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IRREDUCIBLE GE-FILTERS AND GE-MORPHISMS IN GE-ALGEBRAS

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Abstract *The concepts of irreducible GE-filter and under-system are introduced and their properties are investigated in a GE-algebra. Given a GE-filter, the existence of irreducible GE-filter containing it is revealed.Conditions for a GE-filter to be irreducible are given. The irreducible GE-filter is used to characterize weak GE-morphism. The existence of irreducible GE-filter which contains a GE-filter and is disjoint with an under-system is established. Based on GE-morphism, the existence of irreducible GE-filter is discussed. Conditions that allow a weak GE-morphism to be a GE-morphism are explored, and a necessary and sufficient condition for a GE-morphism to be injective is given.*

1 Introduction

BCK-algebras were introduced in 1966 by Y. Imai and K. Iséki (see [8, 9]) as the algebraic semantics for a non-classical logic with only implication. Since then, a number of scholars have investigated generalized notions of BCK-algebras. In the 1950s, L. Henkin and T. Skolem developed Hilbert algebras for research into intuitionistic and other non-classical logics. A. Diego established that Hilbert algebras are a locally finite variety (see [6]). Later, several researchers expanded on Hilbert algebra theory (see [5, 7, 10, 11]). H. S. Kim and Y. H. Kim introduced the concept of BE-algebra as a generalization of a dual BCK-algebra (see [12]). M. B. Prabhakar et al. introduced the notion of ideals in transitive BE-algebras and some characterization theorems of ideals of transitive BE-algebras are derived(see [14]). V. V. Kumar et al. introduced the concept of radical of filters in a BE-algebra and certain properties of these radicals are derived in terms of direct products and homomorphisms (see [13]). A. Rezaei et al. investigated the connections between Hilbert algebras and BE-algebras (see [15]). In the study of algebraic structures, abstraction is an important methodology. R. K. Bandaru et al. introduced the concept of GE-algebras as a generalization of Hilbert algebras and investigated several properties (see [2]). The filter theory is important for the general development of GE-algebras. With this motivation, R. K. Bandaru et al. introduced and investigated the concept of belligerent GE-filters in GE-algebras (see [3]). A. Rezaei et al. introduced and discussed the concept of prominent GE-filters in GE-algebras (see [16]). A. Borumand Saeid et al. introduced the concept of voluntary GE-filters of GE-algebras and investigated its properties (see [4]). S. Z. Song et al. introduced the concept of imploring GE-filters of GE-algebras and discussed its properties (see [18]). Recently, R. K. Bandaru et al. introduced the notion of GE-morphism and established fundamental GE-morphism theorem. They investigated some isomorphism theorems in GE-algebras (see [17]). S. S. Ahn et al. introduced the notion of qualified GE-algebra using weak GE-morphism and its properties are investigated (see [1]).

In this paper, we introduce the concepts of irreducible GE-filter and under-system in a GE-algebra and investigate their properties. We reveal the existence of irreducible GE-filter con-

taining a given GE-filter. We provide conditions for a GE-filter to be irreducible, and we characterize weak GE-morphism by using the irreducible GE-filter. We establish the existence of irreducible GE-filter which contains a GE-filter and is disjoint with an under-system. Based on GE-morphism, we discuss the existence of irreducible GE-filter. We explore conditions that allow a weak GE-morphism to be a GE-morphism, and give a necessary and sufficient condition for a GE-morphism to be injective.

2 Preliminaries

Definition 2.1 ([2]). By a *GE-algebra* we mean a nonempty set $(X$ with a constant 1 and a binary operation "$*$" satisfying the following axioms:
(GE1) $u * u = 1$,
(GE2) $1 * u = u$,
(GE3) $u * (v * w) = u * (v * (u * w))$
for all $u, v, w \in X$.

Definition 2.2 ([2, 3]). A GE-algebra X is said to be

- *transitive* if it satisfies:

$$(\forall a, b, c \in X)\,(a * b \leq (c * a) * (c * b))\,. \tag{2.1}$$

- *left exchangeable* if it satisfies:

$$(\forall a, b, c \in X)\,(a * (b * c) = b * (a * c))\,. \tag{2.2}$$

- *antisymmetric* if the binary relation "$\leq$" is antisymmetric.

Proposition 2.3 ([2]). *Every GE-algebra X satisfies the following items.*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2.3}$$
$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{2.4}$$
$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{2.5}$$
$$(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,. \tag{2.6}$$
$$(\forall u \in X)\,(1 \leq u \;\Rightarrow\; u = 1)\,. \tag{2.7}$$
$$(\forall u, v \in X)\,(u \leq (v * u) * u)\,. \tag{2.8}$$
$$(\forall u, v \in X)\,(u \leq (u * v) * v)\,. \tag{2.9}$$
$$(\forall u, v, w \in X)\,(u \leq v * w \;\Leftrightarrow\; v \leq u * w)\,. \tag{2.10}$$

If X is transitive, then

$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; w * u \leq w * v,\; v * w \leq u * w)\,. \tag{2.11}$$
$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,. \tag{2.12}$$
$$(\forall u, v, w \in X)\,(u \leq v,\; v \leq w \;\Rightarrow\; u \leq w)\,. \tag{2.13}$$

Definition 2.4 ([2]). A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

$$1 \in F, \tag{2.14}$$
$$(\forall u, v \in X)(u \in F,\; u * v \in F \;\Rightarrow\; v \in F). \tag{2.15}$$

Lemma 2.5 ([2]). *In a GE-algebra X, every GE-filter F of X satisfies:*

$$(\forall a, b \in X)\,(a \leq b,\; a \in F \;\Rightarrow\; b \in F)\,. \tag{2.16}$$

Definition 2.6 ([4]). Let F be a subset of a GE-algebra X. The *GE-filter* of X generated by F is denoted by $\langle F\rangle$ and is defined to be the intersection of all GE-filters of X containing F.

Lemma 2.7 ([4]). *If F is a non-empty subset of an antisymmetric left exchangeable GE-algebra X, then $\langle F\rangle$ consists of x's that satisfy the following condition:*

$$(\exists a_1, a_2, \cdots, a_n \in F)\,(a_n * (\cdots * (a_2 * (a_1 * x)) \cdots) = 1)\,, \tag{2.17}$$

that is,

$$\langle F\rangle = \{x \in X \mid a_n * (\cdots * (a_2 * (a_1 * x)) \cdots) = 1 \text{ for some } a_1, a_2, \cdots, a_n \in F\}.$$

In [16], the concept of GE-morphisms in GE-algebras is defined as follows:

Definition 2.8 ([16]). Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras. A mapping $\xi : X \to Y$ is called a *GE-morphism* if it satisfies:

$$(\forall a_1, a_2 \in X)(\xi(a_1 *_X a_2) = \xi(a_1) *_Y \xi(a_2)). \tag{2.18}$$

If a GE-morphism $\xi : X \to Y$ is onto (resp., one-to-one), we say it is a GE-epimorphism (resp., GE-monomorphism). If a GE-morphism $\xi : X \to Y$ is both onto and one-to-one, we say it is a GE-isomorphism.

If $X = Y$ in the GE-morphism $\xi : X \to Y$, we say $\xi : X \to X$ is a GE-endomorphism.

Definition 2.9 ([1]). Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras. A mapping $\xi : X \to Y$ is called a *weak GE-morphism* if it satisfies:

$$(\forall a_1, a_2 \in X)(\xi(a_1 *_X a_2) \leq_Y \xi(a_1) *_Y \xi(a_2)), \tag{2.19}$$

$$1_Y \leq_Y \xi(1_X). \tag{2.20}$$

If $X = Y$, the weak GE-morphism $\xi : X \to X$ is called a weak GE-endomorphism.

If a weak GE-morphism $\xi : X \to Y$ is onto (resp., one-to-one), we say it is a weak GE-epimorphism (resp., weak GE-monomorphism). If a weak GE-morphism $\xi : X \to Y$ is both onto and one-to-one, we say it is a weak GE-isomorphism.

Every GE-morphism is a weak GE-morphism, but not converse (see [1]).

Lemma 2.10 ([1]). *Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras. Given a weak GE-morphism $\xi : X \to Y$, we have*

(i) $\xi(1_X) = 1_Y$.

(ii) $(\forall x, y \in X)\ (x \leq_X y \Rightarrow \xi(x) \leq_Y \xi(y))$.

(iii) $(\forall x, y \in X)\ (\xi(x *_X y) \leq_Y \xi((x *_X y) *_X y) *_Y \xi(y))$.

(iv) *The set* $\mathrm{Ker}(\xi) := \{x \in X \mid \xi(x) = 1_Y\}$*, which is called the kernel of ξ, is a GE-filter of X.*

(v) *The inverse image $\xi^{-1}(F_Y)$ of a GE-filter F_Y of Y under ξ is a GE-filter of X.*

3 Irreducible GE-filters and under-systems

In what follows, let X denote a GE-algebra $(X, *, 1)$ unless otherwise specified.

Definition 3.1. A GE-filter F of X is said to be *irreducible* if $F = G \cap H$ implies $F = G$ or $F = H$ for all GE-filters G and H of X.

Example 3.2. (1) Let $X = \{1, a, b, c, d, e\}$ be a set with a binary operation "$*$" given in the next table:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	c	e
b	1	a	1	d	d	e
c	1	1	b	1	1	e
d	1	1	1	1	1	e
e	1	a	b	c	d	1

Then X is a GE-algebra. Clearly $\{1\}$, $\{1, b\}$, $\{1, e\}$, $\{1, a, b\}$, $\{1, b, e\}$, $\{1, a, b, e\}$ and X are all GE-filters of X, and $\{1, a, b, e\}$ is an irreducible GE-filter of X.

(2) Let $X = \{1, a, b, c, d, e\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	1	1	1
b	1	e	1	1	d	e
c	1	a	1	1	a	a
d	1	1	1	c	1	1
e	1	1	b	1	1	1

Then X is a GE-algebra. We know that $\{1\}$, $\{1, b, c\}$ and X are GE-filters of X, and $\{1, b, c\}$ is an irreducible GE-filter of X.

Theorem 3.3. *Let F be a GE-filter of X and let $x \in X \setminus F$. Then there exists an irreducible GE-filter G of X that satisfies $F \subseteq G$ and $x \notin G$.*

Proof. Consider the set

$$\mathcal{F} := \{H \subseteq X \mid H \text{ is a GE-filter of } X, F \subseteq H, x \notin H\}.$$

Then $\mathcal{F}$ is a poset under the subset relation ($\subseteq$) and every chain of elements in $\mathcal{F}$ has an upper bound. Hence, by Zorn's Lemma, there exists a maximal element G in $\mathcal{F}$. Thus $F \subseteq G$ and $x \notin G$. Let A and B be GE-filters of X such that $G = A \cap B$. If $G \neq A$ and $G \neq B$, then $x \in A \cap B$ by the maximality of G, and so $G \neq A \cap B$. This is a contradiction, and hence $G = A$ or $G = B$. Therefore G is irreducible. □

Note that $\langle a \rangle$ is the GE-filter of X generated by $a \in X$. If $y \nleq x$ in X, then $\langle y \rangle$ is a GE-filter of X which does not contain x. Hence we have the corollary below.

Corollary 3.4. *If $y \nleq x$ in X, then there exists an irreducible GE-filter F of X such that $y \in F$ and $x \notin F$.*

We provide conditions for a GE-filter to be irreducible.

Theorem 3.5. *If a GE-filter G of X satisfies:*

$$(\forall x, y \in X \setminus G)(\exists z \in X \setminus G)(x \leq z,\ y \leq z), \tag{3.1}$$

then G is an irreducible GE-filter of X.

Proof. Assume that G is not irreducible. Then there exist two GE-filters A and B of X such that $G = A \cap B$, $G \neq A$ and $G \neq B$. If $x \in A \setminus G$ and $y \in B \setminus G$, then there exists $z \in X \setminus G$ such that $x \leq z$ and $y \leq z$ by assumption. It follows from Lemma 2.5 that $z \in A \cap B = G$, which is a contradiction. Hence G is irreducible. □

The converse of Theorem 3.5 may not be true as shown in the example below.

Example 3.6. In Example 3.2(1), we can observe that $G = \{1, a, b\}$ is an irreducible GE-filter of X. But G does not satisfies (3.1) because of $e * c = c \neq 1$, $e * d = d \neq 1$, and $c * e = e \neq 1$.

Theorem 3.7. *If a GE-filter F of X satisfies:*

$$(\forall x, y \in X \setminus F)(\exists z \in X \setminus F)(x * z \in F,\ y * z \in F), \tag{3.2}$$

then F is an irreducible GE-filter of X.

Proof. Let G and H be GE-filters of X such that $F = G \cap H$. If $F \neq G$ and $F \neq H$, then there exists $a \in G \setminus F \subseteq X \setminus F$ and $b \in H \setminus F \subseteq X \setminus F$. Hence there exists $c \in X \setminus F$ such that $a * c \in F$ and $b * c \in F$ by (3.2). Since $a \in G$ and $b \in H$, it follows that $c \in G \cap H = F$, a contradiction. Therefore $F = G$ or $F = H$ which shows that F is an irreducible GE-filter of X. □

The converse of Theorem 3.7 may not be true as shown in the example below.

Example 3.8. In Example 3.2(1), we can observe that $G = \{1, a, b\}$ is an irreducible GE-filter of X. But G does not satisfies (3.2) because of $e * c = c \notin F$, $e * d = d \notin F$, and $c * e = e \notin F$.

Theorem 3.9. *Let $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ be GE-algebras. Then a mapping $\xi : X \to Y$ is a weak GE-morphism if and only if the inverse image $\xi^{-1}(G)$ of a GE-filter G of Y under ξ is a GE-filter of X.*

Proof. Assume that ξ is a weak GE-morphism and let G be a GE-filter of Y. Then $1_X \in \xi^{-1}(G)$. Let $x, y \in X$ be such that $x \in \xi^{-1}(G)$ and $x *_X y \in \xi^{-1}(G)$. Then $\xi(x) \in G$ and $\xi(x *_X y) \in G$. It follows from (2.15), Lemma 2.5 and (2.19) that $\xi(y) \in G$, that is, $y \in \xi^{-1}(G)$. Hence $\xi^{-1}(G)$ is a GE-filter of X.

Conversely, suppose that $\xi^{-1}(G)$ is a GE-filter of X for every GE-filter G of Y. If $\xi(1_X) \neq 1_Y$, then there exists a GE-filter B of Y such that $\xi(1_X) \notin B$, i.e., $1_X \notin \xi^{-1}(B)$. This is a contradiction, and so $\xi(1_X) = 1_Y$. If (2.19) is not valid, then $\xi(a_1 *_X a_2) \not\leq \xi(a_1) *_Y \xi(a_2)$ for some $a_1, a_2 \in X$. It follows from Corollary 3.4 that there exists an irreducible GE-filter H of Y such that $\xi(a_1 *_X a_2) \in H$ and $\xi(a_1) *_Y \xi(a_2) \notin H$. Consider the GE-filter $\langle H \cup \{\xi(a_1)\}\rangle$ of Y generated by $H \cup \{\xi(a_1)\}$. If $\xi(a_2) \in \langle H \cup \{\xi(a_1)\}\rangle$, then

$$b_n *_Y (\cdots *_Y (b_1 *_Y (\xi(a_1) *_Y \xi(a_2)) \cdots) = 1_Y \in H$$

for some $b_1, b_2, \cdots, b_n \in H$. Since H is a GE-filter of Y, it follows that $\xi(a_1) *_Y \xi(a_2) \in H$. This is impossible, and thus $\xi(a_2) \notin \langle H \cup \{\xi(a_1)\}\rangle$. Hence there exists an irreducible GE-filter I of Y such that $\langle H \cup \{\xi(a_1)\}\rangle \subseteq I$ and $\xi(a_2) \notin I$ by Theorem 3.3. It follows that $H \subseteq I$, $a_1 \in \xi^{-1}(I)$ and $a_2 \notin \xi^{-1}(I)$. Since $a_1 *_X a_2 \in \xi^{-1}(H) \subseteq \xi^{-1}(I)$ and $\xi^{-1}(I)$ is a GE-filter of X by assumption, we have $a_2 \in \xi^{-1}(I)$. This is a contradiction, and hence (2.19) is valid. Therefore $\xi : X \to Y$ is a weak GE-morphism. □

Definition 3.10. A subset U of X is called an *under-system* of X if it satisfies:

$$(\forall x, y \in X)(y \in U,\, x \leq y \;\Rightarrow\; x \in U), \tag{3.3}$$

$$(\forall x, y \in U)(\exists z \in U)(x \leq z,\, y \leq z). \tag{3.4}$$

Example 3.11. Let $X = \{1, a, b, c, d\}$ be a set with the binary operation " $*$ " given in the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	1
b	1	d	1	c	d
c	1	a	b	1	a
d	1	1	1	c	1

Then X is a GE-algebra and it can be easily observed that the set $U = \{a, b, d\}$ is an under-system of X.

Theorem 3.12. *Let X be a transitive GE-algebra. Given a GE-filter F and an under-system U of X, if F and U are disjoint, then there is an irreducible GE-filter G of X that includes F and is disjoint with U.*

Proof. Let $\mathcal{X}$ denote the set of all GE-filters of X that includes F and is disjoint with U, that is,

$$\mathcal{X} := \{H \in \mathcal{F}(X) \mid F \subseteq H,\, H \cap U = \emptyset\}$$

where $\mathcal{F}(X)$ is the collection of all GE-filters of X. It is clear that the union of a chain of elements of $\mathcal{X}$ is contained in $\mathcal{X}$. Zorn's lemma shows that $\mathcal{X}$ has a maximal element, say G. For every $a, b \in X \setminus G$, consider GE-filters $\langle G \cup \{a\}\rangle$ and $\langle G \cup \{b\}\rangle$ generated by $G \cup \{a\}$ and $G \cup \{b\}$, respectively. Obviously $G \subseteq \langle G \cup \{a\}\rangle \cap \langle G \cup \{b\}\rangle$. If $\langle G \cup \{a\}\rangle \cap U = \emptyset$ or $\langle G \cup \{b\}\rangle \cap U = \emptyset$, then $\langle G \cup \{a\}\rangle \in \mathcal{X}$ or $\langle G \cup \{b\}\rangle \in \mathcal{X}$. This is a contradiction, and so $\langle G \cup \{a\}\rangle \cap U \neq \emptyset$ and $\langle G \cup \{b\}\rangle \cap U \neq \emptyset$. Hence there exist $x, y \in X$ be such that $x \in \langle G \cup \{a\}\rangle \cap U$ and $y \in \langle G \cup \{b\}\rangle \cap U$. It follows that $a * x \in G$ and $b * y \in G$. Since $x, y \in U$ and U is an under-system of X, there exists $z \in U$ such that $x \leq z$ and $y \leq z$. It follows from (2.11) that $a * x \leq a * z$ and $b * y \leq b * z$. Thus $a * z \in G$ and $b * z \in G$. Using Theorem 3.7, we conclude that G is an irreducible GE-filter of X. This completes the proof. □

Lemma 3.13. *Let $\xi : X \to Y$ be a weak GE-morphism of GE-algebras $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$. If Y is transitive and F is an irreducible GE-filter of X, then the set*

$$(\xi(X \setminus F)] := \{y \in Y \mid y \leq \xi(a) \text{ for some } a \in X \setminus F\} \tag{3.5}$$

is an under-system of Y.

Proof. Let $y, b \in Y$ be such that $y \leq b$ and $b \in (\xi(X \setminus F)]$. Then $b \leq \xi(a)$ for some $a \in X \setminus F$. Since Y is transitive, we use (2.13) to get $y \leq \xi(a)$ and it reaches $y \in (\xi(X \setminus F)]$. Let $y, z \in (\xi(X \setminus F)]$. Then there exist $b, c \in X \setminus F$ such that $y \leq \xi(b)$ and $z \leq \xi(c)$. It is clear that $b, c \in (\xi(X \setminus F)]$. Therefore $(\xi(X \setminus F)]$ is an under-system of Y. □

Theorem 3.14. *Let $\xi : X \to Y$ be a GE-morphism of GE-algebras $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ in which Y is transitive. For every irreducible GE-filters F and G of X and Y, respectively, if $\xi^{-1}(G) \subseteq F$, then there exists an irreducible GE-filter H of Y such that $G \subseteq H$ and $\xi^{-1}(H) = F$.*

Proof. Let $\xi : X \to Y$ be a GE-morphism and suppose that $\xi^{-1}(G) \subseteq F$ for every irreducible GE-filters F and G of X and Y, respectively. Consider the GE-filter $\langle G \cup \xi(F)\rangle$ of Y generated by $G \cup \xi(F)$. According to Lemma 3.13, $(\xi(X \setminus F)]$ is an under-system of Y. If $\langle G \cup \xi(F)\rangle$ and $(\xi(X \setminus F)]$ are not disjoint, then there exists $b \in (\xi(X \setminus F)] \cap \langle G \cup \xi(F)\rangle$. Hence $b \leq \xi(a)$ for some $a \in X \setminus F$ and

$$\xi(a_1) *_Y (\cdots *_Y (\xi(a_{n-1}) *_Y (\xi(a_n) *_Y b)) \cdots) \in G \tag{3.6}$$

for some $a_1, \cdots, a_{n-1}, a_n \in F$. Using (2.11) and (2.18), we get

$$\begin{aligned}
&\xi(a_1) *_Y (\cdots *_Y (\xi(a_{n-1}) *_Y (\xi(a_n) *_Y b)) \cdots) \\
&\leq \xi(a_1) *_Y (\cdots *_Y (\xi(a_{n-1}) *_Y (\xi(a_n) *_Y \xi(a))) \cdots) \\
&= \xi(a_1 *_X (\cdots *_X (a_{n-1} *_X (a_n *_X a)) \cdots)).
\end{aligned}$$

Since G is a GE-filter of Y, it follows from (3.6) and Lemma 2.5 that

$$\xi(a_1 *_X (\cdots *_X (a_{n-1} *_X (a_n *_X a)) \cdots)) \in G. \tag{3.7}$$

Hence $a_1 *_X (\cdots *_X (a_{n-1} *_X (a_n *_X a)) \cdots) \in \xi^{-1}(G) \subseteq F$, and so $a \in F$. This is a contradiction, and thus $\langle G \cup \xi(F)\rangle$ and $(\xi(X \setminus F)]$ are disjoint. By Theorem 3.12, there is an irreducible GE-filter H of Y such that $\langle G \cup \xi(F)\rangle \subseteq H$ and $(\xi(X \setminus F)] \cap H = \emptyset$. It follows that $G \subseteq H$ and $\xi(F) \subseteq H$ so that $F \subseteq \xi^{-1}(H)$. If $x \in \xi^{-1}(H)$, then $\xi(x) \in H$ and thus $\xi(x) \notin (\xi(X \setminus F)]$. Hence $x \in F$, which shows that $\xi^{-1}(H) \subseteq F$. This completes the proof. □

We recall that weak GE-morphism may not be GE-morphism (see [1]). So we explore the conditions under which weak GE-morphism can be GE-morphism.

Theorem 3.15. *Let $\xi : X \to Y$ be a weak GE-morphism of GE-algebras $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ in which Y is antisymmetric. Suppose that for every irreducible GE-filters F and G of X and Y, respectively, if $\xi^{-1}(G) \subseteq F$, then there exists an irreducible GE-filter H of Y such that $G \subseteq H$ and $\xi^{-1}(H) = F$. Then ξ is a GE-morphism.*

Proof. Let $a, b \in X$ be such that $\xi(a) *_Y \xi(b) \not\leq_Y \xi(a *_X b)$. Then there exists an irreducible GE-filter G of Y such that $\xi(a) *_Y \xi(b) \in G$ and $\xi(a *_X b) \notin G$ by Corollary 3.4. Since ξ is a weak GE-morphism, it follows from Theorem 3.9 that $\xi^{-1}(G)$ is a GE-filter of X. Let $\langle \xi^{-1}(G) \cup \{a\}\rangle$ be the GE-filter of X generated by $\xi^{-1}(G) \cup \{a\}$. Then $b \notin \langle \xi^{-1}(G) \cup \{a\}\rangle$ because if not then

$a *_X b \in \xi^{-1}(G)$ which is a contradiction. Thus there exists an irreducible GE-filter F of X such that $\langle \xi^{-1}(G) \cup \{a\}\rangle \subseteq F$ and $b \notin F$ by Theorem 3.3, and so $\xi^{-1}(G) \subseteq F$, $a \in F$ and $b \notin F$. It can be seen that there is an irreducible GE-filter H of Y that satisfies $G \subseteq H$ and $\xi^{-1}(H) = F$ due to the assumption given in the theorem. Since $\xi(a) *_Y \xi(b) \in G \subseteq H$ and $\xi(a) \in \xi(F) \subseteq H$, we get $\xi(b) \in H$ by (2.15). This is a contradiction, and so $\xi(x) *_Y \xi(y) \leq_Y \xi(x *_X y)$ for all $x, y \in X$. Therefore ξ is a GE-morphism. □

Theorem 3.16. *Given a GE-morphism $\xi : X \to Y$ of GE-algebras $(X, *_X, 1_X)$ and $(Y, *_Y, 1_Y)$ in which Y is transitive, let*

$$\Gamma := \{G \in \mathcal{IF}(Y) \mid \xi^{-1}(G) \in \mathcal{IF}(X)\} \tag{3.8}$$

where $\mathcal{IF}(X)$ (resp., $\mathcal{IF}(Y)$) is the set of all irreducible GE-filters of X (resp., Y), and consider a mapping $f : \Gamma \to \mathcal{IF}(X)$, $G \mapsto \xi^{-1}(G)$. Then ξ is injective if and only if f is surjective.

Proof. Assume that ξ is injective. Let $F \in \mathcal{IF}(X)$ and consider the GE-filter $\langle \xi(F)\rangle$ of Y. If $\langle \xi(F)\rangle \cap (\xi(X \setminus F)] \neq \emptyset$, then there exists $y \in \langle \xi(F)\rangle \cap (\xi(X \setminus F)]$. Then $y \leq \xi(b)$ for some $b \in X \setminus F$ and

$$\xi(a_1) *_Y (\cdots *_Y (\xi(a_{n-1}) *_Y (\xi(a_n) * y)) \cdots) = 1_Y$$

for some $a_1, \cdots, a_{n-1}, a_n \in F$. It follows from (2.11) that

$$\begin{aligned} 1_Y &= \xi(a_1) *_Y (\cdots *_Y (\xi(a_{n-1}) *_Y (\xi(a_n) * y)) \cdots) \\ &\leq \xi(a_1) *_Y (\cdots *_Y (\xi(a_{n-1}) *_Y (\xi(a_n) * \xi(b))) \cdots) \end{aligned}$$

and so

$$\begin{aligned} &\xi(a_1 *_Y (\cdots *_Y (a_{n-1} *_Y (a_n * b)) \cdots)) \\ &\xi(a_1) *_Y (\cdots *_Y (\xi(a_{n-1}) *_Y (\xi(a_n) * \xi(b))) \cdots) \\ &= 1_Y = \xi(1_X). \end{aligned}$$

Since ξ is injective, we obtain

$$a_1 *_Y (\cdots *_Y (a_{n-1} *_Y (a_n * b)) \cdots) = 1_X \in F,$$

and thus $b \in F$. This is impossible, and therefore $\langle \xi(F)\rangle$ and $(\xi(X \setminus F)]$ are disjoint. Hence there exists $G \in \mathcal{IF}(Y)$ such that $\xi(F) \subseteq G$ and $(\xi(X \setminus F)] \cap G = \emptyset$ by Theorem 3.12. This leads to $\xi^{-1}(G) = F$, and so f is surjective.

Conversely, suppose that f is surjective and let $a, b \in X$ be such that $a \nleq_X b$. Then there exists an irreducible GE-filter F of X such that $a \in F$ and $b \notin F$ by Corollary 3.4. Since f is surjective, we have

$$(\exists\, G \in \Gamma \subseteq \mathcal{IF}(Y))(\xi^{-1}(G) = F).$$

Hence $a \in \xi^{-1}(G)$ and $b \notin \xi^{-1}(G)$, that is, $\xi(a) \in G$ and $\xi(b) \notin G$. It follows that $\xi(a) \nleq_Y \xi(b)$, and hence ξ is injective. □

4 Conclusion

In mathematics, the concept of irreducibility is used in several ways such as in polynomial over a field, in representation theory, in commutative algebra, in matrix, in Markov chain, in the theory of manifolds, in topological space, in universal algebra, and in 3-manifold, etc. The aim of this paper is to introduce the notion of irreducible GE-filter in a GE-algebra and to investigate their properties. We have discussed the existence of irreducible GE-filter containing a given GE-filter. We have provided conditions for a GE-filter to be irreducible, and we have characterized weak GE-morphism by using the irreducible GE-filter. We have defined under-system in GE-algebras and used it to establish the existence of irreducible GE-filter which contains a GE-filter and is disjoint with the under-system. Based on GE-morphism, we have considered the existence of irreducible GE-filter. We have explored conditions that allow a weak GE-morphism to be a GE-morphism, and have provided a necessary and sufficient condition for a GE-morphism to be injective.

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