s(x) \in \Psi(x) + V$, and

(ii) $s(x) \in \Phi(U[x]) + V$.

[19] After *E. Michael* and his celebrated continuous selection theorem [Mi].

Proof. For any given $x \in X$, consider the open neighborhood of x defined by

$$O(x) := \frac{1}{2}U[x] \cap \{x' \in X; \Phi(x') \subset \Phi(x) + \frac{1}{2}V\},$$

with the second set being open due to the upper semicontinuity of Φ.

Let $\Omega = \{\omega\}$ be an open star-refinement of the open cover $\mathcal{O} = \{O(x)\}_{x\in X}$, i.e., for any $\omega \in \Omega$ there is $\overline{x} \in X$ with $st(\omega, \Omega) \subset O(\overline{x})$.

For any $x \in X$, choose $z_x \in \Phi(x) \cap \Psi(x)$ and consider the open cover $\mathcal{D} = \{D_\omega(x)\}_{\omega\in\Omega, x\in\omega}$ of X, where

$$D_\omega(x) := \{x' \in \omega; \Psi(x') \cap (z_x + \frac{1}{2}V) \neq \emptyset\}$$

is open due to the lower semicontinuity of Ψ.

Let $\{\lambda_i\}_{i\in I}$ be a locally finite partition of unity subordinated to the cover $\mathcal{D}$. Hence, for each $i \in I$, there are $\omega_i \in \Omega, x_i \in \omega_i$, with $\lambda_i(x') = 0$ for $x' \notin D_{\omega_i}(x_i)$.

The map $s : X \longrightarrow E$ defined by:

$$s(x) = \sum_{i\in I} \lambda_i(x) z_i, x \in X,$$

with z_i denoting z_{x_i}, is clearly continuous. Moreover, for each $x \in X$ and each index i in the finite set of essential indices $I(x) = \{i \in I; \lambda_i(x) \neq 0\}$, there exists $z_i' \in \Psi(x)$ such that $z_i' \in z_i + \frac{1}{2}V \Leftrightarrow z_i' - z_i \in \frac{1}{2}V$, because $x \in D_{\omega_i}(x_i)$. Thus, by convexity of $\Psi(x)$ and of $V \in \mathcal{N}_E(0)$,

$$\sum_{i\in I(x)} \lambda_i(x) z_i' \in \Psi(x), \text{ and}$$
$$\sum_{i\in I(x)} \lambda_i(x) z_i' - s(x) = \sum_{i\in I(x)} \lambda_i(x)(z_i' - z_i) \in V.$$

In other words, $s(x) \in \Psi(x) + V \neq \emptyset$ for every $x \in X$.

On the other hand, given $x \in X, i \in I(x)$, it follows that $x \in O_{\omega_i}(x_i) \subset V_i$ where $x_i \in \omega_i$. Since Ω is a star-refinement of $\mathcal{O}$, there is $\overline{x} \in X$ such that $x, x_i \in O(\overline{x})$. Therefore, $z_i \in \Phi(x_i) \subset \Phi(\overline{x}) + V$ and $\overline{x} \in U[x]$. The set $\Phi(\overline{x}) + V$ being convex, it follows that $s(x) \in \Phi(\overline{x}) + V \subseteq \Phi(U[x]) + V$. □

We shall also have to justify the passage from the existence of a net of "almost equilibria" to that of an equilibrium for a given map Φ. This of course requires some compactness together with the closedness of the graph of Φ. The next lemma is a mere adaptation of Lemma 2.12 to the equilibrium problem.

Definition 3.13. Given a subset X in a topological vector space E, a map $\Phi : X \rightrightarrows E$ and $U \in \mathcal{N}_E(0)$ an open neighborhood of the origin in E, an element $x_U \in X$ is said to be a U-equilibrium for Φ if

$$U \cap \Phi((x_U + U) \cap X) \neq \emptyset.$$

Lemma 3.14. *If a given usc compact map $\Phi : X \rightrightarrows E$ with closed values has a U-equilibrium in X for every $U \in \mathcal{N}_E(0)$, then Φ has an equilibrium in X.*

Proof. Assume that $\forall x \in X, 0 \notin \Phi(x)$. A topological vector space being regular, for each $x \in X$ there exists $O_x \in \mathcal{N}_E(0)$ and V_x an open neighborhood of the closed set $\Phi(x)$ in E such that $O_x \cap V_x = \emptyset$. Let $U_x \in \mathcal{N}_E(0)$ be symmetric and such that $U_x + U_x \subset O_x$ and, due to the upper semicontinuity of Φ, $\Phi(x + U_x) \subset V_x$. As $cl(\Phi(X))$ is compact in E, $\Phi(X) \subset V := \bigcup_{i=1}^n V_{x_i}$ for some finite subset $\{x_1, \ldots, x_n\}$ of X. For $U := \bigcap_{i=1}^n U_{x_i} \in \mathcal{N}_E(0)$ we have

$$U \cap V = \emptyset \text{ and}, \forall x \in X, \ \Phi((x + U) \cap X) \subseteq \Phi(X) \subset V.$$

That is $U \cap \Phi((x + U) \cap X) = \emptyset$ for all $x \in X$. A contradiction. □

Recall that a metrizable topological space X is an *absolute neighborhood retract* (*ANR* for short) if and only if whenever Y is metrizable and $A \subset Y$ is closed, every continuous mapping $f : A \longrightarrow X$ extends to a continuous mapping $f' : U \longrightarrow X$ defined over an open neighborhood U of A in Y. (If $U = Y$, X is said to be an *absolute retract (AR)*.) The Arens-Eells embedding theorem[20] asserts that any given ANR X is a neighborhood retract of some normed space E in which X is isometrically embedded as a closed subspace, (namely, E is the Banach space of all bounded continuous real-valued functions on X). Thus, given an ANR X, we may assume that X is a closed subset of a normed space $(E, \|.\|)$ with a *neighbourhood retraction* $r : U \longrightarrow X$ with U an open subset of E containing X. (If $U = E$, X is simply saidm to be an absolute retract of E.) For the sake of convenience, we shall make two crucial assumptions on the given ANR X.

(A1) X is compact. (A2) The neighbourhood retraction $r : B(X, \eta) \longrightarrow X$ satisfies the stronger regularity property:

$$\exists k > 0 \text{ such that } \|r(x) - x\| \leq k d_X(x) \text{ for all } x \in B(X, \eta).$$

Definition 3.15. An ANR satisfying (A1) and (A2) above is called a compact $L-$retract.

The class of L-retracts contains numerous subclasses of non-convex sets studied in non-smooth optimization (see [BK, B1] for various examples).

Remark 3.16. Note that (A1) implies that the neighbourhood U of X can be considered to be a uniform neighbourhood $B(X, \eta) := \{x' \in E : d_X(x') < \eta\}$ where $d_X(x') := inf_{x\in X}\|x' - x\|$. Moreover, the neighbourhood retraction $r : B(X, \eta) \longrightarrow X$ is uniformly continuous at X, that is,

$$\forall \varepsilon > 0, \exists 0 < \delta < \varepsilon \text{ such that } \forall x \in B(X, \delta), \forall x' \in X, \text{ we have } \|r(x) - r(x')\| = \|r(x) - x'\| < \varepsilon.$$

The *Euler-Poincaré characteristic* of a compact topological space X is denoted by $\chi(X)$[21].

Theorem 3.17. *Let X be a compact $L-$retract in a normed space E with $\chi(X) \neq 0$. Every tangential map $\Phi \in \mathbf{H}^*(X, E)$ has an equilibrium.*

Proof. Given any $0 < \varepsilon < \eta$, let $\delta < \frac{\varepsilon}{2k+1}$ be as in Remark 3.16, where k is the Lipschitz constant in condition (A2). Due to the properties of d_X^0 mentioned earlier[22], the map $\Psi : X \rightrightarrows E$ defined by the formula:

$$\Psi(x) = \{v \in E; d_X^0(x)(v) < \delta\}, x \in X,$$

has convex values and open graph. Thus, it is lsc. By hypothesis, $\Phi(x) \cap \Psi(x) \neq \emptyset$ for all $x \in X$. In view of Proposition 3.12, there exists a continuous mapping $s : X \longrightarrow X$ satisfying for all $x \in X$:

$$s(x) \in B(\Psi(x), \delta) \cap B(\Phi(B(x, \delta) \cap X), \delta).$$

Being continuous, s is bounded on X, say $\|s(x)\| \leq M$ for some $M > 0$. Choose $\tau > 0$ with $M\tau < \eta$ and consider a sequence $(t_n)_{n\in\mathbf{N}}$ in $(0, \tau], t_n \downarrow 0^+$. For each $n \in \mathbf{N}$, the map $\varphi_n : X \longrightarrow X$ given by:

$$\varphi_n(x) = r(x + t_n s(x)), x \in X,$$

is well-defined since, for each $x \in X, d_X(x+t_n s(x)) \leq \|x+t_n s(x)-x\| = t_n\|s(x)\| \leq \tau M < \delta$, that is $x + t_n s(x) \in B(X, \delta)$. For each $n \in \mathbf{N}$, the homotopy $h_n : X \times [0, 1] \longrightarrow X$ defined by:

$$h(x, \mu) := r(x + \mu t_n s(x)), (x, \mu) \in X \times [0, 1],$$

joins φ_n to the identity on X. Since X is compact and has a non-trivial Euler-Poincaré characteristic $\mathcal{E}(X) = \lambda(1_X)$ [23], it follows from the Lefschetz fixed point theorem (see [G]) that

[20] See [G] for the Arens-Eells theorem and for more on ARs and ANRs.

[21] See [B1, G, DG]. If X is a smooth submanifold of $\mathbb{R}^n$, John Milnor defines $\chi(X)$ as the Brouwer degree of the Gauss mapping $G_X(x)$ = the unit outward normal vector to X at $x \in \partial X$. For non-smooth sets, e.g., X is a compact epi-Lipschitzian subset of $\mathbb{R}^n$ (a special instance of an L-retract), B. Cornet defines the Gauss mapping in terms of proximal normal vectors: $G_X(x) = conv(N_X^C(x) \cap S^{n-1})$ and $\chi(X) = \deg(G_X, int(X), 0)$, the Cellina-Lasota degree. Generally, if X is a compact topological space, the singular cohomology $\{H^q(X)\}$ is a graded linear space of finite type. Denote $\dim_{\mathbb{Q}}(H^q(X)) = \beta_q(X)$ (the qth-Betti number) and define: $\chi(X) := \sum_q(-1)^q\beta_q(X)$. It turns out that $\chi(X) = \lambda(Id_X)$ the Lefschetz number of the identity mapping on X. A non-trivial Euler-Poincaré characteristic is necessary for the existence of equilibria on ANRs.

[22] Recall that $d_X^0(x)(v)$ is upper semicontinuous on $X \times E$ and convex in v; the Clarke tangent cone to X at $x \in X$ is characterized as $T_X^C(x) = \{v \in E; d_X^0(x)(v) = 0\}$.

[23] The Lefschetz number of the identity on X.

φ_n has a fixed point $x_n \in X$. Since X is compact, a subsequence of (x_n) (again denoted by (x_n)) converges to some $\overline{x} \in X$. In view of proposition 3.12, there exists $\overline{v} \in \Psi(\overline{x})$ such that $\|\overline{v} - s(\overline{x})\| < \delta$. Note that

$$\begin{array}{rcl} \|x_n + t_n s(x_n)\| & \leq & \|x_n + t_n\overline{v}\| + t_n\|s(x_n) - s(\bar{x})\| + t_n\|s(\bar{x}) - \overline{v}\| \\ & \Rightarrow & \\ d_X(x_n + t_n s(x_n)) & \leq & d_X(x_n + t_n\overline{v}) + t_n\|s(x_n) - s(\bar{x})\| + t_n\|s(\bar{x}) - \overline{v}\| \end{array}$$

Therefore,

$$\begin{aligned} t_n\|s(x_n)\| = \|t_n s(x_n)\| &= \|x_n + t_n s(x_n) - x_n\| \\ &= \|(x_n + t_n s(x_n) - r(x_n + t_n s(x_n))\| \\ &\leq k d_X(x_n + t_n s(x_n)) \\ \leq k[d_X(x_n + t_n\overline{v}) + t_n\|s(x_n) - s(\bar{x})\| &+ t_n\|s(\bar{x}) - \overline{v}\|] \end{aligned}$$

Thus, for any n,

$$\|s(x_n)\| < k[\frac{1}{t_n} d_X(x_n + t_n\overline{v}) + \|\overline{v} - s(\overline{x})\| + \|s(\overline{x}) - s(x_n)\|].$$

Letting $n \to \infty$, we obtain:

$$\begin{aligned} \|s(\overline{x})\| = \lim_{n\to\infty} \|s(x_n)\| &\leq k(\limsup_{n\to\infty} \tfrac{d_X(x_n + t_n\overline{v})}{t_n} + \|\overline{v} - s(\overline{x})\|) \\ &\leq k(c(\overline{x}, \overline{v}) + \|\overline{v} - s(\overline{x})\|) < 2k\delta < 2k(\varepsilon/(2k+1)) < \varepsilon. \end{aligned}$$

As s is a δ-approximation of Φ, $s(\overline{x}) \in B(\Phi(B(\overline{x}, \delta) \cap X), \delta)$, that is, $\overline{x}$ is a 2ε-almost equilibrium for Φ with ε arbitrarily chosen. Lemma 3.14 ends the proof. □

3.3 From Equilibria to Co-Equilibria

We assume for the sake of convenience that E is a real Hilbert space with inner product $\langle ., . \rangle$ as to have the identification $E' \simeq E$[24]. A *Hilbert space pair* (X, E) consists of a non-empty closed subset X of a real Hilbert space E.

Definition 3.18. Given a Hilbert space pair (X, E), a *co-equilibrium* $\bar{x} \in X$ for a map $\Phi : X \rightrightarrows E$ is an equilibrium point $0 \in \Phi(\bar{x}) - N_X(\bar{x})$ where $N_X(.)$ is a suitably chosen concept of *normal cone* to X.

The existence of an equilibrium to the sum map $\Phi(.) - N_X(.)$ would be facilitated if the map $N_X : X \rightrightarrows E'$ were to be a $\mathbf{K}^*$ map (usc with closed convex values). This is the case if X is Clarke regular with $N_X = N_X^C$ being the Clarke's normal cone (see Remark 2.18 (3)). An obvious alternate point of view is to consider a co-equilibrium $\bar{x} \in X$ for Φ as a coincidence point $\Phi(\bar{x}) \cap N_X(\bar{x}) \neq \emptyset$.

In the case where $N_X(\bar{x}) = T_X(\bar{x})^-$ is the negative polar cone of an associated suitable tangent cone, this coincidence would imply the infsup inequality:

$$\inf_{y\in\Phi(\bar{x})} \sup_{v\in T_X(\bar{x})} \langle y, v \rangle \leq 0.$$

Conversely, $\inf_{y\in\Phi(\bar{x})} \sup_{v\in T_X(\bar{x})} \langle y, v \rangle \leq 0$ implies that $\bar{x}$ is a co-equilibrium for Φ whenever $\Phi(\bar{x})$ is weakly compact. This follows from the fact that the extended real valued function $y \mapsto \sup_{v\in T_X(x)} \langle y, v \rangle$ is lower semicontinuous and convex, hence weakly lower semicontinuous. Consequently, it achieves its infimum on the weakly compact $\Phi(\bar{x})$ at some $\bar{y}$ verifying $\langle \bar{y}, v \rangle \leq 0, \forall v \in T_X(\bar{x})$, i.e. $\bar{y} \in N_X(\bar{x})$.

[24]Naturally, this identification holds not only for Hilbert spaces (Riesz-Fréchet), but also for some more general topological vector spaces. Extending the framework is left to the reader.

Therefore, a co-equilibrium $\bar{x}$ for Φ is also a solution to the quasi-variational inequality:

$$\exists \bar{x}, \bar{p} \in E \text{ such that } \bar{p} \in \Phi(\bar{x}) \text{ and } \langle \bar{p}, v\rangle \leq 0, \forall v \in T_X(\bar{x}).$$

In this section, we describe a simple and generic way to derive the existence of a co-equilibrium from that of an equilibrium.

Definition 3.19. A pair (X, E) consisting of a nonempty subset X in a (real) vector space E has *the equilibrium property* for an abstract class of maps $\mathbf{A}$ if and only if any map in $\mathbf{A}(X, E) := \{\Phi : X \rightrightarrows E : \Phi \in \mathbf{A}\}$ has an equilibrium in X. We write $(X, E) \in \mathcal{E}(\mathbf{A})$.

Theorem 3.20. *Let $(X, E) \in \mathcal{E}(\mathbf{H}^*_\tau)$ be a Hilbert space pair with X sleek. Then, any compact map $\Psi \in \mathbf{H}^*(X, E)$ has a co-equilibrium, i.e., $\exists \bar{x} \in X$ such that* $0 \in \Psi(\bar{x}) - N^C_X(\bar{x})$.

Proof. The image $\Psi(X)$ of Ψ is contained in a closed disk D centered at the origin with radius $M > 0$ in E. Consider the map $\Phi : X \rightrightarrows E$ given by $\Phi(x) := \Psi(x) - (N^C_X(x) \cap D)$. By Proposition 2.20 and since X is sleek, the Clarke's normal cone map $N^C_X : X \rightrightarrows E$ has closed graph.

By Remark 2.1 (1), since the graph of N^C_X is convex and the values $N^C_X(x) \cap D$ are closed, convex, and bounded, hence weakly compact, it follows that the map $x \mapsto N^C_X(x) \cap D$ is uhc with closed convex, and bounded values.

As a linear combination of uhc maps, Φ is also uhc. Being the sum of a compact convex set and a closed bounded convex set, $\Phi(x)$ is closed and convex for each $x \in X$, i.e., $\Phi \in \mathbf{H}^*(X, E)$.

It remains to show that Φ verifies (τ). For any given $x \in \partial X$, since the cone $T^C_X(x)$ is closed and convex, the Moreau decomposition theorem [Mo] implies that any $y \in \Psi(x)$ can be decomposed as a sum $y = y_T + y_N$ with $y_T = \mathit{Proj}_{T^C_X(x)}(y)$ and $y_N = \mathit{Proj}_{N^C_X(x)}(y)$ and $\langle y_N, y_T\rangle = 0$. Hence, $0 = \langle y_N, y_T\rangle = \langle y_N, y - y_N\rangle = \langle y_N, y\rangle - \|y_N\|^2$ and by the Cauchy-Schwarz-Bunyakowsky's inequality $\|y_N\| \leq \|y\| \leq M$, that is $y_T = y - y_N \in \Psi(x) - (N^C_X(x) \cap D)$, i.e., $\Phi(x) \cap T^C_X(x) \neq \emptyset$. The fact that (X, E) has the equilibrium property for $\mathbf{H}^*_\tau$ ends the proof. □

In view of the equilibrium Theorems 3.6 and 3.17, namely:

- Given any pair (X, E) consisting of a non-empty compact convex set X in a locally convex topological vector space E, we have $(X, E) \in \mathcal{E}(\mathbf{H}^*_\tau)$ where $\mathbf{H}^*_\tau$ is the class of $\mathbf{H}^*$ tangential maps (Theorem 3.6).
- Given any pair (X, E) consisting of a non-empty compact L-retract X in a normed space E, we have $(X, E) \in \mathcal{E}(\mathbf{H}^*_\tau)$ where $\mathbf{H}^*_\tau$ is the class of $\mathbf{H}^*$ tangential maps (Theorem 3.17).

Corollary 3.21. *Let (X, E) be a Hilbert space pair. Then every compact-valued map $\Psi \in \mathbf{H}^*(X, E)$ has a co-equilibrium if any one of the following assumptions holds:*
(i) X is convex compact. (ii) X is a sleek compact $L-$retract with $\chi(X) \neq 0$.

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