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ON THE GENERALIZED OF GLOBAL VERSION OF CHEN'S BIHARMONIC CONJECTURE

A. Mohammed Cherif

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Abstract In this paper, we prove that every complete biharmonic submanifold (M, g) satisfies

$$\int_M |H|^2 f(|H|^2)\, v^g < \infty,$$

for some smooth strictly positive increasing real function f, in a Riemannian manifold (N, h) with negative sectional curvature is harmonic, i.e., $H = 0$, where H is the mean curvature vector field of (M, g) in (N, h).

1 Introduction

The energy functional of a smooth map $\varphi : (M, g) \longrightarrow (N, h)$ between two Riemannian manifolds is defined by

$$E(\varphi; D) = \frac{1}{2}\int_D |d\varphi|^2 v^g, \tag{1.1}$$

where D is compact domain of M, $|d\varphi|$ is the Hilbert-Schmidt norm of the differential $d\varphi$, and v^g is the volume element on (M, g). A map φ is called harmonic if it is a critical point of the energy functional (1.1). The Euler Lagrange equation associated to (1.1) is given by (see [2, 5, 11])

$$\tau(\varphi) = \operatorname{Tr}_g \nabla d\varphi = \sum_{i=1}^{m} \left[\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i)\right] = 0, \tag{1.2}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on (M, g), ∇^M is the Levi-Civita connection of (M, g), ∇^{φ} denote the pull-back connection on $\varphi^{-1}TN$, and m is the dimension of M. A natural generalization of harmonic maps is given by integrating the square of the norm of the tension field. More precisely, the bienergy functional of a map $\varphi \in C^\infty(M, N)$ is defined by

$$E_2(\varphi; D) = \frac{1}{2}\int_D |\tau(\varphi)|^2 v^g. \tag{1.3}$$

A map $\varphi \in C^\infty(M, N)$ is called biharmonic if it is a critical point of the bienergy functional, that is, if it is a solution of the Euler Lagrange equation associated to (1.3)

$$\begin{aligned}\tau_2(\varphi) &= -\operatorname{Tr}_g R^N(\tau(\varphi), d\varphi)d\varphi - \operatorname{Tr}_g(\nabla^{\varphi})^2\tau(\varphi) \\ &= -\sum_{i=1}^{m} R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^{m}\left[\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau(\varphi)\right. \\ &\quad \left.-\nabla^{\varphi}_{\nabla^M_{e_i}e_i}\tau(\varphi)\right] = 0, \end{aligned} \tag{1.4}$$

where R^N is the curvature tensor of (N,h) defined by

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z,$$

where ∇^N is the Levi-Civita connection of (N,h), and $X,Y,Z \in \Gamma(TN)$ (see [6, 11]).
Let M be a submanifold in (N,h) of dimension m, $\mathbf{i} : M \hookrightarrow (N,h)$ the canonical inclusion, and let $\{e_i\}_{i=1}^m$ be a local orthonormal frame field with respect to induced Riemannian metric g on M by h. We denote by ∇^N (resp. ∇^M) the Levi-Civita connection of (N,h) (resp. of (M,g)), by B the second fundamental form of the submanifold (M,g), by H the mean curvature vector field of (M,g) in (N,h) (see [2], [10]). The tension field is given by $\tau(\mathbf{i}) = mH$. (M,g) is called a harmonic (resp. biharmonic) submanifold in (N,h) if $\tau(\mathbf{i}) = 0$ (resp. $\tau_2(\mathbf{i}) = 0$).

In [8] and [9], N. Nakauchi and H. Urakawa proved that, if (M,g) is a complete biharmonic submanifold in (N,h) with non-positive sectional curvature, and if

$$\int_M |H|^2\, v^g < \infty,$$

then (M,g) it is harmonic. In this paper, we shall extend the previous result as follows.

Theorem 1.1. *Let (M,g) be a complete biharmonic submanifold in (N,h) with negative sectional curvature. We assume that there exists a smooth function $f : \mathbb{R} \longrightarrow (0,\infty)$ increasing such that*

$$\int_M |H|^2 f(|H|^2)\, v^g < \infty. \tag{1.5}$$

Then (M,g) is harmonic.

2 Proof of Main Theorem

Let H the mean curvature vector field of (M,g) in (N,h), ρ a smooth positive function with compact support on M. We set

$$\xi = \rho^2 f(|H|^2)\,\mathrm{grad}\,|H|^2, \tag{2.1}$$

where f is a real positive smooth function with $f(|H|^2) = f\circ|H|^2$. Let $\{X_i\}_{i=1}^m$ be a geodesic frame field at $x \in M$. We have

$$\mathrm{div}\,\xi = \sum_{i=1}^m X_i\big(\rho^2 f(|H|^2) X_i(|H|^2)\big). \tag{2.2}$$

By using $X\left(f(|H|^2)\right) = f'(|H|^2)X(|H|^2)$ for all $X \in \Gamma(TM)$, where f' is the derivative function of f, we get

$$\begin{aligned}\mathrm{div}\,\xi \;&=\; \sum_{i=1}^m \big[2\rho X_i(\rho) f(|H|^2)X_i(|H|^2) + \rho^2 f'(|H|^2)X_i(|H|^2)X_i(|H|^2)\\ &\quad +\rho^2 f(|H|^2)X_i\left(X_i(|H|^2)\right)\big],\end{aligned} \tag{2.3}$$

it is equivalent to the following equation

$$\begin{aligned}\mathrm{div}\,\xi \;&=\; 2\rho f(|H|^2) g(\mathrm{grad}\,\rho, \mathrm{grad}\,|H|^2) + \rho^2 f'(|H|^2)|\,\mathrm{grad}\,|H|^2|^2\\ &\quad +\rho^2 f(|H|^2)\Delta|H|^2.\end{aligned} \tag{2.4}$$

From the Young's inequality, we obtain

$$-2\rho f(|H|^2)g(\mathrm{grad}\,\rho, \mathrm{grad}\,|H|^2) \le \lambda\rho^2 f(|H|^2)^2|\,\mathrm{grad}\,|H|^2|^2 + \frac{1}{\lambda}|\,\mathrm{grad}\,\rho|^2, \tag{2.5}$$

for all continuous function $\lambda > 0$ on M. By using the biharmonicity condition of (M, g), and since the sectional curvature of (N, h) is negative, we conclude that

$$\Delta|H|^2 \geq 2m|H|^4 + 2|\nabla^{\perp} H|^2, \tag{2.6}$$

where $\nabla^{\perp}$ is the normal connection of (M, g) (see [1, 7]). So, from equation (2.4), and inequality (2.5), we deduce

$$\rho^2\big[f'(|H|^2) - \lambda f(|H|^2)^2\big]|\operatorname{grad}|H|^2|^2 + 2m\rho^2|H|^4 f(|H|^2)$$
$$+2\rho^2 f(|H|^2)|\nabla^{\perp} H|^2 - \operatorname{div}\xi \leq \frac{1}{\lambda}|\operatorname{grad}\rho|^2. \tag{2.7}$$

Take $M_+ = \{x \in M\,,\, |H|_x > 0\}$. We assume that $M_+ \neq \emptyset$. By using the divergence Theorem (see [2]), and the inequality (2.7), we have

$$\int_{M_+} \rho^2\big\{f'(|H|^2) - \lambda f(|H|^2)^2\big\}|\operatorname{grad}|H|^2|^2\, v^g + 2m\int_{M_+} \rho^2|H|^4 f(|H|^2)\, v^g$$
$$+2\int_{M_+} \rho^2 f(|H|^2)|\nabla^{\perp} H|^2\, v^g \leq \int_{M_+} \frac{1}{\lambda}|\operatorname{grad}\rho|^2\, v^g, \tag{2.8}$$

A direct calculation shows that

$$|\operatorname{grad}|H|^2|^2 \leq 4|H|^2|\nabla^{\perp} H|^2, \tag{2.9}$$

on M_+. According to inequalities (2.8) and (2.9) with

$$\lambda = \frac{1}{2|H|^2 f(|H|^2)},$$

on M_+, we obtain the following

$$\int_{M_+} \rho^2 f'(|H|^2)|\operatorname{grad}|H|^2|^2\, v^g + 2m\int_{M_+} \rho^2|H|^4 f(|H|^2)\, v^g$$
$$\leq 2\int_{M_+} |H|^2 f(|H|^2)|\operatorname{grad}\rho|^2\, v^g. \tag{2.10}$$

Let $\rho : M \longrightarrow [0,1]$ be a smooth cut-off function with $\rho = 1$ on $B_R(x)$, $\rho = 0$ off $B_{2R}(x)$ and $|\operatorname{grad}\rho| \leq \frac{2}{R}$. We suppose that the derivative function $f' \geq 0$. From inequality (2.10), we find that

$$m\int_{B_R(x)\cap M_+} |H|^4 f(|H|^2)\, v^g \leq \frac{4}{R^2}\int_{B_{2R}(x)\cap M_+} |H|^2 f(|H|^2)\, v^g. \tag{2.11}$$

Since $\int_M |H|^2 f(|H|^2)\, v^g < \infty$, when $R \to \infty$, we obtain

$$\int_{M_+} |H|^4 f(|H|^2)\, v^g = 0. \tag{2.12}$$

This contradicts our assumption. Hence $M_+ = \emptyset$. Therefore, (M, g) is harmonic.

□

Theorem 1.1 it is considered as an affirmative answer under the condition (1.5) to the generalized of global version of Chen's biharmonic conjecture (see [1, 3, 4, 9]).

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Author information

A. Mohammed Cherif, Faculty of Exact Sciences, University Mustapha Stambouli, Mascara, Algeria.
E-mail: `a.mohammedcherif@univ-mascara.dz`

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